ON L. SCHWARTZ'S BOUNDEDNESS CONDITION FOR KERNELS

T. CONSTANTINESCU AND A. GHEONDEA

ABSTRACT. In previous works we analyzed conditions for linearization of hermitian kernels. The conditions on the kernel turned out to be of a type considered previously by L. Schwartz in the related matter of characterizing the real linear space generated by positive definite kernels. The aim of the present note is to find more concrete expressions of the Schwartz type conditions: in the Hamburger moment problem for Hankel type kernels on the free semigroup, in dilation theory (Stinespring type dilations and Haagerup decomposability), as well as in multi-variable holomorphy. Among other things, we prove that any hermitian holomorphic kernel has a holomorphic linearization, and hence that hermitian holomorphic kernels automatically satisfy L. Schwartz's boundedness condition.

1. INTRODUCTION

We analyzed in [7] conditions under which the linearization functor produces a Kreĭn space from a hermitian kernel, in the spirit of Kolmogorov type decompositions, and subsequently in [8] we generalized this construction to kernels invariant under the action of a semigroup with involution. We also related these constructions with the GNS representations of $*$-algebras, an issue of some recent interest in quantum field theory with indefinite metric ([2], [14], [13], [19]). The conditions on the kernel turned out to be of a type considered previously by L. Schwartz in [18] in the related matter of characterizing those hermitian kernels that are in the real linear space generated by positive definite kernels. These boundedness conditions are rather difficult to be verified, see [2],[13], and their nature is quite obscure, see [19].

The aim of the present note is to find more concrete expressions of the L. Schwartz type conditions. We first note that the invariant Kolmogorov decomposition has a counterpart in the representation theory of semigroups with involution on reproducing kernel Kreĭn spaces. Then, it is explained that the type of invariance that is considered in [8] can be viewed as a Hankel type condition and we apply this to a Hamburger moment problem for the free semigroup on N generators. This is used in order to show how the Schwartz condition is somewhat simplified when it is written for generators of a $*$-algebra.

In Section 4 we first show that the Stinespring dilation of hermitian linear maps fits into the general scheme of invariant Kolmogorov decompositions and we make explicit the connection with completely bounded maps and Wittstock's Theorem [20].

2000 *Mathematics Subject Classification.* Primary: 46E50, 46C20, 46E22, 47A57, 43A35; Secondary: 47B50, 43A65, 46L07, 32A70.

This opens the possibility of defining a class of non-hermitian decomposable kernels, that may successfully replace the missing class of completely bounded kernels, by using a generalization of Haagerup's decomposable linear mappings on C^*-algebras, cf. [12]. An analog of Paulsen's Dilation Theorem for decomposable kernels is obtained in Theorem 4.4.

In Section 5 we show that holomorphic kernels in more than one variable have Kolmogorov decompositions and hence, that hermitian holomorphic kernels automatically satisfy L. Schwartz's boundedness condition. In view of the transcription between Kolmogorov decompositions and reproducing kernel spaces, e.g. see Theorem 2.4, this result is an extension of the result of D. Alpay in [3] proved for one variable holomorphic hermitian kernels.

2. Preliminaries

We briefly review the structure of the Kolmogorov decomposition of invariant hermitian kernels. As it was shown in [18], the natural framework for studying hermitian kernels is given by Kreĭn spaces and for this reason we briefly discuss the necessary terminology.

2.1. **Kreĭn Spaces.** An indefinite inner product space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is called *Kreĭn space* provided that there exists a positive inner product $\langle\cdot,\cdot\rangle$ turning $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ into a Hilbert space and such that $[\xi,\eta] = \langle J\xi,\eta\rangle$, $\xi,\eta \in \mathcal{H}$, for some symmetry J ($J^* = J^{-1} = J$ with respect to the Hilbert space structure) on $\mathcal{H}$. Such a symmetry J is called a *fundamental symmetry* and we will frequently indicate by a lower index the space on which it acts. For two Kreĭn spaces $\mathcal{H}$ and $\mathcal{K}$ we denote by $\mathcal{L}(\mathcal{H},\mathcal{K})$ the set of linear bounded operators from $\mathcal{H}$ to $\mathcal{K}$. For $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ we denote by $T^\sharp \in \mathcal{L}(\mathcal{K},\mathcal{H})$ the adjoint of T with respect to the indefinite inner product $[\cdot,\cdot]$. The Hilbert space adjoint of T with respect to the positive inner products $\langle\cdot,\cdot\rangle$ is denoted by T^*. It is important to note that, if $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are fundamentally symmetries on $\mathcal{H}$ and, respectively, $\mathcal{K}$ then

$$T^\sharp = J_{\mathcal{H}} T^* J_{\mathcal{K}}.$$

We say that $A \in \mathcal{L}(\mathcal{H})$ is a *selfadjoint operator* if $A^\sharp = A$. For example, in terms of fundamental symmetries, this means $J_{\mathcal{H}}A = A^* J_{\mathcal{H}}$. Also, we say that the operator $U \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is *unitary* if $UU^\sharp = I_{\mathcal{K}}$ and $U^\sharp U = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. Equivalently, this means that U is boundedly invertible and $J_{\mathcal{H}}U^{-1} = U^* J_{\mathcal{K}}$. In terms of inner products, this means that U is isometric and surjective.

A special situation occurs for a unitary operator U with domain and range the same Kreĭn space $\mathcal{H}$ if it commutes with some fundamental symmetry $J_{\mathcal{H}}$. Such a unitary operator is called *fundamentally reducible* and it can be characterized in other different ways. For instance, U is fundamentally reducible if and only if it is power bounded.

Most of the difficulties in dealing with operators on Kreĭn spaces are caused by the lack of a well-behaved factorization theory. The concept of induced space turned out to be quite useful in order to deal with this issue. Thus, let $\mathcal{H}$ be a Hilbert space and, for a selfadjoint operator A in $\mathcal{L}(\mathcal{H})$, we define a new inner product $[\cdot,\cdot]_A$ on $\mathcal{H}$ by the formula

$$[\xi,\eta]_A = \langle A\xi,\eta\rangle_{\mathcal{H}}, \quad \xi,\eta\in\mathcal{H}. \tag{2.1}$$

A pair $(\mathcal{K},\Pi)$ consisting of a Kreĭn space $\mathcal{K}$ and a bounded operator $\Pi\in\mathcal{L}(\mathcal{H},\mathcal{K})$ is called a *Kreĭn space induced* by A provided that Π has dense range and the relation

$$[\Pi\xi,\Pi\eta]_{\mathcal{K}} = \langle\xi,\eta\rangle_A \tag{2.2}$$

holds for all $\xi,\eta\in\mathcal{H}$. There are many known examples of induced spaces. A more delicate question is the *uniqueness* of the induced Kreĭn spaces (see [7]).

2.2. **Hermitian Kernels.** We can use the concept of induced space in order to describe the Kolmogorov decomposition of a hermitian kernel. Let X be an arbitrary set. From now on we assume $\mathcal{H}$ is a Hilbert space with inner product denoted by $\langle\cdot,\cdot\rangle$. A *kernel* on X is a mapping K defined on $X\times X$ with values in $\mathcal{L}(\mathcal{H})$. The adjoint K^* of K is defined by the formula $K^*(x,y)=K(y,x)^*$. The kernel K is called *hermitian on* X if $K^*=K$.

Let $\mathcal{F}_0(X,\mathcal{H})$ denote the vector space of all functions on X with values in $\mathcal{H}$ which vanish except on a finite number of points. We associate to K an inner product on $\mathcal{F}_0(X,\mathcal{H})$ by the formula:

$$[f,g]_K = \sum_{x,y\in X}\langle K(x,y)f(y),g(x)\rangle, \quad f,g\in\mathcal{F}_0(X,\mathcal{H}). \tag{2.3}$$

We say that the hermitian kernel $L: X\times X\to\mathcal{L}(\mathcal{H})$ is *positive definite* if the inner product $[\cdot,\cdot]_L$ associated to L by the formula (2.3) is positive. One can introduce a natural partial order on the set of hermitian kernels on X with values in $\mathcal{L}(\mathcal{H})$ as follows: if A, B are hermitian kernels, then $A\leq B$ means $[f,f]_A\leq[f,f]_B$ for all $f\in\mathcal{F}_0(X,\mathcal{H})$.

A *Kolmogorov decomposition* of the hermitian kernel K is a pair $(V;\mathcal{K})$ with the following properties:

KD1 $\mathcal{K}$ is a Kreĭn space with fundamental symmetry J;
KD2 $V=\{V(x)\}_{x\in X}\subset\mathcal{L}(\mathcal{H},\mathcal{K})$ such that $K(x,y)=V(x)^*JV(y)$ for all $x,y\in\mathcal{K}$;
KD3 $\{V(x)\mathcal{H}\mid x\in X\}$ is total in $\mathcal{K}$.

The next result, obtained in [7], settles the question concerning the existence of a Kolmogorov decomposition for a given hermitian kernel.

Theorem 2.1. *Let $K: X\times X\to\mathcal{L}(\mathcal{H})$ be a hermitian kernel. The following assertions are equivalent:*

(1) *There exists a positive definite kernel $L: X\times X\to\mathcal{L}(\mathcal{H})$ such that $-L\leq K\leq L$.*
(2) *K has a Kolmogorov decomposition.*

The condition in assertion (1) of the previous result appeared earlier in the work of L. Schwartz [18] concerning the structure of hermitian kernels. It is easy to see that (1) is also equivalent to the representation of K as a difference of two positive definite kernels. Thus, Theorem 2.1 says that the class of hermitian kernels admitting Kolmogorov decompositions is the same with the class of hermitian kernels in the linear span of the cone of positive definite kernels.

It is convenient for our purpose to review a construction of Kolmogorov decompositions. We assume that there exists a positive definite kernel $L : X \times X \to \mathcal{L}(\mathcal{H})$ such that $-L \le K \le L$. Let $\mathcal{H}_L$ be the Hilbert space obtained by the completion of the quotient space $\mathcal{F}_0(X,\mathcal{H})/\mathcal{N}_L$ with respect to $[\cdot,\cdot]_L$, where $\mathcal{N}_L = \{f \in \mathcal{F}_0(X,\mathcal{H}) \mid [f,f]_L = 0\}$ is the isotropic subspace of the inner product space $(\mathcal{F}_0(X,\mathcal{H}), [\cdot,\cdot]_L)$. Since (1) in Theorem 2.1 is equivalent to

$$|[f,g]_K| \le [f,f]_L^{1/2}\,[g,g]_L^{1/2}$$

for all $f,g \in \mathcal{F}_0(X,\mathcal{H})$ (see Proposition 38, [18]), it follows that $\mathcal{N}_L$ is a subset of the isotropic subspace $\mathcal{N}_K$ of the inner product space $(\mathcal{F}_0(X,\mathcal{H}), [\cdot,\cdot]_K)$. Therefore, $[\cdot,\cdot]_K$ uniquely induces an inner product on $\mathcal{H}_L$, still denoted by $[\cdot,\cdot]_K$, such that (2.2) holds for $f,g \in \mathcal{H}_L$. By the Riesz representation theorem we obtain a selfadjoint contractive operator $A_L \in \mathcal{L}(\mathcal{H}_L)$, referred to as the *Gram operator* of K with respect to L, such that

$$[f,g]_K = [A_L f, g]_L, \quad f,g \in \mathcal{F}_0(X,\mathcal{H}).$$

Let $(\mathcal{K},\Pi)$ be a Kreĭn space induced by A_L. For $\xi \in \mathcal{H}$ and $x \in X$, we define the element $\xi_x = \delta_x\xi \in \mathcal{F}_0(X,\mathcal{H})$ (here δ_x is the Kronecker function delta), that is,

$$\xi_x(y) = \begin{cases} \xi, & y = x; \\ 0, & y \neq x. \end{cases} \tag{2.4}$$

Then we define

$$V(x)\xi = \Pi[\xi_x], \quad x \in X,\ \xi \in \mathcal{H},$$

where $[\xi_x] = \xi_x + \mathcal{N}_L$ denotes the class of ξ_x in $\mathcal{H}_L$ and it can be verified that $(V;\mathcal{K})$ is a Kolmogorov decomposition of the kernel K.

We finally review the uniqueness property of the Kolmogorov decomposition. Two Kolmogorov decompositions $(V_1,\mathcal{K}_1)$ and $(V_2,\mathcal{K}_2)$ of the same hermitian kernel K are *unitarily equivalent* if there exists a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ such that for all $x \in X$ we have $V_2(x) = \Phi V_1(x)$. The following result was obtained in [7]. We denote by $\rho(T)$ the resolvent set of the operator T.

Theorem 2.2. *Let K be a hermitian kernel which has Kolmogorov decompositions. The following assertions are equivalent:*

(1) *All Kolmogorov decompositions of K are unitarily equivalent.*

(2) *For each positive definite kernel L such that $-L \le K \le L$, there exists $\epsilon > 0$ such that either $(0,\epsilon) \subset \rho(A_L)$ or $(-\epsilon,0) \subset \rho(A_L)$, where A_L is the Gram operator of K with respect to L.*

2.3. **Invariant Hermitian Kernels.** We now review some results on the Kolmogorov decomposition of hermitian kernels with additional symmetries. Let ϕ be an action of a unital semigroup S on X. Assume that the $\mathcal{L}(\mathcal{H})$-valued hermitian kernel K has a Kolmogorov decomposition $(V,\mathcal{K})$. The action ϕ is linearized by the following mapping: for any $a\in S$, $x\in X$ and $\xi\in\mathcal{H}$,

$$U(a)V(x)\xi = V(\phi(a,x))\xi. \tag{2.5}$$

We notice that for $a,b\in S$, $x\in X$ and $\xi\in\mathcal{H}$ we have

$$\begin{aligned} U(a)U(b)V(x)\xi &= U(a)V(\phi(b,x))\xi = V(\phi(a,\phi(b,x)))\xi \\ &= V(\phi(ab,x))\xi = U(ab)V(x)\xi. \end{aligned}$$

Therefore, the family $\{U(a)\}_{a\in S}$ is a semigroup of linear operators with a common dense domain $\bigvee_{x\in X} V(x)\mathcal{H}$ (throughout this paper $\bigvee$ denotes the linear space generated by some set, without taking any closure). If K is a positive definite kernel then the previous construction is well-known (see, for instance, [17]). The remaining question, especially in case K is not positive definite, is: what additional conditions on the kernel K should be imposed in order to ensure the boundedness of the operators $U(a)$, $a\in S$? We gave a possible answer in [8], by considering an additional symmetry of the kernel.

Consider the set $B=\{\xi_x \mid \xi\in\mathcal{H}, x\in X\}$ which is a vector space basis of $\mathcal{F}_0(X,\mathcal{H})$. Define for $a\in S$,

$$\psi_a(\xi_x) = \xi_{\phi(a,x)} \tag{2.6}$$

and this mapping can be extended by linearity to a linear mapping, also denoted by ψ_a, from $\mathcal{F}_0(X,\mathcal{H})$ into $\mathcal{F}_0(X,\mathcal{H})$. We say that a positive definite kernel L is *ϕ-bounded* provided that for all $a\in S$, ψ_a is bounded with respect to the seminorm $[\cdot,\cdot]_L^{1/2}$ induced by L on $\mathcal{F}_0(X,\mathcal{H})$. We denote by $\mathcal{B}^+_\phi(X,\mathcal{H})$ the set of positive definite ϕ-bounded kernels on X with values in $\mathcal{L}(\mathcal{H})$.

From now on we assume that S is a unital semigroup with involution, that is, there exists a mapping $\mathfrak{I}: S\to S$ such that $\mathfrak{I}^2=$ the identity on S, and $\mathfrak{I}(ab)=\mathfrak{I}(b)\mathfrak{I}(a)$ for all $a,b\in S$. The following result was obtained in [8].

Theorem 2.3. *Let ϕ be an action of the unital semigroup S with involution $\mathfrak{I}$ on the set X and let K be an $\mathcal{L}(\mathcal{H})$-valued hermitian kernel on X with the property that*

$$K(x,\phi(a,y)) = K(\phi(\mathfrak{I}(a),x),y) \tag{2.7}$$

for all $x,y\in X$ and $a\in S$. The following assertions are equivalent:

(1) *There exists $L\in\mathcal{B}^+_\phi(X,\mathcal{H})$ such that $-L\le K\le L$.*

(2) *K has a Kolmogorov decomposition $(V;\mathcal{K})$ with the property that there exists a representation U of S on $\mathcal{K}$ such that*

$$V(\phi(a,x)) = U(a)V(x) \tag{2.8}$$

for all $x\in X$, $a\in S$. In addition, $U(\mathfrak{I}(a))=U(a)^\sharp$ for all $a\in S$.

(3) *$K=K_1-K_2$ for two positive definite kernels such that $K_1+K_2\in\mathcal{B}^+_\phi(X,\mathcal{H})$.*

(4) $K = K_+ - K_-$ *for two disjoint positive definite kernels such that* $K_+ + K_- \in \mathcal{B}^+_\phi(X,\mathcal{H})$.

2.4. **Reproducing kernel spaces.** We now describe another construction, closely related to the Kolmogorov decomposition of a hermitian kernel. Let K be a hermitian kernel satisfying (2.7) with a Kolmogorov decomposition $(V;\mathcal{K})$ as in Theorem 2.3. Define

$$\mathcal{R} = \{g_f : X \to \mathcal{H} \mid g_f(x) = V^\sharp(x)f,\ f \in \mathcal{K}\}.$$

Then $\mathcal{R}$ is a vector subspace of $\mathcal{F}(X,\mathcal{H})$, the class of functions defined on X with values in $\mathcal{H}$. We define a map $\Phi : \mathcal{K} \to \mathcal{R}$ by

$$\Phi f = g_f, \quad f \in \mathcal{K}.$$

This map is linear and bijective, so that we can define on $\mathcal{R}$ the inner product

$$[g_f, g_h]_\mathcal{R} = [f,h]_\mathcal{K}, \quad f,h \in \mathcal{K}.$$

One checks that $\mathcal{R}$ is a Kreĭn space with respect to this inner product. Also, Φ is a bounded operator between the Kreĭn spaces $\mathcal{K}$ and $\mathcal{R}$, since it is closed and everywhere defined on $\mathcal{K}$, hence it is unitary. Moreover, $\mathcal{R}$ is the closure of the linear space generated by the functions $g_{V(y)\xi}$, $y \in X$ and $\xi \in \mathcal{H}$. These functions are related to the kernel K as follows:

$$g_{V(y)\xi}(x) = V^\sharp(x)V(y)\xi = K(x,y)\xi, \quad x,y \in X,\ \xi \in \mathcal{H}.$$

We will write $g_{y,\xi}$ instead of $g_{V(y)\xi}$, and since $g_{y,\xi}(x) = K(x,y)\xi$, these functions can be defined without using V. Therefore, the space $\mathcal{R}$ has the following *reproducing property*:

$$[g_f(x),\xi]_\mathcal{H} = [g_f, g_{x,\xi}]_\mathcal{R}, \quad x \in X,\ f \in \mathcal{K},\ \xi \in \mathcal{H}. \tag{2.9}$$

We also note that property (2.7) of the kernel K is reflected into a certain symmetry of the elements of $\mathcal{R}$. Thus, we define an operator $\bar{U}(a) \in \mathcal{L}(\mathcal{R})$ by

$$\bar{U}(a) = \Phi U(a) \Phi^\sharp, \quad a \in X,$$

where U is the projective representation of S given by Theorem 2.3. We have

$$\bar{U}(a)g_f = \Phi U(a)\Phi^\sharp g_f = \Phi U(a) f = g_{U(a)f}, \quad a \in X,\ f \in \mathcal{K}.$$

On the other hand, for any $a,x \in X$ and $f \in \mathcal{K}$,

$$\begin{aligned} g_f(\phi(\mathfrak{I}(a),x)) &= V^\sharp(\phi(\mathfrak{I}(a),x))f = V(x)^\sharp U(\mathfrak{I}(a))^\sharp f \\ &= g_{U(a)^\sharp f}(x) \end{aligned}$$

and we deduce that the elements of $\mathcal{R}$ satisfy the relation

$$(\bar{U}(a)g_f)(x) = \sigma(\mathfrak{I}(a),a)^{-1} g_f(\phi(\mathfrak{I}(a),x)).$$

Based on this relation we obtain the following result.

Theorem 2.4. *Let ϕ be an action of the unital semigroup S with involution $\mathfrak{I}$ on the set X and let K be an $\mathcal{L}(\mathcal{H})$-valued hermitian kernel on X with the property that*

$$K(x, \phi(a, y)) = K(\phi(\mathfrak{I}(a), x), y) \tag{2.10}$$

for all $x, y \in X$ and $a \in S$. The following assertions are equivalent:

(1) *There exists $L \in \mathcal{B}^+_\phi(X, \mathcal{H})$ such that $-L \le K \le L$.*

(2) *K has a Kolmogorov decomposition $(V; \mathcal{K})$ with the property that there exists a representation U of S on $\mathcal{K}$ such that $V(\phi(a, x)) = U(a)V(x)$ for all $x \in X$, $a \in S$.*

(3) *There exists a Kreĭn space $\mathcal{R}$ such that*

(a) $\mathcal{R} \subset \mathcal{F}(X, \mathcal{H})$.

(b) *The set $\{g_{x,\xi} \mid x \in X, \xi \in \mathcal{H}\}$ is total in $\mathcal{R}$.*

(c) $[f(x), \xi]_{\mathcal{H}} = [f, g_{x,\xi}]_{\mathcal{R}}$ *for all $f \in \mathcal{R}, \xi \in \mathcal{H}, x \in X$.*

(d) *There exists a representation $\bar{U}$ of S on $\mathcal{R}$ such that*

$$(\bar{U}(a)f)(x) = f(\phi(\mathfrak{I}(a), x))$$

for all $a \in S$, $x \in X$ and $f \in \mathcal{R}$.

Proof. The implication (2) $\Rightarrow$ (3) was already proved above. In order to prove (3) $\Rightarrow$ (2) we define the linear mapping $\bar{V}(x)$ from $\mathcal{H}$ into $\mathcal{R}$ by the formula:

$$\bar{V}(x)\xi = g_{x,\xi}, \quad x \in X, \ \xi \in \mathcal{H}.$$

The property (c) shows that $\bar{V}(x)$ is a closed operator and by the closed graph theorem we deduce that $\bar{V}(x) \in \mathcal{L}(\mathcal{H}, \mathcal{R})$. From (b) and (c) we deduce that $(\bar{V}; \mathcal{R})$ is a Kolmogorov decomposition of K. Finally,

$$\begin{aligned}(\bar{V}(\phi(a, x))\xi)(y) &= g_{\phi(a,x),\xi}(y) = K(y, \phi(a, x))\xi \\ &= K(\phi(\mathfrak{I}(a), y), x)\xi \\ &= g_{x,\xi}(\phi(\mathfrak{I}(a), y)) \\ &= (\bar{U}(a)\bar{V}(x)\xi)(y).\end{aligned}$$

This completes the proof. □

3. Hankel type kernels

In this section we interpret the invariance property (2.7) as a Hankel condition. To see this, let $S = \mathbb{N}$ be the additive semigroup of natural numbers (including 0) and the action ϕ is given by right translation. If K satisfies (2.7), then $K(n, p + m) = K(p + n, m)$ for $m, n, p \in \mathbb{N}$ and K is a so-called *Hankel kernel.* We can extend this example to a noncommutative setting as follows. Let $S = \mathbb{F}^+_N$ be the unital free semigroup on N generators $g_1, \dots, g_N$ with lexicographic order $\prec$. The empty word is the identity element and the length of the word σ is denoted by $|\sigma|$. The length of the empty word is 0. There is a natural involution on $\mathbb{F}^+_N$ given by $\mathfrak{I}(g_1 \dots g_k) = g_k \dots g_1$

as well as a natural action of $\mathbb{F}_N^+$ on itself by juxtaposition, $\phi(\sigma,\tau)=\sigma\tau$, $\sigma,\tau\in\mathbb{F}_N^+$. The condition (2.7) means in this case that

$$K(\sigma,\beta\tau)=K(\mathfrak{I}(\beta)\sigma,\tau) \tag{3.1}$$

for $\beta,\sigma,\tau\in\mathbb{F}_N^+$. It was noticed in [6] that kernels as above appear in connection with orthogonal polynomials in N indeterminates satisfying the relations $Y_k^*=Y_k$, $k=1,\dots,N$.

Let $\mathcal{P}_N^0$ be the algebra of polynomials in N non-commuting indeterminates $Y_1,\dots,Y_N$ with complex coefficients. For any $\sigma=g_{j_1}g_{j_2}\cdots g_{j_l}\in\mathbb{F}_N^+$, where $j_p\in\{1,2,\dots,N\}$ for all $p=1,\dots,l$, $l=|\sigma|$, we denote $Y_\sigma=Y_{g_{j_1}}Y_{g_{j_2}}\cdots Y_{g_{j_l}}$. With this notation, each element $P\in\mathcal{P}_N^0$ can be uniquely written as

$$P=\sum_{\sigma\in\mathbb{F}_N^+}c_\sigma Y_\sigma, \tag{3.2}$$

where $(c_\sigma)_{\sigma\in\mathbb{F}_N^+}\subset\mathbb{C}$ has finite support.

An involution $*$ on $\mathcal{P}_N^0$ can be introduced as follows: $Y_k^*=Y_k$, $k=1,\dots,N$; on monomials, $(Y_\sigma)^*=Y_{\mathfrak{I}(\sigma)}$; and, in general, if P has the representation as in (3.2) then

$$P^*=\sum_{\sigma\in\mathbb{F}_N^+}\overline{c}_\sigma Y_\sigma^*.$$

Thus, $\mathcal{P}_N^0$ is a unital, associative, $*$-algebra over $\mathbb{C}$. A linear functional Z on $\mathcal{P}_N^0$ is called *hermitian* if $Z(P^*)=\overline{Z(P)}$ for $P\in\mathcal{P}_N^0$.

A convenient subclass of hermitian functionals, called *GNS functionals*, is given by those functionals admitting GNS data. A triplet $(\pi,\mathcal{K},\Omega)$ is called a *GNS data* associated to Z if π is a hermitian closable representation of $\mathcal{P}_N^0$ on a Kreĭn space $\mathcal{K}$ and $\Omega\in\mathcal{D}(\pi)$, the domain of π, such that $Z(P)=[\pi(P)\Omega,\Omega]_{\mathcal{K}}$ for $P\in\mathcal{P}_N^0$, and $\bigvee_{P\in\mathcal{P}_N^0}\pi(P)\Omega=\mathcal{D}(\pi)$ (see [2], [13]). The numbers $s_\sigma=Z(Y_\sigma)$, $\sigma\in\mathbb{F}_N^+$, are called the *moments* of Z.

Conversely, to any family of complex numbers $\Sigma=(s_\sigma)_{\sigma\in\mathbb{F}_N^+}$, we can associate the kernel

$$K_\Sigma(\sigma,\tau)=s_{\mathfrak{I}(\sigma)\tau},\quad \sigma,\tau\in\mathbb{F}_N^+, \tag{3.3}$$

and it is easy to see that this kernel satisfies (3.1).

The following is a Hamburger type description of moments.

Theorem 3.1. *The complex numbers s_σ, $\sigma\in\mathbb{F}_N^+$, are the moments of a GNS functional on $\mathcal{P}_N^0$ if and only if there exists a positive definite kernel L on $\mathbb{F}_N^+$ such that $-L\le K\le L$, where $K(\sigma,\tau)=s_{\mathfrak{I}(\sigma)\tau}$, $\sigma,\tau\in\mathbb{F}_N^+$.*

Proof. This result is just another facet of Theorem 2.1. Assume first that the numbers s_σ, $\sigma\in\mathbb{F}_N^+$, are the moments of a GNS functional on $\mathcal{P}_N^0$. Let $(\pi,\mathcal{K},\Omega)$ be a GNS data associated to Z. Define $V:\mathbb{F}_N^+\to\mathcal{L}(\mathbb{C},\mathcal{K})$ by the formula:

$$V(\sigma)\lambda=\pi(Y_\sigma)(\lambda\Omega),\quad \sigma\in\mathbb{F}_N^+,\lambda\in\mathbb{C}.$$

We deduce that for $\sigma, \tau \in \mathbb{F}_N^+$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} V(\sigma)^\sharp V(\tau)\lambda\overline{\nu} &= [V(\tau)\lambda, V(\sigma)\nu]_{\mathcal{K}} = [\pi(Y_\tau)(\lambda\Omega), \pi(Y_\sigma)(\nu\Omega)]_{\mathcal{K}} \\ &= \lambda\overline{\nu}[\pi(Y_\sigma^* Y_\tau)\Omega, \Omega]_{\mathcal{K}} = \lambda\overline{\nu} Z(Y_{I(\sigma)\tau}) = \lambda\overline{\nu} K(\sigma, \tau). \end{aligned}$$

Also, the set $\{V(\sigma)\lambda \mid \sigma \in \mathbb{F}_N^+, \lambda \in \mathbb{C}\}$ is total in $\mathcal{K}$, so that $(V, \mathcal{K})$ is a Kolmogorov decomposition of K. By Theorem 2.1, there exists a positive definite kernel L on $\mathbb{F}_N^+$ such that $-L \leq K \leq L$.

Conversely, let $(V, \mathcal{K})$ be a Kolmogorov decomposition of K. Define $\Omega = V(\emptyset)$ and

$$\pi(Y_\sigma)\Omega = V(\sigma), \quad \sigma \in \mathbb{F}_N^+.$$

We notice that $\bigvee_{P \in \mathcal{P}_N^0} \pi(P)\Omega = \bigvee_{\sigma \in \mathbb{F}_N^+} V(\sigma)\mathbb{C}$, we define $\mathcal{D}(\pi) = \bigvee_{\sigma \in \mathbb{F}_N^+} V(\sigma)\mathbb{C}$, and we can extend π to $\mathcal{P}_N^0$ by linearity. Clearly, $\mathcal{D}(\pi)$ is invariant under $\pi(P)$, $P \in \mathcal{P}_N^0$, and $\pi(P)\pi(Q) = \pi(PQ)$. Also, for $k, k' \in \mathcal{D}(\pi)$,

$$\begin{aligned} [\pi(Y_\sigma)k, k']_{\mathcal{K}} &= [\pi(Y_\sigma)\sum_{k=1}^{n} c_k \pi(Y_{\tau_k})\Omega, \sum_{j=1}^{m} d_j \pi(Y_{\tau_j'})\Omega]_{\mathcal{K}} \\ &= \sum_{k,j=1}^{n,m} c_k \overline{d}_j [\pi(Y_\sigma)\pi(Y_{\tau_k})\Omega, \pi(Y_{\tau_j'})\Omega]_{\mathcal{K}} \\ &= \sum_{k,j=1}^{n,m} c_k \overline{d}_j [V(\sigma\tau_k)1, V(\tau_j')1]_{\mathcal{K}} = \sum_{k,j=1}^{n,m} c_k \overline{d}_j V(\tau_j')^\sharp V(\sigma\tau_k) \\ &= \sum_{k,j=1}^{n,m} c_k \overline{d}_j K(\tau_j', \sigma\tau_k) = \sum_{k,j=1}^{n,m} c_k \overline{d}_j K(\mathfrak{I}(\sigma)\tau_j', \tau_k) \\ &= [k, \pi(Y_{\mathfrak{I}(\sigma)})k']_{\mathcal{K}}, \end{aligned}$$

which shows that the domain of $\pi(Y_\sigma)^\sharp$ contains $\mathcal{D}(\pi)$ and

$$\pi(Y_\sigma)^\sharp | \mathcal{D}(\pi) = \pi(Y_{\mathfrak{I}(\sigma)}) = \pi(Y_\sigma^*).$$

We can extend this argument and show that the same is true for any $P \in \mathcal{P}_N^0$, so that $(\pi, \mathcal{K}, \Omega)$ is a GNS data for $Z(P) = [\pi(P)\Omega, \Omega]_{\mathcal{K}}$, $P \in \mathcal{P}_N^0$. The moments of Z are

$$\begin{aligned} Z(Y_\sigma) &= [\pi(Y_\sigma)\Omega, \Omega]_{\mathcal{K}} = [V(\sigma)1, V(\emptyset)1]_{\mathcal{K}} \\ &= V(\emptyset)^\sharp V(\sigma) = K(\emptyset, \sigma) = s_{I(\emptyset)\sigma} = s_\sigma. \end{aligned}$$

□

As a consequence of the previous result and Theorem 2.2, we deduce a uniqueness condition for the solvability of the Hamburger moment problem for GNS functionals.

Theorem 3.2. *Let s_σ, $\sigma \in \mathbb{F}_N^+$, be the moments of some GNS functional on $\mathcal{P}_N^0$, and consider the kernel $K(\sigma, \tau) = s_{\mathfrak{I}(\sigma)\tau}$, $\sigma, \tau \in \mathbb{F}_N^+$. Then, there exists a unique GNS functional on $\mathcal{P}_N^0$ with moments s_σ, if and only if for each positive definite kernel L on $\mathbb{F}_N^+$ such that $-L \leq K \leq L$, there exists $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A_L)$ or $(-\epsilon, 0) \subset \rho(A_L)$, where A_L is the Gram operator of K with respect to L.*

4. Dilations and decomposition of kernels

In this section we show that Theorem 2.3 provides a general framework for a version of the Stinespring theorem and for decompositions of hermitian linear maps. Let $\mathcal{A}$ be a unital $*$-algebra, $\mathcal{H}$ a Hilbert space, and let $T : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a linear hermitian map. A *Stinespring dilation* of T is, by definition, a triplet $(\pi, \mathcal{K}, B)$ such that:

SD1 $\mathcal{K}$ is a Kreĭn space with a fundamental symmetry J, and $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$;

SD2 $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{K})$ is a selfadjoint (that is, $\pi(a^*) = \pi(a)^\sharp = J\pi(a)^*J$ for all $a \in \mathcal{A}$) representation, such that $T(a) = B^*J\pi(a)B$, for all $a \in \mathcal{A}$.

If, in addition,

SD3 $\{\pi(a)B\mathcal{H} \mid a \in \mathcal{A}\}$ is total in $\mathcal{K}$,

then the Stinespring dilation is called *minimal.*

We consider the action ϕ of $\mathcal{A}$ on itself defined by

$$\phi(a, x) = xa^*, \quad x, a \in \mathcal{A}, \tag{4.1}$$

and a hermitian kernel is associated to T by the formula

$$K_T(x, y) = T(xy^*), \quad x, y \in \mathcal{A}. \tag{4.2}$$

It readily follows that K_T is ϕ-invariant, that is,

$$K_T(x, \phi(a, y)) = T(xay^*) = K_T(\phi(a^*, x), y), \quad a, x, y \in \mathcal{A}. \tag{4.3}$$

Proposition 4.1. *Given a minimal Stinespring dilation $(\pi, \mathcal{K}, B)$ of the hermitian linear map $T\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$, let*

$$V(x) = \pi(x^*)JB, \quad x \in \mathcal{A}, \tag{4.4}$$

where J is a fundamental decomposition of $\mathcal{K}$. Then $(V, \mathcal{K})$ is an invariant Kolmogorov decomposition of the hermitian kernel K_T.

In addition, (4.4) establishes a bijective correspondence between the set of minimal Stinespring dilations of T and the set of invariant Kolmogorov decompositions of K_T.

Proof. Let $(\pi, \mathcal{K}, B)$ be a minimal Stinespring dilation of T and define $(V, \mathcal{K})$ as in (4.4). Then

$$V(x)^*JV(y) = B^*J\pi(x^*)^*J\pi(y^*)JB = B^*J\pi(xy^*)JB = T(xy^*) = K_T(x, y), \quad x, y \in \mathcal{A}.$$

Since $\bigvee_{a\in\mathcal{A}} \pi(a)B\mathcal{H} = \bigvee_{x\in\mathcal{A}} V(x)\mathcal{H}$ it follows that $(V, \mathcal{K})$ is a Kolmogorov decomposition of T. Let us note that, by the definition of V,

$$\pi(a)V(x) = \pi(a)\pi(x^*)JB = \pi(ax^*)JB = V(xa^*) \quad a, x \in \mathcal{A},$$

and hence, letting $U = \pi$, it follows that the Kolmogorov decomposition $(V, \mathcal{K})$ is ϕ-invariant.

Conversely, let $(V, \mathcal{K})$ be an invariant Kolmogorov decomposition of the hermitian kernel K_T, that is, there exists a hermitian representation $U\colon \mathcal{A} \to \mathcal{L}(\mathcal{K})$ of the multiplicative semigroup $\mathcal{A}$ with involution $*$, such that

$$U(a)V(x) = V(xa^*), \quad a, x \in \mathcal{A}.$$

Define $\pi = U$ and $B = JV(1)$. Since T is linear it follows easily that π is also linear, hence a selfadjoint representation of the $*$-algebra $\mathcal{A}$ on the Kreĭn space $\mathcal{K}$. Then, taking into account that $V(a) = U(a^*)B$ for all $a \in \mathcal{A}$, it follows

$$T(a) = V(a)^* JV(1) = B^* U(a^*)^* JB = B^* JU(a)B, \quad a \in \mathcal{A},$$

and since $\bigvee_{a\in\mathcal{A}} \pi(a)B\mathcal{H} = \bigvee_{x\in\mathcal{A}} V(x)\mathcal{H}$ we thus proved that $(\pi, \mathcal{K}, B)$ is a minimal Stinespring dilation of T. One easily check that the mapping defined in (4.4) is the inverse of the mapping associating to each invariant Kolmogorov decomposition $(V, \mathcal{K})$ the minimal Stinespring dilation $(\pi, \mathcal{K}, B)$ as above. □

Proposition 4.1 reduces the existence of Stinespring dilations of hermitian maps T to the existence of invariant Kolmogorov decompositions for the hermitian kernel K_T defined as in (4.2). Now the following result is just an application of Theorem 2.3.

Theorem 4.2. *Let $\mathcal{A}$ be a unital $*$-algebra and let $T : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a linear hermitian map. The following assertions are equivalent:*

(1) *There exists a positive definite kernel $L \in \mathcal{B}^+_\phi(\mathcal{A}, \mathcal{H})$, ϕ given by (4.1), such that $-L \leq K_T \leq L$.*

(2) *T has a minimal Stinespring dilation.*

We show now that Wittstock's Decomposition Theorem [20] and Paulsen's Dilation Theorem [16] fit into the framework of invariant Kolmogorov decompositions, more precisely, the following result shows that in case $\mathcal{A}$ is a C^*-algebra, Schwartz's boundedness condition for hermitian kernels represents an extension of the concept of completely bounded map. We use standard terminology from the theory of operator spaces, e.g. see [16] and [10].

Theorem 4.3. *Let $\mathcal{A}$ be a unital C^*-algebra and let $T : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a linear hermitian map. The following assertions are equivalent:*

(1) *T is completely bounded.*

(2) *There exists a completely positive map $S : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $-S \leq T \leq S$.*

(3) *There exists a Hilbert space $\mathcal{K}$ with a symmetry J, a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ commuting with J, and a bounded operator $B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that*

$$T(a) = B^* J\pi(a)B, \quad a \in \mathcal{A},$$

and $\bigvee_{a\in\mathcal{A}} \pi(a)B\mathcal{H}$ is dense in $\mathcal{K}$.

(4) *$T = T_+ - T_-$ for two completely positive maps T_+ and T_-.*

Proof. In the following we let $\mathcal{U}(\mathcal{A})$ be the unitary group of $\mathcal{A}$. Then $\mathcal{U}(\mathcal{A})$ has the involution $\mathfrak{I}(a) = a^{-1} = a^*$ and acts on $\mathcal{A}$ by $\phi(a, x) = xa^* = xa^{-1}$.

(1) ⇒ (2) We use Paulsen's off-diagonal technique. Briefly, assume that T is completely bounded. By Theorem 7.3 in [16], there exist completely positive maps ϕ_1 and ϕ_2 such that the map

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \phi_1(a) & T(b) \\ T(c^*)^* & \phi_2(d) \end{bmatrix}$$

is completely positive. Define $S(a)=\frac{1}{2}(\phi_1(a)+\phi_2(a))$, which is a completely positive map. We can check that $-S\leq T\leq S$. First, let $a\geq 0$, $a\in\mathcal{A}$. Then $\begin{bmatrix} a & \pm a \\ \pm a & a\end{bmatrix}\geq 0$, so that $\begin{bmatrix} \phi_1(a) & \pm T(a) \\ \pm T(a) & \phi_2(a)\end{bmatrix}\geq 0$. In particular, for $\xi\in\mathcal{H}$,

$$\langle\begin{bmatrix} \phi_1(a) & \pm T(a) \\ \pm T(a) & \phi_2(a)\end{bmatrix}\begin{bmatrix}\xi\\ \xi\end{bmatrix},\begin{bmatrix}\xi\\ \xi\end{bmatrix}\rangle\geq 0,$$

equivalently, $\langle(\phi_1(a)\pm 2T(a)+\phi_2(a))\xi,\xi\rangle\geq 0$. Therefore, $S\pm T$ are positive maps. The argument can be extended in a straightforward manner (using the so-called canonical shuffle as in [16]) to show that $S\pm T$ are completely positive maps.

(2) $\Rightarrow$ (3) Since S is completely positive, the kernel K_S is positive definite and satisfies $-K_S\leq K_T\leq K_S$. Also,

$$K_S(x,\phi(a,y))=K_S(\phi(a^{-1},x),y),\quad a\in\mathcal{U}(\mathcal{A}),\ x,y\in\mathcal{A}.$$

By Theorem 4.3 in [8], there exists a Kolmogorov decomposition $(V,\mathcal{K})$ of K_T and a fundamentally reducible representation U of $\mathcal{U}(\mathcal{A})$ on $\mathcal{K}$ such that

$$U(a)V(x)=V(xa^{-1}),\quad a\in\mathcal{U}(\mathcal{A}),\ x\in\mathcal{A}.$$

Let J be a fundamental symmetry on $\mathcal{K}$ such that $U(a)J=JU(a)$ for all $a\in\mathcal{U}(\mathcal{A})$. Then U is also a representation of $\mathcal{U}(\mathcal{A})$ on the Hilbert space $(\mathcal{K},\langle\cdot,\cdot\rangle_J)$. Also, for $a\in\mathcal{U}(\mathcal{A})$,

$$T(a)=K_T(a,1)=V(a)^*JV(1)=V(1)^*JU(a)V(1).$$

Since $\mathcal{A}$ is the linear span of $\mathcal{U}(\mathcal{A})$ and T is linear, U can be extended by linearity to a representation π of $\mathcal{A}$ on $\mathcal{K}$ commuting with J and such that

$$T(a)=V(1)^*J\pi(a)V(1)$$

holds for all $a\in\mathcal{A}$. Also, $\bigvee_{a\in\mathcal{U}(\mathcal{A})}U(a)V(1)\mathcal{H}=\bigvee_{a\in\mathcal{U}(\mathcal{A})}V(a^{-1})\mathcal{H}=\bigvee_{a\in\mathcal{U}(\mathcal{A})}V(a)\mathcal{H}$ and using once again the fact that $\mathcal{A}$ is the linear span of $\mathcal{U}(\mathcal{A})$, we deduce that $\bigvee_{a\in\mathcal{A}}U(a)V(1)\mathcal{H}$ is dense in $\mathcal{K}$.

(3)$\Rightarrow$(4) We define for $a\in\mathcal{A}$,

$$V(a)=\pi(a^*)B.$$

Then $V(a)$ is in $\mathcal{L}(\mathcal{H},\mathcal{K})$ and $\bigvee_{a\in\mathcal{A}}V(a)\mathcal{H}=\bigvee_{a\in\mathcal{A}}\pi(a)B\mathcal{H}$. $\mathcal{K}$ becomes a Kreĭn space by setting $[x,y]_{\mathcal{K}}=\langle Jx,y\rangle$, $x,y\in\mathcal{K}$. Also, for $\xi,\eta\in\mathcal{H}$,

$$\begin{aligned}\langle V(x)^*JV(y)\xi,\eta\rangle &=\langle JV(y)\xi,V(x)\eta\rangle=\langle J\pi(y^*)B\xi,\pi(x^*)B\eta\rangle\\ &=\langle J\pi(xy^*)B\xi,B\eta\rangle=\langle T(xy^*)\xi,\eta\rangle\\ &=\langle K_T(x,y)\xi,\eta\rangle,\end{aligned}$$

so that $(V,\mathcal{K})$ is a Kolmogorov decomposition of K_T. Let $J=J_+-J_-$ be the Jordan decomposition of J and define the hermitian kernels

$$K_\pm(x,y)=V(x)^*J_\pm V(y).$$

One can check that

$$K_{\pm}(x, \phi(a, y)) = K_{\pm}(\phi(a^{-1}, x), y)$$

for all $x, y \in \mathcal{A}$ and $a \in \mathcal{U}(\mathcal{A})$. For $x \in \mathcal{A}$ define

$$T_{\pm}(x) = K_{\pm}(x, 1).$$

Then $T_{\pm}(x) = BJ\pi(x^*)^* J_{\pm} V(1)$ are linear maps on $\mathcal{A}$ and for $x \in \mathcal{A}$, $y \in \mathcal{U}(\mathcal{A})$, we get

$$\begin{aligned} K_{T_{\pm}}(x, y) &= T_{\pm}(xy^{-1}) = K_{\pm}(xy^{-1}, 1) \\ &= K_{\pm}(\phi(y, x), 1) = K_{\pm}(x, \phi(y^{-1}, 1)) \\ &= K_{\pm}(x, y). \end{aligned}$$

Since $K_{T_{\pm}}$ and $K_{\pm}$ are antilinear in the second variable and $\mathcal{A}$ is the linear span of $\mathcal{U}(\mathcal{A})$, it follows that

$$K_{T_{\pm}}(x, y) = K_{\pm}(x, y)$$

for all $x, y \in \mathcal{A}$. This implies that $T_{\pm}$ are disjoint completely positive maps such that $T = T_+ - T_-$. The implication (3)⇒(1) follows from Theorem 2.3. □

Theorem 4.3 suggests how to extend the concept of decomposition to arbitrary kernels. In the following we repeatedly use the following observation: if L is a positive definite kernel on X and with values in $\mathcal{L}(\mathcal{H})$, and $T \in \mathcal{L}(\mathcal{H})$, then the kernel T^*LT is positive definite. Thus, if K is a hermitian kernel, L is a positive definite kernel, both on a set X and with entries in $\mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, then for any $x, y \in X$

$$\begin{aligned} &\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^* \begin{bmatrix} L(x,y) & K(x,y) \\ K(x,y) & L(x,y) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &\qquad = 2 \begin{bmatrix} L(x,y) + K(x,y) & 0 \\ 0 & L(x,y) - K(x,y) \end{bmatrix}. \end{aligned}$$

Thus, $\begin{bmatrix} L & K \\ K & L \end{bmatrix}$ is positive definite if and only if both $L + K$ and $L - K$ are positive definite, that is, if and only if the Schwartz condition $-L \leq K \leq L$ holds.

Following U. Haagerup [12], this observation and Theorem 4.3 suggest the following definition: a kernel $K : X \times X \to \mathcal{L}(\mathcal{H})$ is called *decomposable* if there exist two positive definite kernels $L_1, L_2 : X \times X \to \mathcal{L}(\mathcal{H})$ such that the kernel $\begin{bmatrix} L_1 & K \\ K^* & L_2 \end{bmatrix}$ is positive definite. Clearly, a kernel K is decomposable if and only if it is a linear combination of positive definite kernels. Actually, it is easy to see that any decomposable kernel can be written as linear combination of at most four positive definite kernels. The next result can be viewed as an analog of V. Paulsen dilation theorem [16].

Theorem 4.4. *The kernel K is decomposable if and only if there is a Hilbert space $\mathcal{K}$, a mapping $V : X \to \mathcal{L}(\mathcal{H}, \mathcal{K})$, and a contraction U on $\mathcal{K}$ such that $K(x, y) = V(x)^* U V(y)$ for all $x, y \in X$, and the set $\{V(x)h \mid x \in X, h \in \mathcal{H}\}$ is total in $\mathcal{K}$.*

Proof. If $K(x,y)=V(x)^*UV(y)$ for $x,y\in X$ and some contraction U, then consider the positive definite kernels $L_1(x,y)=L_2(x,y)=V(x)^*V(y)$. We deduce that

$$\begin{aligned}\begin{bmatrix} L_1(x,y) & K(x,y) \\ K^*(x,y) & L_2(x,y)\end{bmatrix} &= \begin{bmatrix} V(x)^*V(y) & V(x)^*UV(y) \\ V(x)^*U^*V(y) & V(x)^*V(y)\end{bmatrix} \\ &= \begin{bmatrix} V(x)^* & 0 \\ 0 & V(x)^*\end{bmatrix}\begin{bmatrix} I & U \\ U^* & I\end{bmatrix}\begin{bmatrix} V(y) & 0 \\ 0 & V(y)\end{bmatrix}.\end{aligned}$$

Since U is a contraction, the matrix $\begin{bmatrix} I & U \\ U^* & I\end{bmatrix}$ is positive. Next, take $x_1,\dots,x_n\in X$ and after reshuffling, the matrix

$$\left[\begin{bmatrix} L_1(x_i,x_j) & K(x_i,x_j) \\ K^*(x_i,x_j) & L_2(x_i,x_j)\end{bmatrix}\right]_{i,j=1}^n$$

can be written in the form

$$\left(\bigoplus_{i=1}^n (V(x_i)^*\oplus V(x_i)^*\right)\left(\begin{bmatrix} I & U \\ U^* & I\end{bmatrix}\otimes\begin{bmatrix} I & I & \dots & I \\ I & I & \dots & I \\ \vdots & \vdots & \ddots & \\ I & I & & I\end{bmatrix}\right)\left(\bigoplus_{j=1}^n (V(x_j)\oplus V(x_j)\right),$$

which shows that the kernel $\begin{bmatrix} L_1(x,y) & K(x,y) \\ K^*(x,y) & L_2(x,y)\end{bmatrix}$ is positive definite.

Conversely, assume that K is decomposable. We consider the real and imaginary parts of K,

$$K_1(x,y)=\frac{1}{2}(K(x,y)+K^*(x,y)), \tag{4.5}$$

$$K_2(x,y)=\frac{1}{2\mathrm{i}}(K(x,y)-K^*(x,y)), \tag{4.6}$$

therefore K_1, K_2 are hermitian kernels and $K=K_1+\mathrm{i}K_2$. Since K is decomposable, there exist positive definite kernels L_1 and L_2 such that the kernel $\begin{bmatrix} L_1 & K \\ K^* & L_2\end{bmatrix}$ is positive definite. Therefore,

$$\begin{aligned}&\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}^*\begin{bmatrix} L_1 & K \\ K^* & L_2\end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix} \\ &\qquad= \begin{bmatrix} L_1+K^*+K+L_2 & L_1+K-K^*-L_2 \\ L_1+K^*-K-L_2 & L_1-K^*-K+L_2\end{bmatrix}\end{aligned}$$

is also a positive definite kernel, which implies that $-\frac{1}{2}(L_1+L_2)\leq K_1\leq \frac{1}{2}(L_1+L_2)$. Similarly,

$$\begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{bmatrix}^* \begin{bmatrix} L_1 & K \\ K^* & L_2 \end{bmatrix} \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{bmatrix} = \begin{bmatrix} L_1+\mathrm{i}K^*-\mathrm{i}K+L_2 & \mathrm{i}L_1-K^*+K+\mathrm{i}L_2 \\ -\mathrm{i}L_1-K+K^*+\mathrm{i}L_2 & L_1-\mathrm{i}L_1-\mathrm{i}K^*+\mathrm{i}K+L_2 \end{bmatrix}$$

is a positive definite kernel, which gives that $-\frac{1}{2}(L_1+L_2)\leq K_2\leq \frac{1}{2}(L_1+L_2)$. Since $\frac{1}{2}(L_1+L_2)$ is a positive definite kernel, we deduce from Theorem 2.1 that both K_1 and K_2 have Kolmogorov decompositions, say

$$K_i(x,y)=V_i(x)^*J_1V_i(y), \quad i=1,2,$$

where $V_i : X\to \mathcal{L}(\mathcal{H},\mathcal{K}_i)$ and J_i are fundamental symmetries on $\mathcal{K}_i$, $i=1,2$. It follows that

$$\begin{aligned} K(x,y) &= V_1(x)^*J_1V_1(y)+\mathrm{i}V_2(x)^*J_2V_2(y) \\ &= \begin{bmatrix} V_1(x) \\ V_2(x) \end{bmatrix}^* \begin{bmatrix} J_1 & 0 \\ 0 & \mathrm{i}J_2 \end{bmatrix} \begin{bmatrix} V_1(y) \\ V_2(y) \end{bmatrix}. \end{aligned}$$

Define $V'(x)=\begin{bmatrix} V_1(x) \\ V_2(x) \end{bmatrix} : \mathcal{H}\to\mathcal{K}_1\oplus\mathcal{K}_2$ and let $\mathcal{K}$ be the closure in $\mathcal{K}_1\oplus\mathcal{K}_2$ of the linear space generated by the elements of the form $V'(x)h$, $x\in X$ and $h\in\mathcal{H}$. Finally, define $V(x)=P_{\mathcal{K}}V'(x)$, where $P_{\mathcal{K}}$ denotes the orthogonal projection of $\mathcal{K}_1\oplus\mathcal{K}_2$ onto $\mathcal{K}$. Then the set $\{V(x)h \mid x\in X, h\in\mathcal{H}\}$ is total in $\mathcal{K}$ and $K(x,y)=V(x)^*P_{\mathcal{K}}\begin{bmatrix} J_1 & 0 \\ 0 & \mathrm{i}J_2 \end{bmatrix}P_{\mathcal{K}}V(y)$. The operator $U=P_{\mathcal{K}}\begin{bmatrix} J_1 & 0 \\ 0 & \mathrm{i}J_2 \end{bmatrix}P_{\mathcal{K}}$ is a contraction and the proof is concluded. □

5. Holomorphic kernels

There are many examples of hermitian kernels which are holomorphic on some domain in the complex plane, see for instance [5]. In all these cases it is known that the kernels are associated with reproducing kernel Kreĭn (Hilbert) spaces, and D. Alpay in [3] proved a general result in this direction. Our goal is to extend the result in [3] to the multi-variable case. In view of the transcription between Kolmogorov decompositions and reproducing kernel spaces, e.g. see the invariant version in Theorem 2.4, we actually show that the original idea of the proof in [3], which goes back to [18], can be adapted to this multi-variable setting.

We first review the well-known example of the Szegö kernel (see [4]). Let $\mathcal{G}$ be a Hilbert space and denote by $B_r(\xi)$ the open ball of radius r and center ξ, $B_r(\xi)=\{\eta\in\mathcal{G}\mid \|\eta-\xi\|<r\}$. We write B_r instead of $B_r(0)$. For $\xi,\eta\in B_1$, define

$$S(\xi,\eta)=\frac{1}{1-\langle\eta,\xi\rangle}, \tag{5.1}$$

and note that S is a positive definite kernel on B_1. We now describe its Kolmogorov decomposition. Let

$$F(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \mathcal{G}^{\otimes n},$$

be the Fock space associated to $\mathcal{G}$, where $\mathcal{G}^{\otimes 0} = \mathbb{C}$ and $\mathcal{G}^{\otimes n}$ is the n-fold tensor product of $\mathcal{G}$ with itself. Let

$$P_n = (n!)^{-1} \sum_{\pi \in S_n} \hat{\pi} \tag{5.2}$$

be the orthogonal projection of $\mathcal{G}^{\otimes n}$ onto its symmetric part, where

$$\hat{\pi}(\xi_1 \otimes \ldots \otimes \xi_n) = \xi_{\pi^{-1}(1)} \otimes \ldots \otimes \xi_{\pi^{-1}(n)}$$

and π is an element of the permutation group S_n on n symbols. The symmetric Fock space is $F^s(\mathcal{G}) = (\bigoplus_{n=0}^{\infty} P_n)F(\mathcal{G})$. For $\xi \in B_1$ set $\xi^{\otimes 0} = 1$ and let $\xi^{\otimes n}$ denote the n-fold tensor product $\xi \otimes \ldots \otimes \xi$, $n \geq 1$. Note that

$$\|\bigoplus_{n\geq 0} \xi^{\otimes n}\|^2 = \sum_{n\geq 0} \|\xi^{\otimes n}\|^2 = \sum_{n\geq 0} \|\xi\|^{2n} = \frac{1}{1-\|\xi\|^2}.$$

Hence $\bigoplus_{n\geq 0} \xi^{\otimes n} \in F^s(\mathcal{G})$ and we can define the mapping V_S from B_1 into $F^s(\mathcal{G})$,

$$V_S(\xi) = \bigoplus_{n\geq 0} \xi^{\otimes n}, \quad \xi \in \mathcal{G}. \tag{5.3}$$

Lemma 5.1. *The pair $(V_S, F^s(\mathcal{G}))$ is the Kolmogorov decomposition of the kernel S.*

Proof. $V_S(\xi)$ is also viewed as a bounded linear operator from $\mathbb{C}$ into $F^s(\mathcal{G})$ by $V_S(\xi)\lambda = \lambda V_S(\xi)$, $\lambda \in \mathbb{C}$, so that, for $\xi, \eta \in B_1$,

$$\begin{aligned} V_S(\xi)^* V_S(\eta) &= \langle V_S(\eta), V_S(\xi)\rangle \\ &= \sum_{n\geq 0} \langle \eta^{\otimes n}, \xi^{\otimes n}\rangle \\ &= \sum_{n\geq 0} \langle \eta, \xi\rangle^n = \frac{1}{1-\langle \eta, \xi\rangle} = S(\xi, \eta). \end{aligned}$$

The set $\{V_S(\xi) \mid \xi \in B_1\}$ is total in $F^s(\mathcal{G})$ since for $n \geq 1$ and $\xi \in \mathcal{G}$ we have $\frac{d^n}{dt^n} V(t\xi)|_{t=0} = n!\xi^{\otimes n}$. $\square$

The reproducing kernel Hilbert space associated to the Szegö kernel S, see (5.1), is given by the completion of the linear space generated by the functions $s_\eta = S(\cdot, \eta)$, $\eta \in B_1$, with respect to the inner product defined by

$$\langle s_\eta, s_\xi\rangle = S(\xi, \eta).$$

Note that there exists a unitary operator $\mathcal{F}$ from the reproducing kernel Hilbert space associated to the Szegö kernel S onto $F^s(\mathcal{G})$ such that $\mathcal{F}s_\xi = V_S(\xi)$, $\xi \in B_1$.

We now explore the fact that S is a holomorphic kernel. We use the terminology and results from [15], [9] for holomorphic functions in infinite dimensions. Thus, we say that a function f defined on the open subset $\mathcal{O}$ of $\mathcal{G}$ is holomorphic if f is continuous on $\mathcal{O}$ and for all $\eta \in \mathcal{O}$, $\xi \in \mathcal{G}$, the mapping $\lambda \to f(\eta+\lambda\xi)$ is holomorphic on the open set $\{\lambda \in \mathbb{C} \mid \eta + \lambda\xi \in \mathcal{O}\}$. One easily checks that V_S is holomorphic on B_1, therefore $S(\xi, \cdot)$ is holomorphic on B_1 for each fixed $\xi \in B_1$. We also notice that the reproducing kernel Hilbert space associated to S consists of anti-holomorphic functions on B_1. It is somewhat more convenient to replace this space by a Hilbert space of holomorphic functions on B_1. Thus, define the holomorphic function $a_\xi = S(\xi, \cdot)$ on B_1 for each $\xi \in B_1$, and let $H^2(\mathcal{G})$ denote the completion of the linear space generated by the functions a_ξ, $\xi \in B_1$, with respect to the inner product defined by

$$\langle a_\xi, a_\eta \rangle = S(\xi, \eta), \quad \xi, \eta \in B_1.$$

We notice that $H^2(\mathcal{G})$ is an anti-unitary copy of the reproducing kernel Hilbert space of S.

We say that a hermitian kernel K, defined on an open subset $\mathcal{O}$ of $\mathcal{G}$, is *holomorphic on* $\mathcal{O}$ if $K(\xi, \cdot)$ is holomorphic on $\mathcal{O}$ for each fixed $\xi \in \mathcal{O}$.

Theorem 5.2. *Let K be a holomorphic hermitian kernel on B_r, with $r > 0$, and valued in $\mathcal{L}(\mathcal{G})$ for some Hilbert space $\mathcal{G}$. Then there exists $0 < r' \leq r$ and a Kolmogorov decomposition $(V; \mathcal{K})$ of $K|B_{r'} \times B_{r'}$, such that $V \colon B_{r'} \to \mathcal{L}(\mathcal{G}, \mathcal{K})$ is holomorphic.*

Proof. Let K be a holomorphic kernel on B_r, $r > 0$. Since K is hermitian, it follows that $K(\cdot, \eta)$ is anti-holomorphic on B_r for each $\eta \in B_r$. It is convenient to reformulate this fact as follows. Let $\{e_\alpha\}_{\alpha \in A}$ be an orthonormal basis for $\mathcal{G}$. We define the mapping

$$\xi = \sum_{\alpha \in A} \langle \xi, e_\alpha \rangle e_\alpha \to \sum_{\alpha \in A} \overline{\langle \xi, e_\alpha \rangle} e_\alpha = \xi^*,$$

so that the function $f(\xi, \eta) = K(\xi^*, \eta)$ is separately holomorphic on $B_r \times B_r$. By Hartogs' Theorem ([15], Theorem 36.8), f is holomorphic on $B_r \times B_r$. By ([15], Proposition 8.6), f is locally bounded. We first suppose that $r > 1$. Hence there exist $1 < \rho < r$ and $C > 0$ such that:

$$|K(\xi, \eta)| \leq C \quad \text{for} \quad \xi, \eta \in B_\rho \tag{5.4}$$

and

$$K(\xi^*, \eta) = \sum_{m \geq 0} p_m(\xi, \eta) \tag{5.5}$$

uniformly on B_ρ, where each p_m, $m \geq 0$, is an m-homogeneous continuous polynomial on $\mathcal{G} \times \mathcal{G}$. That is (see [15] or [9], Chapter 1), there exists a continuous linear mapping A_m on $P_m((\mathcal{G} \times \mathcal{G})^{\otimes m})$ such that

$$p_m(\xi, \eta) = A_m((\xi, \eta)^{\otimes m}) \tag{5.6}$$

for all $\xi, \eta \in \mathcal{G}$.

Using Cauchy Inequalities, [9], Proposition 3.2, for B_ρ, we deduce

$$\|A_m\| \le \|p_m\| \le C\frac{1}{\rho^m}, \tag{5.7}$$

hence

$$\sum_{m\ge 0}\|A_m\|^2 \le C\sum_{m\ge 0}\frac{1}{\rho^{2m}} = C\frac{1}{1-1/\rho^2} = C' < \infty. \tag{5.8}$$

The previous facts are valid with respect to the norm $\|(\xi,\eta)\| = \max\{\|\xi\|,\|\eta\|\}$. Since this norm is equivalent to the Hilbert norm $\|(\xi,\eta)\| = \sqrt{\|\xi\|^2+\|\eta\|^2}$, we deduce that each A_m is also continuous with respect to this Hilbert norm on $\mathcal{G}\times\mathcal{G}$. By Riesz representation theorem, there exist $a_m \in P_m(\mathcal{G}\times\mathcal{G})^{\otimes m}$, $m \ge 0$, such that

$$A_m((\xi,\eta)^{\otimes m}) = \langle(\xi,\eta)^{\otimes m}, a_m\rangle \tag{5.9}$$

and

$$\|a_m\| = \|A_m\| \tag{5.10}$$

(with $a_0 = A_0 \in \mathbb{C}$). Since $P_m(\mathcal{G}\times\mathcal{G})^{\otimes m}$ is isometrically isomorphic to $(P_m\mathcal{G}^{\otimes m})^{\oplus(m+1)}$, we deduce that there are $a_m^k \in P_m\mathcal{G}^{\otimes m}$, $k = 0,\ldots,m$, such that

$$\langle(\xi,\eta)^{\otimes m}, a_m\rangle = \sum_{k=0}^{m}\langle b_m^k(\xi,\eta), a_m^k\rangle \tag{5.11}$$

and

$$\sum_{k=0}^{m}\|a_m^k\|^2 = \|a_m\|^2, \tag{5.12}$$

where $b_0^0 = 1$ and $b_m^k(\xi,\eta) = \xi^{\otimes(m-k)}\otimes\eta^{\otimes k}$, $m\ge 1$, $k = 0,\ldots,m$.

We now show that for each fixed $\xi \in B_1$, $g_\xi(\eta) = K(\xi,\eta)$ belongs to $H^2(\mathcal{G})$. By (5.5), (5.6), (5.9), and (5.11),

$$K(\xi,\eta) = \sum_{m\ge 0}\sum_{k=0}^{m}\langle b_m^k(\xi^*,\eta), a_m^k\rangle,$$

and the series converges absolutely on η by (5.7). Reordering to m-homogeneous terms in η,

$$K(\xi,\eta) = \sum_{k\ge 0}\sum_{m\ge k}\langle b_m^k(\xi^*,\eta), a_m^k\rangle.$$

For fixed ξ define $F_k(\eta)=\sum_{m\geq k}\langle b^k_m(\xi^*,\eta),a^k_m\rangle$. By Schwarz inequality,

$$\begin{aligned}\|F_k(\eta)\|^2&\leq (\sum_{m\geq k}\|b^k_m(\xi^*,\eta)\|^2)(\sum_{m\geq k}\|a^k_m\|^2)\\ &=\|\eta\|^{2k}(\sum_{m\geq k}\|\xi\|^{2(m-k)})(\sum_{m\geq k}\|a^k_m\|^2)\\ &=\frac{\|\eta\|^{2k}}{1-\|\xi\|^2}\sum_{m\geq k}\|a^k_m\|^2,\end{aligned}$$

which implies

$$\|F_k\|\leq \frac{1}{1-\|\xi\|^2}\sum_{m\geq k}\|a^k_m\|^2.$$

Finally,

$$\|g_\xi\|^2_{H^2(\mathcal{G})}=\sum_{k\geq 0}\|F_k\|^2\leq \frac{1}{1-\|\xi\|^2}\sum_{k\geq 0}\sum_{m\geq k}\|a^k_m\|^2.$$

By (5.8), (5.10), and (5.12), we deduce that

$$\|g_\xi\|^2_{H^2(\mathcal{G})}\leq \frac{1}{1-\|\xi\|^2}\sum_{m\geq 0}\|A_m\|^2\leq C'\frac{1}{1-\|\xi\|^2}.$$

This shows that $g_\xi\in H^2(\mathcal{G})$ and, more than that, the formula

$$\mathcal{P}a_\xi=g_\xi$$

gives a bounded linear operator $\mathcal{P}$ on $H^2(\mathcal{G})$, such that

$$K(\xi,\eta)=g_\xi(\eta)=(\mathcal{P}a_\xi)(\eta)=\langle \mathcal{P}a_\xi,a_\eta\rangle_{H^2(\mathcal{G})}.$$

This implies that $\mathcal{P}$ is selfadjoint and let $\mathcal{P}=\mathcal{P}_+-\mathcal{P}_-$ be its Jordan decomposition, where $\mathcal{P}_\pm$ are positive operators on $H^2(\mathcal{G})$. Then K can be written as the difference of two positive definite kernels. By [18] and Theorem 2.3, K has a Kolmogorov decomposition, and it is easy to see that this Kolmogorov decomposition has the required holomorphy property. In case $r\leq 1$, a scaling argument as in [3] concludes the proof. □

As mentioned, the above proof is based on the same idea as in [3], which goes back to [18]. An interesting aspect of this idea is that once again the Szegö kernel S has a certain universality property, that is, any holomorphic kernel on B_r, $r>1$, is the image of S through a bounded selfadjoint operator on $H^2(\mathcal{G})$. A different kind of universality property of S, related to the solution of the Nevanlinna-Pick interpolation problem, was established in [1].

Finally we apply Theorem 5.2 to show that non-hermitian holomorphic kernels are decomposable. A kernel $K\colon \mathcal{O}\times\mathcal{O}\to\mathcal{L}(\mathcal{G})$, where $\mathcal{O}$ is an open subset of some Hilbert space $\mathcal{G}$, is *holomorphic* on $\mathcal{O}$ if, for any fixed $\xi\in\mathcal{O}$, the function $K(\xi,\cdot)\colon\mathcal{O}\to\mathcal{L}(\mathcal{G})$ is holomorphic and, for any fixed $\eta\in\mathcal{O}$, the function $K(\cdot,\eta)\colon\mathcal{O}\to\mathcal{L}(\mathcal{G})$ is anti-holomorphic.

Corollary 5.3. *Let K be a holomorphic kernel on B_r, with $r > 0$, and valued in $\mathcal{L}(\mathcal{G})$ for some Hilbert space $\mathcal{G}$. Then there exists $0 < r' \leq r$, a Hilbert space $\mathcal{K}$, a holomorphic mapping $V : B_{r'} \to \mathcal{L}(\mathcal{G}, \mathcal{K})$, and a contraction U on $\mathcal{K}$, such that $K(\xi, \eta) = V(\xi)^* U V(\eta)$ for all $\xi, \eta \in B_{r'}$, and the set $\{V(\xi)h \mid \xi \in B_{r'}, h \in \mathcal{G}\}$ is total in $\mathcal{K}$.*

Proof. We consider the real part K_1 (4.5) and, respectively, the imaginary part K_2 (4.6) of K and note that both are holomorphic hermitian kernels. Then we apply Theorem 5.2 to produce holomorphic Kolmogorov decompositions of K_1 and K_2 on a possibly smaller, but nontrivial, ball $B_{r'}$ in $\mathcal{G}$ and, proceeding as in the proof of Theorem 4.4, we obtain the decomposition of K as required. □

References

[1] J. Agler, J. E. McCarthy: Complete Nevanlinna-Pick kernels, *J. Funct. Anal.*, **175**(2000), 111–124.

[2] S. Albeverio, H. Gottschalk, J.-L. Wu: Models of local relativistic quantum fields with indefinite metric (in all dimensions), *Commun. Math. Phys.*, **184**(1997), 509–531.

[3] D. Alpay: Some remarks on reproducing kernel Kreĭn spaces, *Rocky Mountain J. Math.*, **21**(1991), 1189–1205.

[4] W. Arveson: Subalgebras of C^*-algebras. III. Multivariable operator theory, *Acta Math.*, **181**(1998), 159–228.

[5] D. Alpay, A. Dijksma, J. Rovnyak, H. de Snoo: *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Birkhäuser Verlag, Basel, 1997.

[6] T. Constantinescu: Orthogonal polynomials in several variables. I, lanl, math.FA/0205333.

[7] T. Constantinescu, A. Gheondea: Representations of hermitian kernels by means of Kreĭn spaces, *Publ. RIMS. Kyoto Univ.*, **33**(1997), 917–951.

[8] T. Constantinescu, A. Gheondea: Representations of hermitian kernels by means of Kreĭn spaces II. Invariant kernels, *Commun. Math. Phys.*, **216**(2001), 409–430.

[9] S. Dineen: *Complex analysis on infinite-dimensional spaces*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 1999.

[10] E. Effros, Zh.-J. Ruan: *Operator Spaces*, London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York 2000.

[11] D. E. Evans, J. T. Lewis: *Dilations of Irreducible Evolutions in Algebraic Quantum Theory*, Dublin Institute for Advanced Studies, Dublin, 1977.

[12] U. Haagerup: Injectivity and decomposition of completely bounded maps, in *Operator Algebras and their Connections with Topology and Ergodic Theory (Buşteni, 1983)*, pp. 170–222, Lecture Notes in Math., Vol. 1132, Springer, Berlin 1985.

[13] G. Hofmann, On GNS representations on inner product spaces I. The structure of the representation space, *Commun. Math. Phys.*, **191**(1998), 299-323.

[14] G. Morchio, D. Pierotti, F. Strocchi: Infrared and vacuum structure in two-dimensional local quantum field theory models. The massless scalar field, *J. Math. Phys.*, **31**(1990), 1467-1477.

[15] J. Mujica: *Complex analysis in Banach spaces*, North-Holland, Amsterdam, 1986.

[16] V. I. Paulsen: *Completely bounded maps and dilations*, Pitman Research Notes in Math., 146, Longman, Wiley, New York, 1986.

[17] K. R. Parthasaraty, K. Schmidt: *Positive-Definite Kernels, Continous Tensor Products and Central Limit Theorems of Probability Theory*, Lecture Notes in Mathematics, Vol. **272**, Springer-Verlag, Berlin 1972.

[18] L. Schwartz: Sous espace Hilbertiens d'espaces vectoriel topologiques et noyaux associés (noyaux reproduisants), *J. Analyse Math.*, **13**(1964), 115–256.

[19] F. Strocchi: *Selected Topics on the General Properties of Quantum Field Theory*, Lecture Notes Phys. 51, World Scientific, 1993.
[20] G. Wittstock: Ein operatorwertiger Hahn-Banach Satz, *J. Functional Analysis*, **40**(1981), 127–150.

Department of Mathematics, University of Texas at Dallas, Box 830688 Richardson, TX 75083-0688, U. S. A.

E-mail address: tiberiu@utdallas.edu

Institutul de Matematică al Academiei Române, C.P. 1-764, 70700 Bucureşti, România

E-mail address: gheondea@imar.ro