REPORT ON THE PAPER "AN INJECTIVITY THEOREM"

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We are interested in the following *lifting problem*: given a Cartier divisor L on a complex variety X and a closed subvariety $Y \subset X$, when is the restriction map

$$\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L))$$

surjective? The standard method is to consider the short exact sequence

$$0 \to \mathcal{I}_Y(L) \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L) \to 0,$$

which induces a long exact sequence in cohomology

$$0 \to \Gamma(X, \mathcal{I}_Y(L)) \to \Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L)) \to H^1(X, \mathcal{I}_Y(L)) \stackrel{\alpha}{\to} H^1(X, \mathcal{O}_X(L)) \cdots$$

The restriction is surjective if and only if α is injective. In particular, if $H^1(X, \mathcal{I}_Y(L)) = 0$.

If X is a nonsingular proper curve, Serre duality answers completely the lifting problem: the restriction map is not surjective if and only if $L \sim K_X + Y - D$ for some effective divisor D such that $D - Y$ is not effective. In particular, $\deg L \le \deg(K_X + Y)$. If $\deg L > \deg(K_X + Y)$, then $H^1(X, \mathcal{I}_Y(L)) = 0$, and therefore lifting holds.

If X is a nonsingular projective surface, only sufficient criteria for lifting are known (see [29]). If H is a general hyperplane section induced by a Veronese embedding of sufficiently large degree (depending on L), then $\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(H, \mathcal{O}_H(L))$ is an isomorphism (Enriques-Severi-Zariski). If H is a hyperplane section of X, then $H^i(X, \mathcal{O}_X(K_X + H)) = 0$ $(i > 0)$ (Picard-Severi).

These classical results were extended by Serre [24] as follows: if X is affine and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F}) = 0$ $(i > 0)$. If X is projective, H is ample and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F}(mH)) = 0$ $(i > 0)$ for m sufficiently large.

Kodaira [17] extended Picard-Severi's result as follows: if X is a projective complex manifold, and H is an ample divisor, then $H^i(X, \mathcal{O}_X(K_X + H)) = 0$ $(i > 0)$. This vanishing remains true over a field of characteristic zero, but may fail in positive characteristic (Raynaud [23]). Kodaira's vanishing is central in the classification theory of complex algebraic varieties, but one has to weaken the positivity of H to apply it successfully: it still holds if H is only semiample and big (Mumford [20], Ramanujam [21]), or if $K_X + H$ is replaced by $\lceil K_X + H \rceil$ for a $\mathbb{Q}$-divisor H which is nef and big, whose fractional part is supported by a normal crossings divisor (Ramanujam [22], Miyaoka [19], Kawamata [16], Viehweg [28]). Recall that the round up of a real number x is $\lceil x \rceil = \min\{n \in \mathbb{Z}; x \le n\}$, and the round up of a $\mathbb{Q}$-divisor $D = \sum_E d_E E$ is $\lceil D \rceil = \sum_E \lceil d_E \rceil E$.

The first lifting criterion in the absence of bigness is due to Tankeev [27]: if X is proper nonsingular and $Y \subset X$ is the general member of a free linear system, then the restriction

$$\Gamma(X, \mathcal{O}_X(K_X + 2Y)) \to \Gamma(Y, \mathcal{O}_Y(K_X + 2Y))$$

is surjective. Kollár [18] extended it to the following injectivity theorem: if H is a semiample divisor and $D \in |m_0 H|$ for some $m_0 \ge 1$, then the homomorphism

$$H^q(X, \mathcal{O}_X(K_X + mH)) \to H^q(X, \mathcal{O}_X(K_X + mH + D))$$

is injective for all $m \geq 1, q \geq 0$. Esnault and Viehweg [12, 13] removed completely the positivity assumption, to obtain the following injectivity result: let L be a Cartier divisor on X such that $L \sim_{\mathbb{Q}} K_X + \sum_i b_i E_i$, where $\sum_i E_i$ is a normal crossings divisor and $0 \leq b_i \leq 1$ are rational numbers. If D is an effective divisor supported by $\sum_{0<b_i<1} E_i$, then the homomorphism

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

is injective, for all q. The original result [13, Theorem 5.1] was stated in terms of roots of sections of powers of line bundles, and restated in this logarithmic form in [2, Corollary 3.2]. It was used in [1, 2] to derive basic properties of log varieties and quasi-log varieties.

The main result of this paper (Theorem 2.3) is that Esnault-Viehweg's injectivity remains true even if some components E_i of D have $b_i = 1$. In fact, it reduces to the special case when all $b_i = 1$, which has the following geometric interpretation:

Theorem 0.1. *Let X be a proper nonsingular variety, defined over an algebraically closed field of characteristic zero. Let Σ be a normal crossings divisor on X, let $U = X \setminus \Sigma$. Then the restriction homomorphism*

$$H^q(X, \mathcal{O}_X(K_X + \Sigma)) \to H^q(U, \mathcal{O}_U(K_U))$$

is injective, for all q.

Combined with Serre vanishing on affine varieties, it gives:

Corollary 0.2. *Let X be a proper nonsingular variety, defined over an algebraically closed field of characteristic zero. Let Σ be a normal crossings divisor on X such that $X \setminus \Sigma$ is contained in an affine open subset of X. Then*

$$H^q(X, \mathcal{O}_X(K_X + \Sigma)) = 0$$

for $q > 0$.

If $X \setminus \Sigma$ itself is affine, this vanishing is due to Esnault and Viehweg [13, page 5]. It implies the Kodaira vanishing theorem.

We outline the structure of this paper. After some preliminaries in Section 1, we prove the main injectivity result in Section 2. The proof is similar to that of Esnault-Viehweg, except that we do not use duality. It is an immediate consequence of the Atiyah-Hodge Lemma and Deligne's degeneration of the logarithmic Hodge to de Rham spectral sequence. In Section 3, we obtain some vanishing theorems for sheaves of logarithmic forms of intermediate degree. The results are the same as in [13], except that the complement of the boundary is only contained in an affine open subset, instead of being itself affine. They suggest that injectivity may extend to forms of intermediate degree (Question 7.1). In section 4, we introduce the *locus of totally canonical singularities* and the *non-log canonical locus* of a log variety. The latter has the same support as the subscheme structure for the non-log canonical locus introduced in [1], but the scheme structure usually differ (see Remark 4.4). In Section 5, we partially extend the injectivity theorem to the category of log varieties. The open subset to which we restrict is the locus of totally canonical singularities of some log structure. We can only prove the injectivity for the first cohomology group. The idea is to descend injectivity from a log resolution, and to make this work for higher cohomology groups one needs vanishing theorems or at least the degeneration of the Leray spectral sequence for a certain resolution. We do not pursue this here. In Section 6, we establish the *lifting property of* $\Gamma(X, \mathcal{O}_X(L)) \to \Gamma(Y, \mathcal{O}_Y(L))$

for a Cartier divisor $L \sim_{\mathbb{R}} K_X + B$*, with* Y *the non-log canonical locus of* X (Theorem 6.2). We give two applications for this unexpected property. For a proper generalized log Calabi-Yau variety, we show that the non-log canonical locus is connected and intersects every lc center (Theorem 6.3). And we obtain an extension theorem from a union of log canonical centers, in the log canonical case (Theorem 6.4). We expect this extension to play a key role in the characterization of the restriction of log canonical rings to lc centers. In Section 7 we list some questions that appeared naturally during this work.

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