HODGE THEORY
HOMEWORK 2

1) Prove that every closed r-form on $\mathbb{R}^n$ is exact (*Poincare's Lemma*) as follows. First, prove it for $r = 0$. Then assume $r \geq 1$ and let α a r-form on $\mathbb{R}^n$ such that $d\alpha = 0$. Write $\alpha = \sum_{|I|=r} a_I dx_I$ in the usual basis and define

$$\beta = \sum_{I} \sum_{k=1}^{r} \left((-1)^k \int_0^1 a_I(tx) t^{k-1} dt \right) x_{i_k} dx_{i_1} \wedge \cdots \wedge \widehat{d_{x_{i_k}}} \wedge \cdots \wedge dx_{i_r},$$

where the hat means omit the term. Then show that $d\beta = \alpha$.

2) Use the Mayer-Vietoris sequence to compute the de Rham cohomology of the following spaces:

(a) $M = \mathbb{R}^2 \setminus \{a\}$ for $a \in \mathbb{R}^2$.
(b) $M = \mathbb{R}^2 \setminus \{a, b\}$ where $a, b \in \mathbb{R}^2$ are distinct.
(c) $M = \mathbb{R}^2 \setminus \{a_1, \ldots, a_n\}$ where $a_i \in \mathbb{R}^2$ are all distinct.
(d) $M = \mathbb{S}^1 \times \mathbb{S}^1$ the 2-dimensional torus.
(e) $M = \mathbb{R}^n \setminus \{0\}$.

3) Let α, β differential forms on M. Show that: If α and β are closed, then $\alpha \wedge \beta$ is also closed; If either α or β are exact, then $\alpha \wedge \beta$ is also exact. Conclude that the map:

$$H^*(M) \times H^*(M) \ni ([\alpha], [\beta]) \to [\alpha] \cup [\beta] := [\alpha \wedge \beta] \in H^*(M)$$

is well defined, i.e. does not depend on the choice of representatives α, β for the cohomology classes $[\alpha], [\beta]$. This map is called the *cup product*. Prove that $H^*(M)$ with the usual addition and the multiplication defined by $\cup$ is a ring. Is it a commutative ring?

4) Let M_1 and M_2 be differentiable manifolds and $p_i : M_1 \times M_2 \to M_i$ the projections, $i = 1, 2$. Show that the map π defined by:

$$H^*(M_1) \otimes H^*(M_2) \ni [\alpha] \otimes [\beta] \to [\alpha] \times [\beta] := [p_1^*\alpha] \cup [p_2^*\beta] \in H^*(M_1 \times M_2)$$

is well-defined. This map is called the *cross product*. Prove that π is a ring isomorphism.

5) Assuming the cohomology spaces of the n-sphere known, determine the cohomology ring $H^*(M)$ for the following manifolds: (a) $M = \mathbb{S}^n$, (b) $M = \mathbb{S}^1 \times \mathbb{S}^1$, and (c) $M = \mathbb{S}^2 \times \mathbb{S}^2$.