HODGE THEORY

HOMEWORK 1

The algebra of quaternions \( \mathbb{H} \) can be identified with \( \mathbb{R}^4 \) by writing \( q \in \mathbb{H} \) in the form \( q = x_1 + ix_2 + jx_3 + kx_4 \), where the \( x_i \) are real numbers. The addition in \( \mathbb{H} \) is component-wise, while the multiplication uses that \( i^2 = j^2 = k^2 = -1 \) and \( ij = k, jk = i, ki = j \). Moreover \( \mathbb{H} \) can also be identified with \( \mathbb{C}^2 \) by writing \( q = z_1 + jz_2 \), where \( z_1 = x_1 + ix_2 \) and \( z_2 = x_3 + ix_4 \). Define the conjugate of \( q \) by \( \bar{q} = x_1 - ix_2 - jx_3 - kx_4 \) and the norm \( |q|^2 = q\bar{q} = x_1^2 + x_2^2 + x_3^2 + x_4^2 \). Furthermore \( \mathbb{H} \) can be identified with a complex matrix group by

\[
q = x_1 + ix_2 + jx_3 + kx_4 \rightarrow \begin{pmatrix} z_1 & z_2 \\
-\overline{z_2} & \overline{z_1} \end{pmatrix}.
\]

1) Let \( S^n = \{ x = (x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1 \} \) be the \( n \)-dimensional unit sphere in \( \mathbb{R}^{n+1} \).

(a) Let \( N = (0, \ldots, 0, 1) \) and \( S = (0, \ldots, 0, -1) \) be the north and resp. south pole on \( S^n \) and set \( U_N = S^n \setminus \{ N \} \) and \( U_S = S^n \setminus \{ S \} \). Show that the atlas of coordinates \( \{(U_N, \pi_N), (U_S, \pi_S)\} \), where \( \pi_N : U_N \rightarrow \mathbb{R}^n, \pi_S : U_S \rightarrow \mathbb{R}^n \) are the stereographic projections from \( N \), resp. \( S \), onto the hyperplane \( \mathbb{R}^n = \{ x_{n+1} = 0 \} \) gives \( S^n \) a structure of differentiable manifold.

(b) Let \( U_i^+ = \{ x_i > 0 \} \cap S^n \) and \( U_i^- = \{ x_i < 0 \} \cap S^n \). Define local coordinates \( h_i^\pm : U_i^\pm \rightarrow \mathbb{R}^n \) by \( (x_1, \ldots, x_{n+1}) \rightarrow (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \). Show that the atlas of coordinates \( \{(U_i^\pm, h_i^\pm)\}_{1 \leq i \leq n+1} \) gives \( S^n \) a structure of differentiable manifold.

(c) Show that the differential structures on \( S^n \) from (a) and (b) are equivalent. Two atlases are equivalent if the union of their sets of charts forms an atlas. This differentiable manifold structure on \( S^n \) is called standard.

(d) Let \( A, B \) be antipodal points on \( S^4 \), and \( U = S^4 \setminus \{ A \} \) and \( V = S^4 \setminus \{ B \} \). Glue trivial sphere bundles \( U \times S^3 \) and \( V \times S^3 \) together to get a 3-sphere bundle \( \Sigma^7 \) over \( S^4 \) as follows.

Under stereographic projection \( U \) and \( V \) are homemorphic to \( \mathbb{R}^4 \), and \( U \cap V \) is homemorphic to \( \mathbb{R}^4 \setminus \{0\} \). Identify \( \mathbb{R}^4 \) with the quaternions \( \mathbb{H} \), and \( \mathbb{R}^4 \setminus \{0\} \) with non-zero quaternions \( \mathbb{H}^* \), and \( S^3 \) with the unitary quaternions. Choose an odd number \( k \) such that \( k^2 \neq 1 \mod 7 \). Define \( \tau : \mathbb{H}^* \times S^3 \rightarrow \mathbb{H}^* \times S^3 \) by \( \tau(q, x) = \left( \frac{q}{|q|^2}, \frac{q^m x q^n}{|q|} \right) \), where \( m = (1+k)/2 \) and \( n = (1-k)/2 \) and multiplication and norm \( |\cdot| \) are taken in \( \mathbb{H} \). Show that \( \tau \) is a smooth map, and \( \Sigma^7 \) is homeomorphic to \( S^7 \). The manifold \( \Sigma^7 \) is Milnor’s exotic 7-sphere. One can show that the smooth structure on \( \Sigma^7 \) is different from the standard one.
(e) Show that the tangent space $T_x S^n$ at $x \in S^n$ can be identified with the subspace $x^\perp$ of $\mathbb{R}^{n+1}$ of all vectors orthogonal to $x$. Give a concrete description of $T S^n$ as a submanifold of $S^n \times \mathbb{R}^{n+1}$.

Show that the tangent bundles $T S^n$ are trivial for $n = 1$ and $n = 3$, by giving $n$ linearly independent sections (i.e. vector fields) of $T S^n$. Use the fact that, for $S^1$ are the unitary complex numbers, respectively that $S^3$ are the unitary quaternions. Is the fiber bundle $T S^2$ trivial?

2) Denote by $SO(n)$ the subgroup of $GL(n, \mathbb{R})$ of all orthogonal matrices of determinant 1:

$$SO(n) = \{ A \in GL(n, \mathbb{R}) \mid A^T A = I, \det(A) = 1 \}.$$ 

Denote by $U(n)$ the subgroup of $GL(n, \mathbb{C})$ of all unitary matrices:

$$U(n) = \{ A \in GL(n, \mathbb{C}) \mid \bar{A}^T A = I \},$$

and by $SU(n)$ the subgroup of $GL(n, \mathbb{C})$ of all unitary matrices of determinant 1:

$$SU(n) = \{ A \in GL(n, \mathbb{C}) \mid \bar{A}^T A = I, \det(A) = 1 \},$$

(a) Show that $S^1, \mathbb{RP}^1, SO(2)$ and $U(1)$ are diffeomorphic.

(b) Show that $S^2$ and $\mathbb{CP}^1$ are diffeomorphic.

(c) Show that $S^3$ and $SU(2)$ are diffeomorphic (use the quaternions $\mathbb{H}$).

(d) Show that $\mathbb{RP}^3$ and $SO(3)$ are diffeomorphic (use the quaternions $\mathbb{H}$).

3) (a) Show that the following maps are (principal) fiber bundles with fiber $S^1$.

Write local trivialisations and transition functions:

(i) the Hopf map $h : S^3 \to S^2$ given by $(z_0, z_1) \mapsto (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$.

(ii) the map $g : \mathbb{RP}^3 \to \mathbb{CP}^1$ given by $[x_0, x_1, x_2, x_3] \mapsto [x_0 + ix_1, x_2 + ix_3]$.

(b) Show that the natural projection $\mathbb{CP}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a holomorphic fiber bundle with fiber $\mathbb{C}^*$. The restriction to the sphere $S^{2n+1} = \{ z = (z_0, \ldots, z_n) \mid |z_0|^2 + \cdots + |z_n|^2 = 1 \} \subset \mathbb{C}^{n+1}$ gives the Hopf bundle $S^{2n+1} \to \mathbb{CP}^n$ with fiber the circle $S^1$. Write local trivialisations and transition functions for both fibrations.

4) The algebra of octonions $\mathbb{O}$ is the product algebra $\mathbb{H} \times \mathbb{H}$. The conjugate of an element $u = (p, q)$ in $\mathbb{O}$ is $\bar{u} = (\bar{p}, -q)$. The real numbers $\mathbb{R}$ can be identified with the subset $\{ u \in \mathbb{O} \mid \bar{u} = u \}$ of $\mathbb{O}$. The pure imaginary numbers in $\mathbb{O}$ are defined as $\mathbb{I} = \{ u \in \mathbb{O} \mid \bar{u} = -u \}$. Denote by $| \cdot |$ the norm on $\mathbb{O}$ associated to the inner-product $\langle u, v \rangle = \frac{1}{2}(\bar{u}v + v\bar{u})$. Let $S$ be the unit sphere in $\mathbb{I}$ defined by $S = \{ u \in \mathbb{I} \mid |u| = 1 \}$. It easy to see that $\mathbb{I}$ can be identified with $\mathbb{R}^7$ and $S$ with the 6-dimensional sphere $S^6$.

For each $u \in S$ define map $J_u : T_u S \to \mathbb{O}$, $v \mapsto vu$, where $T_u S$ is identified with subspace $u^\perp \cap \mathbb{I}$. Show that $J_u$ maps $T_u S$ to itself and defines an almost complex structure on $S$. 
5) Consider $S^{2p+1}$ as the set of all points $z = (z_0, \ldots, z_p)$ such that $\sum_{0 \leq k \leq p} z_k \bar{z}_k = 1$, and $S^{2q+1}$ as the set of all points $w = (w_0, \ldots, w_q)$ such that $\sum_{0 \leq k \leq p} w_j \bar{w}_j = 1$. Define  

$$U_{kj} = \{(z, w) \in S^{2p+1} \times S^{2q+1} \mid z_k w_j \neq 0 \}.$$  

Let $\tau$ be a complex number such that $\text{Im}(\tau) \neq 0$. Define $p + q + 1$ coordinate functions $u_{hk} = \frac{z_h}{z_k}$ for $h \neq k, 0 \leq h \leq p$ and $v_{lj} = \frac{w_l}{w_j}$ for $l \neq j, 0 \leq l \leq q$; and also $t_{kj} = \frac{1}{2\pi i} (\log z_k + \tau \log w_j)$, where equality is taken modulo 1 and $\tau$. Thus $t_{jk}$ defines a point on the torus $T = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$, the quotient of $\mathbb{C}$ by the translations by 1 and by $\tau$. Let $\phi_{kj} : U_{kj} \to \mathbb{C}^{p+q} \times T$ be the map defined by the functions $u_{hk}, v_{lj}, t_{kj}$. Prove that it is a homeomorphism.

Show that the system of coordinates $\{(U_{kj}, \phi_{kj})\}$ gives a structure of complex manifold on the product of spheres $S^{2p+1} \times S^{2q+1}$, where $p, q > 0$. 