HODGE THEORY
HOMEWORK 1

The algebra of quaternions $\mathbb{H}$ can be identified with $\mathbb{R}^4$ by writing $q \in \mathbb{H}$ in the form $q = x_1 + ix_2 + jx_3 + kx_4$, where the x_i are real numbers. The addition in $\mathbb{H}$ is component-wise, while the multiplication uses that $i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j$. Moreover $\mathbb{H}$ can also be identified with $\mathbb{C}^2$ by writing $q = z_1 + jz_2$, where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$. Define the conjugate of q by $\bar{q} = x_1 - ix_2 - jx_3 - kx_4$ and the norm $|q|^2 = q\bar{q} = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Furthermore $\mathbb{H}$ can be identified with a complex matrix group by

$$q = x_1 + ix_2 + jx_3 + kx_4 \to \begin{pmatrix} z_1 & z_2 \\ -z_2 & \bar{z}_1 \end{pmatrix}.$$

1) Let $\mathbb{S}^n = \{x = (x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ be the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.
(a) Let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ be the north and resp. south pole on $\mathbb{S}^n$ and set $U_N = \mathbb{S}^n \setminus \{N\}$ and $U_S = \mathbb{S}^n \setminus \{S\}$. Show that the atlas of coordinates $\{(U_N, \pi_N), (U_S, \pi_S)\}$, where $\pi_N : U_N \to \mathbb{R}^n, \pi_S : U_S \to \mathbb{R}^n$ are the stereographic projections from N, resp. S, onto the hyperplane $\mathbb{R}^n = \{x_{n+1} = 0\}$ gives $\mathbb{S}^n$ a structure of differentiable manifold.
(b) Let $U_i^+ = \{x_i > 0\} \cap \mathbb{S}^n$ and $U_i^- = \{x_i < 0\} \cap \mathbb{S}^n$. Define local coordinates $h_i^\pm : U_i^\pm \to \mathbb{R}^n$ by $(x_1, \ldots, x_{n+1}) \to (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$. Show that the atlas of coordinates $\{(U_i^\pm, h_i^\pm)\}_{1 \le i \le n+1}$ gives $\mathbb{S}^n$ a structure of differentiable manifold.
(c) Show that the differential structures on $\mathbb{S}^n$ from (a) and (b) are equivalent. Two atlases are equivalent if the union of their sets of charts forms an atlas. This differentiable manifold structure on $\mathbb{S}^n$ is called standard.
(d) Let A, B be antipodal points on $\mathbb{S}^4$, and $U = \mathbb{S}^4 \setminus \{A\}$ and $V = \mathbb{S}^4 \setminus \{B\}$. Glue trivial sphere bundles $U \times \mathbb{S}^3$ and $V \times \mathbb{S}^3$ together to get a 3-sphere bundle Σ^7 over $\mathbb{S}^4$ as follows.

Under stereographic projection U and V are homemorphic to $\mathbb{R}^4$, and $U \cap V$ is homemorphic to $\mathbb{R}^4 \setminus \{0\}$. Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, and $\mathbb{R}^4 \setminus \{0\}$ with non-zero quaternions $\mathbb{H}^*$, and $\mathbb{S}^3$ with the unitary quaternions. Choose an odd number k such that $k^2 \not\equiv 1 \mod 7$. Define $\tau : \mathbb{H}^* \times \mathbb{S}^3 \to \mathbb{H}^* \times \mathbb{S}^3$ by $\tau(q, x) = \left(\frac{q}{|q|^2}, \frac{q^m x q^n}{|q|} \right)$, where $m = (1+k)/2$ and $n = (1-k)/2$ and multiplication and norm $|\cdot|$ are taken in $\mathbb{H}$. Show that τ is a smooth map, and Σ^7 is homeomorphic to $\mathbb{S}^7$. The manifold Σ^7 is Milnor's exotic 7-sphere. One can show that the smooth structure on Σ^7 is different from the standard one.

(e) Show that the tangent space $T_x\mathbb{S}^n$ at $x \in \mathbb{S}^n$ can be identified with the subspace $x^\perp$ of $\mathbb{R}^{n+1}$ of all vectors orthogonal to x. Give a concrete description of $T\mathbb{S}^n$ as a submanifold of $\mathbb{S}^n \times \mathbb{R}^{n+1}$.

Show that the tangent bundles $T\mathbb{S}^n$ are trivial for $n = 1$ and $n = 3$, by giving n linearly independent sections (i.e. vector fields) of $T\mathbb{S}^n$. Use the fact that, for $\mathbb{S}^1$ are the unitary complex numbers, respectively that $\mathbb{S}^3$ are the unitary quaternions. Is the fiber bundle $T\mathbb{S}^2$ trivial?

2) Denote by $SO(n)$ the subgroup of $\mathrm{GL}(n, \mathbb{R})$ of all orthogonal matrices of determinant 1:

$$SO(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I, \, \det(A) = 1\}.$$

Denote by $U(n)$ the subgroup of $\mathrm{GL}(n, \mathbb{C})$ of all unitary matrices:

$$U(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{A}^T A = I\},$$

and by $SU(n)$ the subgroup of $\mathrm{GL}(n, \mathbb{C})$ of all unitary matrices of determinant 1:

$$SU(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{A}^T A = I, \, \det(A) = 1\},$$

(a) Show that $\mathbb{S}^1, \mathbb{RP}^1, SO(2)$ and $U(1)$ are diffeomorphic.
(b) Show that $\mathbb{S}^2$ and $\mathbb{CP}^1$ are diffeomorphic.
(c) Show that $\mathbb{S}^3$ and $SU(2)$ are diffeomorphic (use the quaternions $\mathbb{H}$).
(d) Show that $\mathbb{RP}^3$ and $SO(3)$ are diffeomorphic (use the quaternions $\mathbb{H}$).

3) (a) Show that the following maps are (principal) fiber bundles with fiber $\mathbb{S}^1$. Write local trivialisations and transition functions:
(i) the Hopf map $h : \mathbb{S}^3 \to \mathbb{S}^2$ given by $(z_0, z_1) \to (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$.
(ii) the map $g : \mathbb{RP}^3 \to \mathbb{CP}^1$ given by $[x_0, x_1, x_2, x_3] \to [x_0 + ix_1, x_2 + ix_3]$.
(b) Show that the natural projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is a holomorphic fiber bundle with fiber $\mathbb{C}^*$. The restriction to the sphere $\mathbb{S}^{2n+1} = \{z = (z_0, \ldots, z_n) \mid |z_0|^2 + \cdots + |z_n|^2 = 1\} \subset \mathbb{C}^{n+1}$ gives the Hopf bundle $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ with fiber the circle $\mathbb{S}^1$. Write local trivialisations and transition functions for both fibrations.

4) The algebra of octonions $\mathbb{O}$ is the product algebra $\mathbb{H} \times \mathbb{H}$. The conjugate of an element $u = (p, q)$ in $\mathbb{O}$ is $\bar{u} = (\bar{p}, -q)$. The real numbers $\mathbb{R}$ can be identified with the subset $\{u \in \mathbb{O} \mid \bar{u} = u\}$ of $\mathbb{O}$. The pure imaginary numbers in $\mathbb{O}$ are defined as $\mathbb{I} = \{u \in \mathbb{O} \mid \bar{u} = -u\}$. Denote by $|\cdot|$ the norm on $\mathbb{O}$ associated to the inner-product $\langle u, v \rangle = \frac{1}{2}(\bar{u}v + \bar{v}u)$. Let $\mathbb{S}$ be the unit sphere in $\mathbb{I}$ defined by $\mathbb{S} = \{u \in \mathbb{I} \mid |u| = 1\}$. It easy to see that $\mathbb{I}$ can be identified with $\mathbb{R}^7$ and $\mathbb{S}$ with the 6-dimensional sphere $\mathbb{S}^6$.

For each $u \in \mathbb{S}$ define map $J_u : T_u\mathbb{S} \to \mathbb{O}$, $v \to vu$, where $T_u\mathbb{S}$ is identified with subspace $u^\perp \cap \mathbb{I}$. Show that J_u maps $T_u\mathbb{S}$ to itself and defines an almost complex structure on $\mathbb{S}$.

5) Consider $\mathbb{S}^{2p+1}$ as the set of all points $z = (z_0, \ldots, z_p)$ such that $\sum_{0\leq k\leq p} z_k\bar{z}_k = 1$, and $\mathbb{S}^{2q+1}$ as the set of all points $w = (w_0, \ldots, w_q)$ such that $\sum_{0\leq k\leq p} w_j\bar{w}_j = 1$. Define

$$U_{kj} = \{(z, w) \in \mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1} \mid z_k w_j \neq 0\}.$$

Let τ be a complex number such that $\mathrm{Im}(\tau) \neq 0$. Define $p + q + 1$ coordinate functions $u_{hk} = \frac{z_h}{z_k}$ for $h \neq k, 0 \leq h \leq p$ and $v_{lj} = \frac{w_l}{w_j}$ for $l \neq j, 0 \leq l \leq q$; and also $t_{kj} = \frac{1}{2\pi i}(\log z_k + \tau \log w_j)$, where equality is taken modulo 1 and τ. Thus t_{jk} defines a point on the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, the quotient of $\mathbb{C}$ by the translations by 1 and by τ. Let $\phi_{kj} : U_{kj} \to \mathbb{C}^{p+q} \times \mathbb{T}$ be the map defined by the functions u_{hk}, v_{lj}, t_{kj}. Prove that it is a homeomorphism.

Show that the system of coordinates $\{(U_{kj}, \phi_{kj})\}$ gives a structure of complex manifold on the product of spheres $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$, where $p, q > 0$.