INTRODUCTION TO HODGE THEORY

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Abstract. This course will present the basics of Hodge theory aiming to familiarize students with an important technique in complex and algebraic geometry. We start by reviewing complex manifolds, Kahler manifolds and the de Rham theorems. We then introduce Laplacians and establish the connection between harmonic forms and cohomology. The main theorems are then detailed: the Hodge decomposition and the Lefschetz decomposition. The Hodge index theorem, Hodge structures and polarizations are discussed. The non-compact case is also considered. Finally, time permitted, rudiments of the theory of variations of Hodge structures are given.

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1. Introduction

The goal of these lectures is to explain the existence of special structures on the cohomology of Kahler manifolds, namely, the Hodge decomposition and the Lefschetz decomposition, and to discuss their basic properties and consequences.

A Kahler manifold is a complex manifold equipped with a Hermitian metric whose imaginary part, which is a 2-form of type (1,1) relative to the complex structure, is closed. This 2-form is called the Kahler form of the Kahler metric.

Smooth projective complex manifolds are special cases of compact Kahler manifolds. As complex projective space (equipped, for example, with the Fubini-Study metric) is a Kahler manifold, the complex submanifolds of projective space equipped with the induced metric are also Kahler. We can indicate precisely which members of the set of Kahler manifolds are complex projective, thanks to Kodaira's theorem:

Theorem 1.1. *A compact complex manifold admits a holomorphic embedding into complex projective space if and only if it admits a Kahler metric whose Kahler form is of integral class.*

We are essentially interested in the class of Kahler manifolds, without particularly emphasising projective manifolds. The reason is that our goal here is to establish the existence of the Hodge decomposition and the Lefschetz decomposition on the cohomology of such a manifold, and for this, there is no need to assume that the Kahler class is integral. However, the Lefschetz decomposition will be defined on the rational cohomology only in the projective case, and this is already an important reason to restrict ourselves, later, to the case of projective manifolds.

If X is a complex manifold, the tangent space to X at each point x is equipped with a complex structure J_x. The data consisting of this complex structure at each point is what is known as the underlying almost complex structure. The J_x provide a decomposition

$$T_xX \otimes \mathbb{C} = T_xX^{1,0} \oplus T_xX^{0,1}, \tag{1.1}$$

where $T_xX^{1,0}$ is the vector space of complexified tangent vectors u such that $J_xu = iu$ and $T_xX^{0,1}$ is the complex conjugate of $T_xX^{1,0}$. From the point of view of the complex structure, i.e. of the local data of holomorphic coordinates, the vector fields of type $(0,1)$ are those which kill the holomorphic functions.

The decomposition (1.1) induces a similar decomposition on the bundles of complex differential forms

$$\Lambda^kT^*X \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}T^*X, \tag{1.2}$$

where

$$\Lambda^{p,q}T^*X = \Lambda^pT^*X^{1,0} \otimes \Lambda^qT^*X^{0,1}$$

and

$$T^*X \otimes \mathbb{C} = T^*X^{1,0} \oplus T^*X^{0,1},$$

is the dual decomposition of (1.1).

The decomposition (1.2) has the property of Hodge symmetry

$$\overline{\Lambda^{p,q}T^*X} = \Lambda^{q,p}T^*X,$$

where complex conjugation acts naturally on $\Lambda^k T^*X \otimes \mathbb{C}$.

If we let $\mathcal{E}^k(X)_{\mathbb{C}}$ denote the space of complex differential forms of degree on X, i.e. the $\mathcal{C}^\infty$-sections of the vector bundle $\Lambda^k T^*X \otimes \mathbb{C}$, then we also have the exterior differential

$$d : \mathcal{E}^k(X)_{\mathbb{C}} \to \mathcal{E}^{k+1}(X)_{\mathbb{C}},$$

which satisfies $d \circ d = 0$. We then define the k^{th} de Rham cohomology group of X by

$$H^k(X, \mathbb{C}) = \frac{\operatorname{Ker}(d : \mathcal{E}^k(X)_{\mathbb{C}} \to \mathcal{E}^{k+1}(X)_{\mathbb{C}})}{\operatorname{Im}(d : \mathcal{E}^{k-1}(X)_{\mathbb{C}} \to \mathcal{E}^k(X)_{\mathbb{C}})}.$$

The main theorem proved in these notes is the following.

Theorem 1.2. *Let $H^{p,q}(X) \subset H^k(X, \mathbb{C})$ be the set of classes which are representable by a closed form α which is of type (p, q) at every point x in the decomposition (1.2). Then we have a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X). \tag{1.3}$$

Note that by definition, we have the *Hodge symmetry*

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$

where complex conjugation acts naturally on $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$. Here $H^k(X, \mathbb{R})$ is defined by replacing the complex differential forms by real differential forms in the above definition.

This theorem immediately gives constraints on the cohomology of a Kahler manifold, which reveal the existence of compact complex manifolds which are not Kahler. For example, the decomposition (1.3) and the Hodge symmetry imply that the dimensions $b_k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$, called the *Betti numbers*, are even for odd k, property not satisfied by Hopf surfaces.

Example 1.3. The *Hopf surfaces* are the quotients of $\mathbb{C}^2 \setminus \{0\}$ by the fixed-point-free action of a group isomorphic to $\mathbb{Z}$, where a generator g acts via $g(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$ where the λ_i, are non-zero complex numbers of modulus strictly less than 1. These surfaces are compact, equipped with the quotient complex structures, and their π_1 is isomorphic to $\mathbb{Z}$ since $\mathbb{C}^2 \setminus \{0\}$ is simply connected. Thus, their first Betti number is equal to 1, which implies that they are not Kahler.

The *Lefschetz decomposition* is another decomposition of the cohomology of a compact Kahler manifold X, this time of topological nature. It depends only on the cohomology class of the Kahler form $[\omega] \in H^2(X, \mathbb{R})$. The exterior product on differential forms satisfies Leibniz' rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$, so the exterior product with ω sends *closed forms* (i.e. forms killed by d) to closed forms and *exact forms* (i.e. forms in the image of d) to exact forms. Thus it induces an operator, called the *Lefschetz operator*,

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R}).$$

The following theorem is sometimes called the *hard Lefschetz theorem*.

Theorem 1.4. *For every $k \leq n = \dim X$, the map*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R}) \tag{1.4}$$

is an isomorphism.

Remark 1.5. Note that the spaces on the right and on the left are of the same dimension by Poincare duality, which is valid for all compact oriented manifolds.

A very simple consequence of the above isomorphism is the following result, which is an additional topological constraint satisfied by Kahler manifolds.

Corollary 1.6. *The morphism*

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$$

is injective for $k \leq n = \dim X$. Thus, the odd Betti numbers $b_{2k-1}(X)$ increase with k for $2k-1 \leq n$, and similarly, the even Betti numbers $b_{2k}(X)$ increase for $2k \leq n$.

An algebraic consequence of Lefschetz' theorem is the Lefschetz decomposition, which as we noted earlier is particularly important in the case of projective manifolds. Let us define the *primitive cohomology* of a compact Kahler manifold X by

$$H^k(X, \mathbb{R})_{\text{prim}} := \text{Ker}(L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R}))$$

for $k \leq n$. One can extend this definition to the cohomology of degree $> n$ by using the isomorphism (1.4).

Theorem 1.7. *The natural map*

$$i : \bigoplus_{k-2r \geq 0} H^{k-2r}(X, \mathbb{R})_{\text{prim}} \to H^k(X, \mathbb{R})$$

$$(\alpha_r) \to \sum L^r \alpha_r$$

is an isomorphism for $k \leq n$.

Once again, we can extend this decomposition to the cohomology of degree $> n$ by using the isomorphism (1.4).

Let us now express the main principle of Hodge theory, which has immense applications. The study of the cohomology of Kahler manifolds and the proof of the Theorems (eq:ihdec) and (1.4), which are the main content of these lectures, are among the most important applications, but the principle applies in various other situations. We restrict ourselves here to giving an explanation of the main idea, which is the notion of a harmonic form, and the application of the theory of elliptic operators which makes it possible to represent the cohomology classes by harmonic forms, but we will omit the proof of the fundamental theorem on elliptic operators, which uses estimations and notions from analysis (Sobolev spaces), which are in different directions from the aims of this book. The delicate point consists in passing from spaces of L^2 differential forms, in which the Hodge decomposition is algebraically obvious, to spaces of $\mathcal{C}^\infty$ differential forms. One of the problems we encounter is the fact that the operators considered here are differential operators, and thus do not define continuous operators on the spaces of L^2 forms.

The idea that we want to explain here is the following: using the metric on X, we can define the L^2 metric on the spaces of differential forms

$$(\alpha, \beta)_{L^2} = \int_X \langle \alpha, \beta \rangle_x \, \mathrm{Vol},$$

where α, β are differential forms of degree k and the scalar product $\langle \alpha, \beta \rangle_x$ at a point $x \in X$ is induced by the evaluation of the forms at the point x and by the metric at the point x.

The operator $d : \mathcal{E}^k(X) \to \mathcal{E}^{k+1}(X)$ is a differential operator, and we can construct its formal adjoint $d^* : \mathcal{E}^k(X) \to \mathcal{E}^{k-1}(X)$, which is also a differential operator, and satisfies the identity

$$(\alpha, d\beta)_{L^2} = (d^*\alpha, \beta)_{L^2} \,,$$

for $\alpha \in \mathcal{E}^k(X), \beta \in \mathcal{E}^{k-1}(X)$. This adjunction relation only makes d^* into a formal adjoint, since these operators are not defined on the Hilbert space of L^2 differential forms, which is the completion of $\mathcal{E}^*(X)$ for the L^2 metric.

The idea of Hodge theory consists in using the adjoint d^* to write the decompositions

$$\mathcal{E}^k(X) = \mathrm{Im}\, d \oplus \mathrm{Im}\, d^{\perp} = \mathrm{Im}\, d \oplus \mathrm{Ker}\, d^*, \quad \mathcal{E}^k(X) = \mathrm{Ker}\, d \oplus \mathrm{Ker}\, d^{\perp} = \mathrm{Ker}\, d^* \oplus \mathrm{Im}\, d,$$

and finally, using the inclusion $\mathrm{Im}\, d \subset \mathrm{Ker}\, d$,

$$\mathcal{E}^k(X) = \mathrm{Im}\, d \oplus \mathrm{Im}\, d^* \oplus \mathrm{Ker}\, d \cap \mathrm{Ker}\, d^*.$$

Of course, these identities, which would be valid on finite-dimensional spaces or Hilbert spaces since the operator d has closed image there, require the analysis mentioned above in order to justify them here. Apart from this issue, if we accept these identities, we see that the space

$$\mathcal{H}^k := \mathrm{Ker}\, d \cap \mathrm{Ker}\, d^* \subset \mathcal{E}^k(X)$$

of *harmonic forms* projects bijectively onto $H^k(X, \mathbb{R})$ (or $H^k(X, \mathbb{C})$ if we study the cohomology with complex coefficients), since it is a supplementary space of $\mathrm{Im}\, d$ inside $\mathrm{Ker}\, d$.

Another characterisation of harmonic forms uses the *Laplacian*

$$\Delta_d = dd^* + d^*d.$$

Indeed, it is very easy to see that we have

$$\mathcal{H}^k = \mathrm{Ker}\, \Delta_d.$$

The operator Δ_d is an *elliptic operator*. This property of a differential operator can be read directly from its *symbol*, which is essentially its homogeneous term of largest order (which is 2 for the Laplacian). The decompositions written above are special cases of the decomposition associated to an elliptic operator.

The Hodge decomposition (1.3) is obtained by combining the Hodge theory sketched above and the study of the properties of the Laplacian of a Kahler manifold. We have already mentioned various operators acting on the spaces of differential forms of a Kahler manifold, namely d, L and their formal adjoints d^*, Λ for the L^2 metric. Moreover, the complex structure makes it possible to decompose d as

$$d = \partial + \bar{\partial},$$

where the *Dolbeault operator* $\bar{\partial}$ sends $\alpha \in \mathcal{E}^{p,q}(X)$ to the component of bidegree $(p, q+1)$ of $d\alpha$. Here $\alpha \in \mathcal{E}^{p,q}(X)$ is the space of differential forms of bidegree (p,q) at every point of X; it is also the space of sections of the bundle $\Lambda^{p,q}T^*X$, which appears in the decomposition (1.2) given by the complex structure. The differential operators ∂ and $\bar{\partial}$ are differential operators of order 1, and have formal adjoint operators ∂^* and $\bar{\partial}^*$.

The *Kahler identities* establish commutation relations between these operators. For example, we have the identity

$$[\Lambda, \partial] = i\bar{\partial}^*,$$

and the other identities follow from this one via passage to the complex conjugate or to the adjoint. From these identities, and from the fact that L commutes with d while ∂ and $\bar{\partial}$ anticommute, we deduce the following result.

Theorem 1.8. *The Laplacians $\Delta_d, \Delta_\partial$ and $\Delta_{\bar{\partial}}$ associated to the operators d, ∂ and $\bar{\partial}$ respectively satisfy the equalities*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{1.5}$$

We deduce that the harmonic forms for d are also harmonic for ∂ and $\bar{\partial}$, and in particular are also ∂ and $\bar{\partial}$-closed. Finally, as the operators ∂ and $\bar{\partial}$ are bihomogeneous (of bidegree $(1,0)$ and $(0,1)$ respectively) for the bigraduation of the spaces of differential forms given by the decomposition (1.2), it follows easily that each of the Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$ is bihomogeneous of bidegree $(0,0)$, i.e. preserves the forms of type (p,q) for every bidegree (p,q). The same then holds for Δ_d by the equality (1.5). The Hodge decomposition is then obtained simply by the decomposition of the harmonic forms as sums of forms of type (p,q):

Corollary 1.9. *Let X be a compact Kahler manifold. If α is a harmonic form (for the Laplacian associated to the operator d and to the metric), its components of type (p,q) are harmonic. Thus, we have a decomposition*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X), \tag{1.6}$$

where $\mathcal{H}^{p,q}(X)$ is the space of harmonic forms of type (p,q) at every point of X.

The Hodge decomposition (1.3) is obtained by combining the theorem of representation of cohomology classes by harmonic forms with the decomposition (1.6). The Lefschetz decomposition is also an easy consequence of the decomposition (1.6). Indeed, we first show that Theorem 1.4 holds for the operator L acting on differential forms. Furthermore, the Kahler identities show that L commutes with the Laplacian, so that the operators L^r send harmonic forms to harmonic forms, and once the theorem is proved on the level of forms, it remains valid on the level of harmonic forms, and thus also on cohomology classes.

The Hodge decomposition (1.3) gives an extremely interesting structure when it is combined with the integral structure on the cohomology $H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. For this equality, which follows from the change of coefficients theorem, one must adopt a different definition of cohomology, which does not make use of differential forms. For one possible definition, we can introduce the singular cohomology $H^k_{\text{sing}}(X,\mathbb{Z})$. We start from the complex $(C_*(X), \partial : C_k(X) \to C_{k-1}(X))$ of singular chains, where $C_k(X)$ is the free abelian group generated by the continuous maps from the simplex Δ_k of dimension k to X. The map ∂ is given by the restriction to the boundary

$$\partial\phi = \sum_i (-1)^i \phi_{|\Delta_{k,i}}$$

where $\Delta_{k,i}$ is the i^{th} face of Δ_k. The complex $(C^*_{\text{sing}}(X), d)$ of singular cochains is then defined as the dual complex of $(C_*(X), \partial)$. Its cohomology is the singular cohomology $H^*_{\text{sing}}(X,\mathbb{Z})$. We have the following theorem, due to de Rham.

Theorem 1.10. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we have $H^k(X,\mathbb{K}) = H^k_{\text{sing}}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}$.*

If we consider the complex of differentiable chains, we can prove this theorem by using the natural map from $\mathcal{E}^k(X)$ to $C^*_{\text{sing}}(X)$ given by integration:

$$\alpha \to \left(\phi \to \int_{\Delta_k} \phi^*\alpha\right).$$

A much more conceptual proof of de Rham's theorem can be given by using the language of sheaf theory. Sheaf cohomology will be used, for example, in the Hodge decomposition, to describe the spaces $H^{p,q}$ as the Dolbeault cohomology groups $H^q(X,\Omega^p_X)$, which are defined for every complex manifold X as the q^{th} cohomology group of X with values in the sheaf Ω^p_X of holomorphic differential forms of degree p. We note, however, that this identification is valid only in the Kahler case. In general, without the Kahler hypothesis, we cannot identify $H^q(X,\Omega^p_X)$ with the space of cohomology classes of degree $p+q$ which are representable by a closed form of type (p,q) at every point.

Let Γ be the functor of global sections Γ of the category of sheaves of abelian groups on X to the category of abelian groups. We show using Poincare's theorem that the sheaves of differential forms form a Γ-acyclic resolution of the constant sheaf $\mathbb{C}_X$ (often written $\mathbb{C}$) of stalk $\mathbb{C}$, so that the space $H^k(X,\mathbb{C})$ defined above must be understood as the k^{th} cohomology group of X with values in $\mathbb{C}_X$. Similarly, we can interpret the singular cohomology as the cohomology of the complex of global sections of a Γ-acyclic resolution of the constant sheaf of stalk $\mathbb{Z}$. Thus, we have $H^*_{\text{sing}}(X,\mathbb{Z}) = H^k(X,\mathbb{Z})$ canonically. De Rham's theorem thus reduces to proving a change of coefficients theorem for the cohomology of the sheaves $H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, which is not difficult.

These different interpretations of the cohomology, corresponding to different resolutions, are all equally important, since they carry different types of information. For example, the Hodge decomposition of the cohomology of a Kahler manifold requires the de Rham version of the cohomology, while that of the integral structure requires another version, singular or Cech for example.

2. Holomorphic Functions of Many Variables

In this section, we recall the main properties of holomorphic functions of several complex variables. The $\mathbb{C}$-valued holomorphic functions of the complex variables $z_1, \ldots, z_n$ those whose differential is $\mathbb{C}$-linear, or equivalently, those which are annihilated by the operators $\frac{\partial}{\partial \bar{z}_l}$. It follows immediately from this definition that the set of holomorphic functions forms a ring, and that the composition of two holomorphic functions is holomorphic. The following theorem, however, requires a certain amount of work.

Theorem 2.1. *The holomorphic functions of the complex variables $z_1, \ldots, z_n$ are complex analytic, i.e. they locally admit expansions as power series in the variables z_l.*

This result is an easy consequence of Cauchy's formula in several variables, which is a generalisation of the formula

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where f is a holomorphic function defined in a disk of radius > 1, and $|z| < 1$.

Cauchy's formula can also be used to prove Riemann's theorem of analytic continuation:

Theorem 2.2. *Let f be a bounded holomorphic function on the pointed disk. Then f extends to a holomorphic function on the whole disk.*

And also Hartogs' theorem:

Theorem 2.3. *Let f be a holomorphic function defined on the complement of an analytic subset of codimension 2 in a ball B of $\mathbb{C}^n, n \geq 2$. Then f extends to a holomorphic function on B.*

Hartogs' theorem enables us to show that a holomorphic section of a complex vector bundle over a complex manifold is defined everywhere if it is defined on the complement of an analytic subset of codimension 2.

2.4. **Holomorphic functions of one variable.** Let $U \subset \mathbb{C} = \mathbb{R}^2$ be an open set, and $f : U \to \mathbb{C}$ a $\mathcal{C}^1$ map. Let x, y be the linear coordinates on $\mathbb{R}^2$ such that $z = x + iy$ is the canonical linear complex coordinate on $\mathbb{C}$. Consider the complex-valued differential form

$$dz = dx + idy \in \mathrm{Hom}_{\mathbb{R}}(TU, \mathbb{C}) = \Omega_{U,\mathbb{R}} \otimes \mathbb{C}.$$

Clearly dz and its complex conjugate $d\bar{z}$ form a basis of $\Omega_{U,\mathbb{R}} \otimes \mathbb{C}$ over $\mathbb{C}$ at every point of U, since

$$2dx = dz + d\bar{z}, 2idy = dz - d\bar{z}.$$

The complex differential form $df \in \mathrm{Hom}_{\mathbb{R}}(TU, \mathbb{C})$ can thus be uniquely written

$$d_u f = \frac{\partial f}{\partial z}(u) dz + \frac{\partial f}{\partial \bar{z}}(u) d\bar{z},$$

where the complex-valued functions $\frac{\partial f}{\partial z}(\cdot)$, $\frac{\partial f}{\partial \bar{z}}(\cdot)$ are continuous. We obviously have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

We can also consider the above decomposition of df as the decomposition of $df \in \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts:

Lemma 2.5. *We have $\frac{\partial f}{\partial \bar{z}}(u) = 0$ if and only if the $\mathbb{R}$-linear map $d_u f : T_u U = \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear, i.e. is equal to multiplication by a complex number, which is then equal to $\frac{\partial f}{\partial z}(u)$.*

Definition 2.6. The function f is said to be *holomorphic* if it satisfies one of the equivalent conditions of lemma 2.5 at every point of U.

Lemma 2.7. *If f is holomorphic and does not vanish on U, then $1/f$ is holomorphic. Similarly, if f, g are holomorphic, fg and $f + g$ and $g \circ f$ (when g is defined on the image of f) are all holomorphic.*

In particular, we will use the following corollary.

Corollary 2.8. *If f is holomorphic on U, the map g defined by*

$$g(z) = \frac{f(z)}{z - a}$$

is holomorphic on $U \setminus \{a\}$.

2.9. **Stokes'formula.** Let α be a differential k-form on an n-dimensional manifold U. If $x_1, \ldots, x_n$ are local coordinates on U, we can write

$$\alpha = \sum_I \alpha_I dx_I,$$

where the indices I parametrise the totally ordered subsets $i_1 < \cdots < i_k$ of $\{1, ..., n\}$, with $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. We can then define the continuous $(k+1)$-form

$$d\alpha = \sum_{I,i} \frac{\partial \alpha_I}{\partial x_i} dx_i \wedge dx_I; \tag{2.1}$$

we check that it is independent of the choice of coordinates. This follows from the more general fact that if V is an m-dimensional manifold and $\phi : V \to U$ is a differential map given in local coordinates by $\phi^* x_i := x_i \circ \phi = \phi(y_1, \ldots, y_m)$, then for every differential form $\alpha = \sum_I \alpha_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, we can define its inverse image

$$\phi^* \alpha = \sum_I \alpha_I \circ \phi \; d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

Moreover, this image inverse satisfies

$$d(\phi^* \alpha) = \phi^*(d\alpha),$$

where the coordinates y_i (and the formulae (2.1)) are used on the left, while the coordinates x_i, are used on the right.

A differential k-form a can be integrated over the compact oriented k-dimensional submanifolds of U with boundary, or over the image of such manifolds under differentiable maps.

To begin with, let us recall that a k-dimensional *manifold with boundary* is a topological space covered by open sets U_i which are homeomorphic, via certain maps h_i to open subsets of $\mathbb{R}^k$ or to $(0,1] \times V$, where V is an open set of $\mathbb{R}^{k-1}$. We require the transition functions $h_i \circ h_j^{-1}$ to be differentiable on $h_j(U_i \cap U_j)$. When $h_j(U_i \cap U_j)$ contains points on the boundary of U_j, i.e. is locally isomorphic to $(0,1] \times V$, where V is an open set of $\mathbb{R}^{k-1}$, $h_j(U_i \cap U_j)$ must also be locally isomorphic to $(0,1] \times W$, where W is an open set of $\mathbb{R}^{k-1}$, and the differentiable map $h_i \circ h_j^{-1}$ must locally extend to a diffeomorphism of a neighbourhood in $\mathbb{R}^k$ of $(0,1] \times V$ to a neighbourhood of $(0,1] \times W$, inducing a diffeomorphism from $1 \times V$ to $1 \times W$. In particular, the boundary of M, which we denote by ∂M and which is defined, with the preceding notation, as the union of the $h_i^{-1}(1 \times V)$, is a closed set of M which possesses an induced differentiable manifold structure.

The manifold with boundary M is said to be *oriented* if the diffeomorphisms $h_i \circ h_j^{-1}$ have positive Jacobian. The boundary of M is then also naturally oriented by the charts $1 \times V$, where V is an open set of $\mathbb{R}^{k-1}$ as above, since the induced transition diffeomorphisms $h_i \circ h_j^{-1} \mid_{1\times V}: V \to W$ also have positive Jacobian.

If M is k-dimensional manifold with boundary and $\phi : M \to U$ is a differentiable map (along the boundary of M, which is locally isomorphic to $(0,1] \times V$, we require ϕ to extend locally to a differentiable map on a neighbourhood $(0, 1+\epsilon) \times V$ of $\{1\} \times V$), then for every differentiable k-form α, we have the inverse image $\beta = \phi^*\alpha$ defined above, which is a differentiable k-form on M. If moreover M is oriented and compact, such a form can be integrated over M as follows. Let ρ_i be a partition of unity subordinate to a covering of M by open sets U_i as above, which we may assume to be diffeomorphic to $(0,1] \times (0,1)^{k-1}$ or to $(0,1)^k$. Then $\beta = \sum_i \rho_i \beta$ on extends to a differentiable form on $[0,1]^k$. Setting $\rho_i\beta = g_i(x_1,\ldots,x_k)dx_1 \wedge \cdots \wedge dx_k$, we then define

$$\int_M \beta = \sum_i \int_{U_i} \rho_i\beta, \quad \int_{U_i} \rho_i\beta = \int_0^1 \ldots \int_0^1 g_i(x_1,\ldots,x_k)dx_1 \ldots dx_k.$$

The change of variables formula for multiple integrals and the fact that the authorised variable changes have positive Jacobians ensure that $\int_M \beta$ is well-defined independently of the choice of oriented charts, i.e. of local orientation-preserving coordinates.

Remark 2.10. If we change the orientation of M, i.e. if we compose all the charts with local diffeomorphisms of $\mathbb{R}^k$ with negative Jacobians, the integrals $\int_M \phi^*\alpha$ change sign. This follows from the change of variables formula for multiple integrals, which uses only the absolute value of the Jacobian, whereas the change of variables formula for differential forms of maximal degree uses the Jacobian itself.

Suppose now that α is a $(k-1)$-form on U. Then, as $\phi \mid_{\partial M}$ is differentiable and ∂M is a compact oriented manifold of dimension $k-1$, we can compute the integral $\int_{\partial M} \phi^*\alpha$. Moreover, we can integrate the differential $d\phi^*\alpha = \phi^*d\alpha$ over M. We then have

Theorem 2.11. *(Stokes' formula) The following equality holds:*

$$\int_M \phi^* d\alpha = \int_{\partial M} \phi^*\alpha.$$

In particular, if $d\alpha = 0$, *we have* $\int_{\partial M} \phi^* \alpha = 0$.

We will use Stokes' formula very frequently throughout this text. In particular, it will enable us to pair the de Rham cohomology with the singular homology. The following consequence will be particularly useful.

Corollary 2.12. *If* α *is a differential form of degree* $n-1$ *on a compact* n*- dimensional manifold without boundary, then* $\int_M d\alpha = 0$.

2.13. **Cauchy's formula.** We propose to apply Stokes' formula, using the following lemma.

Lemma 2.14. *Let* $f : U \to \mathbb{C}$ *be a holomorphic map. Then the complex differential form* $f dz$ *is closed.*

We thus also have the following.

Corollary 2.15. *If* f *is holomorphic on* U, *the differential form* $\frac{f}{z-z_0} dz$ *is closed on* $U \setminus \{z_0\}$.

Suppose now that U contains a closed disk D. For every $z_0 \in D$, let D_ϵ be the open disk of radius ϵ centred at $_0$ which is contained in D for sufficiently small ϵ. Then $D \setminus D_\epsilon$ is a manifold with boundary, whose boundary is the union of the circle ∂D and the circle of centre z_0 and radius ϵ, the first with its natural orientation, the second with the opposite orientation. For holomorphic f, Stokes' formula and previous corollary then give the equality

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f(z)}{z - z_0} dz.$$

Furthermore, we have the following.

Lemma 2.16. *If* f *is a function which is continuous at* z_0, *then*

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f(z)}{z - z_0} dz = f(z_0).$$

Combining thus lemma and the above equality, we now have

Theorem 2.17. *(Cauchy's formula) Let* f *be a holomorphic function on* U *and* D *a closed disk contained in* U. *Then for every point* z_0 *in the interior of* D, *we have the equality*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

2.18. **Holomorphic functions of several variables.**

2.18.1. **Cauchy's formula and analyticity.** Let U be an open set of $\mathbb{C}^n$, and let $f : U \to \mathbb{C}$ be a differentiable map. For $u \in U$, we have a canonical identification $T_u U = \mathbb{C}^n$. We can thus generalise the notion of a holomorphic function to higher dimensions.

Definition 2.19. The function f is said to be *holomorphic* if for every $u \in U$, the differential $d_u f : \mathrm{Hom}(T_u U, \mathbb{C}) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ is $\mathbb{C}$-linear.

It is easy to prove that lemma 2.7 remains true in higher dimensions. Furthermore, we have the three following characterisations of holomorphic functions.

Theorem 2.20. *The following three properties are equivalent for a differentiable function*

(i) f *is holomorphic.*

(ii) *In the neighbourhood of each point* $z_0 \in U$, f *admits an expansion as a power series of the form* $f(z_0 + z) = \sum_I a_I z^I$, *where* I *runs through the set of the* n*-tuples of integers* $(i_1, \dots, i_n)$ *with* $i_k \geq 0$, *and* $z^I := z_1^{i_1} \dots z_n^{i_n}$. *The coefficients of the series satisfy the following property: there exist* $R_1 > 0, \dots, R_n > 0$ *such that the power series* $\sum_I a_I r^I$ *converges for every* $r_1 < R_1, \dots, r_n < R_n$.

(iii) *If* $D = \{(\zeta_1, \dots, \zeta_n) || \zeta_i - a_i| \leq \epsilon_i\}$ *is a poly disk contained in* U, *then for every* $z = (z_1, \dots, z_n) \in D^\circ$, *we have the equality*

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_i - a_i| = \epsilon_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \dots \wedge \frac{d\zeta_n}{\zeta_n - z_n}.$$

In the preceding formula, the integral is taken over a product of circles, equipped with the orientation which is the product of the natural orientations.

Because of property (ii), holomorphic functions are also known as complex analytic functions. Property (iii) is Cauchy's formula in several variables. One can prove it by induction on the dimension, using Cauchy's formula in a single variable. One can also directly apply Stokes' formula, using the following analogue of lemma 2.14.

Lemma 2.21. *If* f *is holomorphic, then the differential form* $f(z) dz_1 \wedge \dots \wedge dz_n$ *is closed.*

Let us give some applications of theorem 2.20. To begin with, we have

Theorem 2.22. *(The maximum principle) Let* f *be a holomorphic function on an open subset* U *of* $\mathbb{C}^n$. *If* $|f|$ *admits a local maximum at a point* $u \in U$, *then* f *is constant in the neighbourhood of this point.*

Another essential application is the principle of analytic continuation.

Theorem 2.23. *Let* U *be a connected open set of* $\mathbb{C}^n$, *and* f *a holomorphic function on* U. *If* f *vanishes on an open set of* U, *then* f *is identically zero.*

Let us now give some subtler applications of Cauchy's formula or its generalisations. These theorems show that the possible singularities of a holomorphic function cannot exist unless the function is not bounded (Riemann), and is not defined on the complement of an analytic subset of codimension 2 (Hartogs).

Theorem 2.24. *(Riemann) Let* f *be a holomorphic function on* $U \setminus \{z | z_1 = 0\}$, *where* U *is an open set of* $\mathbb{C}^n$. *Then if* f *is locally bounded on* U, f *extends to a holomorphic map on* U.

To conclude this section, we will mention the following version of Hartogs' extension theorem which implies the more general theorem mentioned above.

Theorem 2.25. *Let* U *be an open set of* $\mathbb{C}^n$ *and* f *a holomorphic function defined on* $U \setminus \{z | z_1 = z_2 = 0\}$. *Then* f *extends to a holomorphic function on* U.

2.26. **The equation $\frac{\partial g}{\partial \bar z} = f$.** The following theorem will play an essential role in the proof of the local exactness of the operator $\bar\partial$.

Theorem 2.27. *Let f be a differentiable function on an open set of $\mathbb{C}$. Then, locally on this open set, there exists a differentiable function g (defined up to the addition of a holomorphic function), such that*

$$\frac{\partial g}{\partial \bar z} = f.$$

Proof. As the statement is local, we may assume that f has compact support, and thus is defined on $\mathbb{C}$. Now set

$$g(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta.$$

This is a singular integral. By definition, it is equal to the limit, as $\epsilon \to 0$, of the integrals

$$g(z) = \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta,$$

where D_ϵ is a disk of radius ϵ centred at z. It is easy to see that this limit exists (the function $\frac{f(\zeta)}{\zeta - z}$ is L^1). Making the change of variable $\zeta' = \zeta - z$, we also have

$$g(z) = \lim_{\epsilon\to 0} g_\epsilon(z),\ g_\epsilon(z) = \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D'_\epsilon} \frac{f(\zeta' + z)}{\zeta'} d\zeta' \wedge d\bar\zeta',$$

The convergence of the $g_\epsilon(z)$ when $\epsilon \to 0$ is uniform in z. Moreover, we can differentiate under the integral sign the (non-singular) integral defining g_ϵ

$$\frac{\partial g_\epsilon}{\partial \bar z} = \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D'_\epsilon} \frac{\partial f(\zeta' + z)}{\partial \bar z}\frac{d\zeta' \wedge d\bar\zeta'}{\zeta'},$$

As $\frac{\partial f(\zeta'+z)}{\partial \bar z}$ is differentiable, the functions $\frac{\partial g_\epsilon}{\partial \bar z}$ converge uniformly, and we conclude that g is differentiable and satisfies

$$\frac{\partial g}{\partial \bar z} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial f(\zeta' + z)}{\partial \bar z}\frac{d\zeta' \wedge d\bar\zeta'}{\zeta'},$$

Thus, it remains to show the equality $\frac{\partial g}{\partial \bar z} = f$. Changing back to $\zeta = \zeta' + z$, we have

$$\frac{\partial g}{\partial \bar z} = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\epsilon} \frac{\partial f(\zeta)}{\partial \bar\zeta}\frac{d\zeta \wedge d\bar\zeta}{\zeta - z}, \tag{2.2}$$

Now, on $\mathbb{C}\setminus D_\epsilon$ we have the equality $\frac{\partial f(\zeta)}{\partial \bar\zeta}\frac{d\zeta\wedge d\bar\zeta}{\zeta - z} = -d\left(\frac{f d\zeta}{\zeta - z}\right)$. Stokes' formula thus gives

$$\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\epsilon} \frac{\partial f(\zeta)}{\partial \bar\zeta}\frac{d\zeta \wedge d\bar\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D_\epsilon} f(\zeta)\frac{d\zeta}{\zeta - z}. \tag{2.3}$$

Using lemma 2.16 and the equalities (2.2), (2.3) we have thus proved the desired equality. □

3. Complex Manifolds

We introduce and study the notion of a complex structure on a differentiable or complex manifold. A complex manifold X of (complex) dimension n is a differentiable manifold locally equipped with complex-valued coordinates (called holomorphic coordinates) $z_1, \dots, z_n$, such that the diffeomorphisms from an open set of $\mathbb{C}^n$ to an open set of $\mathbb{C}^n$ given by coordinate changes are holomorphic. By the definition of a holomorphic transformation, we then see that the structure of a complex vector space on the tangent space T_xX given by the identification $T_xX = \mathbb{C}^n$ induced by the holomorphic coordinates $z_1, \dots, z_n$ does not depend on the choice of holomorphic coordinates. The tangent bundle TX of a complex manifold X is thus equipped with the structure of a complex vector bundle. Such a structure is called an almost complex structure. The Newlander-Nirenberg theorem characterises the almost complex structures induced as above by a complex structure. We also introduce holomorphic vector bundles over a complex manifold. These vector bundles are those whose "transition matrices" are holomorphic. It turns out that we can define a differential operator (the Dolbeault operator) $\bar{\partial}$ on the space of sections of such a vector bundle E, and more generally, on the space of differential forms with values in such a bundle. The holomorphic sections σ of E are then characterised by the equation $\bar{\partial}\sigma = 0$. One can show that the Dolbeault operator satisfies the condition $\bar{\partial} \circ \bar{\partial} = 0$, and that the complex defined in this way is locally exact. This will be used later to represent the cohomology of X with values in the sheaf of holomorphic sections of E using $\bar{\partial}$-closed differential forms with coefficients in E.

3.1. **Manifolds and vector bundles.**

3.1.1. **Definitions.** A *topological manifold* is a topological space X equipped with a covering by open sets U_i, which are homeomorphic, via maps h_i called "local charts", to open sets of $\mathbb{R}^n$. One can show that such an n is necessarily independent of i when X is connected; n is then called the *dimension* of X.

Definition 3.2. A C^∞ *differentiable manifold* X is a topological manifold equipped with a system of local charts $h_i : U_i \to \mathbb{R}^n$ such that the open sets U_i cover X, and the change of chart morphisms $h_j \circ h_i^{-1} : h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$ are C^∞ maps.

A C^∞ *differentiable function* on such a manifold (or on an open set) is a function f such that for each U_i, the function $f \circ h_i^{-1}$ is C^∞ differentiable. A map $f : X \to Y$ between differentiable manifolds is C^∞ *differentiable map* if $g_j \circ f \circ h_i^{-1}$ are C^∞ differentiable.

A *real* (resp. *complex*) topological *vector bundle of rank m* over a topological space X is a topological space E equipped with a map $\pi : E \to X$ such that for an open cover $\{U_i\}$ of X, we have "local trivialisation" homeomorphisms

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^m (\text{resp. } U_i \times \mathbb{C}^m),$$

such that:

(i) $pr_1 \circ \phi_i = \pi$

(ii) The transition functions $\phi_j \circ \phi_i^{-1} : \phi(\pi^{-1}(U_i \cap U_j)) \to \phi(\pi^{-1}(U_i \cap U_j))$ are $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) on each fibre $x \times \mathbb{R}^m$ (resp. $x \times \mathbb{C}^m$).

Such a transformation $U_i \cap U_j \times \mathbb{R}^m \to U_i \cap U_j \times \mathbb{R}^m$ must respect the first projection, by condition (i) above, and is thus described by a real $m \times m$ matrix, whose coefficients, by continuity, are continuous functions of $x \in U_i \cap U_j$. (In the complex case, we must consider complex matrices.) These matrices are called *transition matrices*.

Definition 3.3. If X is a differentiable manifold, a vector bundle E over X is equipped with a differentiable structure if we are given local trivialisations whose transition matrices are C^∞.

Remark 3.4. The bundle E is then equipped with the structure of a C^∞ manifold for which π is C^∞ as well as the local trivialisations.

A *section* of a vector bundle $\pi : E \to X$ is a map $\sigma : X \to E$ such that $\pi \circ \sigma = \mathrm{Id}_X$. This section is said to be continuous, resp. differentiable, if a is continuous, resp. differentiable. If $\pi : E \to X$ is a vector bundle and $x \in X$, we write $E_x := \pi^{-1}(x)$. It is canonically a vector space, with structure given by any of the trivialisations of E in the neighbourhood of x; E_x is called the *fibre* of E at the point x.

A vector bundle $\pi : E \to X$ is said to be *trivial* if it admits a global trivialisation $\phi : E = X \times \mathbb{R}^n$. Equivalently, E must admit n global sections which provide a basis of the fibre E_x at each point. These sections are given by at $\sigma_i = \phi^{-1} \circ \tilde{e_i}$, where $\tilde{e_i} : X \to X \times \mathbb{R}^n$ is given by $e_i(x) = (x, e_i)$, where the e_i form the standard basis of $\mathbb{R}^n$.

Let $\pi_E : E \to X$ and $\pi_F : F \to X$ be vector bundles over X. A *morphism* $\psi : E \to F$ of vector bundles is a continuous map such that $\pi_F \circ \psi = \pi_E$, and ψ is linear on each fibre. This means that in local trivialisations, ψ becomes linear ($\mathbb{C}$-linear in the case of complex bundles) on the fibres $x \in \mathbb{R}^n$; his definition is independent of the choice of the open set containing x, since the transition functions are also linear on the fibres. We have an analogous definition for differentiable bundles.

Given a vector bundle E, we can define its dual E^* and its exterior powers $\Lambda^k E$, which are differentiable if E is. The points of E^* are the linear forms on the fibres of $\pi_E : E \to X$; E^* admits a natural trivialisation when E is trivialised, and the transition matrices of E^* are the inverses of the transposes of the transition matrices of E. Similarly, the points of $\Lambda^k E$ can be identified with the alternating $\mathbb{K}$-linear forms on the fibres of $\pi_E^* : E^* \to X$.

3.4.1. **The tangent bundle.** If X is a differentiable manifold, the *tangent bundle* TX of X is a differentiable bundle of rank $n = \dim X$ which we can define as follows. If X is covered by open sets U_i equipped with diffeomorphisms h_i to open sets of R^n, then TX is covered by open sets $U_i \times \mathbb{R}^n$, where the identifications (or transition morphisms) between $U_i \cap U_j \times \mathbb{R}^n \subset U_i \times \mathbb{R}^n$ and $U_i \cap U_j \times \mathbb{R}^n \subset U_j \times \mathbb{R}^n$ are given by

$$(x, v) \to (x, \phi_{ij*}(x)v).$$

Here $h_{ij} = h_i \circ h_j^{-1}$ is the transition diffeomorphism between the open sets $h_i(U_i \cap U_j)$ and $h_j(U_i \cap U_j)$ of R^n, and $h_{ij*}(x)$ is its Jacobian matrix at the point x. A section of the tangent bundle of a differentiable manifold is called a *vector field*.

There exist two intrinsic ways of describing the elements of the tangent bundle. The points of the tangent bundle can be identified with equivalence classes of differentiable maps $\gamma : [-\epsilon, \epsilon] \to X$ (for an $\epsilon \in \mathbb{R}, \epsilon > 0$ varying with γ) for the equivalence relation

$$\gamma \equiv \gamma' \text{ if and only if } \gamma(0) = \gamma'(0),\ \frac{d\gamma}{dt}(0) = \frac{d\gamma'}{dt}(0).$$

The second equality in this definition makes sense in any local chart for X in the neighbourhood of $\gamma(0)$. We call these equivalence classes "jets of order 1". To check that the set defined in this way has the structure of a vector bundle introduced earlier, it suffices to note that the jets of order 1 of an open set U of $\mathbb{R}^n$ can be identified, via the map $\gamma \to (\gamma(0), \dot{\gamma}(0))$, with $U \times \mathbb{R}^n$, and that a diffeomorphism $\psi : U \to V$ between two open sets of $\mathbb{R}^n$ induces the isomorphism (ψ, ψ_*) between the spaces of jets of order 1 of U and V.

Another definition of the tangent vectors, i.e. of the elements of the tangent bundle, consists in identifying them with the derivations of the algebra of the real differentiable functions on X with values in $\mathbb{R}$ supported at a point $x \in X$. This means that we consider the linear maps $\delta : C^\infty(X) \to \mathbb{R}$ satisfying Leibniz' rule

$$\delta(fg) = f(x)\delta(g) + g(x)\delta(f)$$

for a point $x \in X$. The equivalence between the two definitions is realised by the map which to a jet γ associates the derivation $\delta_\gamma(f) = \frac{d(\gamma \circ f)}{dt}(0)$.

Definition 3.5. A *differential form of degree* k is a section of $\Lambda^k T^* X$. We write $\deg \alpha$ for the degree of such a form α.

In general, we write $\Omega_{X,\mathbb{R}}$ the bundle of real differential 1-forms, and $\Omega_{X,\mathbb{C}}$ for its complexification $\mathrm{Hom}(\Omega_{X,\mathbb{R}}, \mathbb{C})$. Similarly, the bundle of real (resp. complex) k-forms is written $\Omega^k_{X,\mathbb{R}}$ (resp. $\Omega^k_{X,\mathbb{C}}$). We see immediately that if f is a real differentiable function on X, then df is a C^∞ section of $\Omega_{X,\mathbb{R}}$. We also see that if $x_1, \dots, x_n$ are local coordinates defined on an open set $U \subset X$, then the $dx_I = dx_{i_l} \wedge \cdots \wedge dx_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$ provide a basis of the fibre of $\Omega^k_{X,\mathbb{R}}$ at each point of the open set U. Indeed, by the definition of TX, the coordinates x_i provide a local trivialisation of TX, where the corresponding local basis is given at each point $x \in U$ by the derivations $\frac{\partial}{\partial x_i}(x)$. The dx_i simply form the dual basis of $\Omega_{X,\mathbb{R}}$ at each point of U.

3.5.1. **Complex manifolds.** Let X be a differentiable manifold of dimension $2n$.

Definition 3.6. We say that X is equipped with a *complex structure* if X is covered by open sets U_i which are diffeomorphic, via maps called h_i, to open sets of C^n, in such a way that the transition diffeomorphisms $h_j \circ h_i^{-1} : h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$ are holomorphic.

The (complex) *dimension* of X is by definition equal to n. On a complex manifold, a map $f : U \to \mathbb{C}$ defined on an open set U is said to be holomorphic if $f \circ h_i^{-1}$ is holomorphic on $h_i(U \cap U_i)$. Once again, this definition does not depend on the choice of chart, since the change of chart morphisms is holomorphic and compositions of holomorphic functions are also holomorphic.

We can also define the notion of a *holomorphic vector bundle.*

Definition 3.7. A differentiable complex vector bundle $\pi : E \to X$ over a complex manifold X is said to be equipped with a holomorphic structure if we have trivialisations

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^m,$$

such that the transition matrices $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ have holomorphic coefficients.

The above trivialisations will be called "holomorphic trivialisations". If E is a holomorphic vector bundle, E is in particular a complex manifold such that π is holomorphic. Indeed, we can assume, in the definition above, that the U_i are charts, i.e. identified via h_i with open sets of $\mathbb{C}^n$; then the $(h_i \times \mathrm{Id}_{\mathbb{C}^n}) \circ \phi_i$ give charts for E whose transition functions are clearly holomorphic.

A holomorphic section of a holomorphic vector bundle $\pi : E \to X$ over an open set U of X is a section $\sigma : X \to E$ of π which is a holomorphic map. For example, a holomorphic local trivialisation ϕ_i of E as above is given by the choice of a family of holomorphic sections of , whose values at each point x of U_i form a basis of the fibre E_x over $\mathbb{C}$.

Example 3.8. The holomorphic tangent bundle. This bundle is defined exactly like the real tangent bundle of a differentiable manifold. Given a system of charts $h_i : U_i \to V_i \subset \mathbb{C}^n$, we define TX as the union of the $U_i \times \mathbb{C}^n$, glued by identifying $U_i \cap U_j \times \mathbb{C}^n \subset U_i \times \mathbb{C}^n$ and $U_i \cap U_j \times \mathbb{C}^n \subset U_j \times \mathbb{C}^n$ via

$$(x, v) \to (x, h_{ij*}(x)v).$$

Here, the holomorphic Jacobian matrix h_{ij*} is the matrix with holomorphic coefficients $\frac{\partial h_{ij}^k}{\partial z_l}$, where $h_{ij} = h_i \circ h_j^{-1}$. We can also, as for the real tangent bundle, define the *holomorphic tangent bundle* as the set of complex-valued derivations of the $\mathbb{C}$-algebra of holomorphic functions, or as the set of jets of order 1 of holomorphic maps from the complex disk to X.

3.9. **Integrability of almost complex structures.**

3.9.1. *Tangent bundle of a complex manifold.* Let X be a complex manifold, and let $h_k : U_i \to \mathbb{C}^n$ be holomorphic local charts. Then the real tangent bundle $TU_{k\mathbb{R}}$ can be identified, via the differential h_{k*} , with $U_k \times \mathbb{C}^n$. Moreover, the change of chart morphisms $h_k \circ h_j^{-1}$ are holomorphic by hypothesis, i.e. have $\mathbb{C}$-linear differentials, for the natural identifications: $T_x\mathbb{C}^n = \mathbb{C}^n$, $x \in \mathbb{C}^n$. It follows that the $\mathbb{R}$-linear operators $J_k : TU_{k\mathbb{R}} \to TU_{k\mathbb{R}}$ identified with $1 \times i$ acting on $U_k \times \mathbb{C}^n$, glue together on $U_k \cap U_j$ and define a global endomorphism, written J, of the bundle $TX_{\mathbb{R}}$. Obviously J satisfies the identity $J^2 = -1$; thus J defines the structure of a $\mathbb{C}$-vector space of rank n on each fibre $T_xX_{\mathbb{R}}$. The differentiability of J even shows that $TX_{\mathbb{R}}$ is thus equipped with the structure of a differentiable complex vector bundle. This leads us to introduce the following definition.

Definition 3.10. An *almost complex structure* on a differentiable manifold X is an endomorphism J of the real tangent bundle $TX_{\mathbb{R}}$ such that $J^2 = -1$; equivalently, it is the structure of a complex vector bundle on $TX_{\mathbb{R}}$.

We saw that a complex structure on X naturally induces an almost complex structure.

Definition 3.11. An almost complex structure J on a manifold X is said to be *integrable* if there exists a complex structure on X which induces J.

In the case of a complex manifold, the relation between $TX_{\mathbb{R}}$ seen as a complex vector bundle and the holomorphic tangent bundle TX of X is as follows: the bundle TX is generated, in a chart U, by the elements

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$

which are naturally elements of $TU_{\mathbb{R}} \otimes \mathbb{C}$. Thus, in fact, we have an inclusion of complex vector bundles

$$TX \subset TX_{\mathbb{R}} \otimes \mathbb{C}.$$

Moreover, for an almost complex manifold (X, J), the complexified tangent bundle $TX_{\mathbb{R}} \otimes \mathbb{C}$ contains a complex vector subbundle, denoted by $T^{1,0}X$ and defined as the bundle of eigenvectors of J for the eigenvalue i. As a real vector bundle, $T^{1,0}X$ is naturally isomorphic to $TX_{\mathbb{R}}$ via the application Re (real part) which to a complex field $v + iw$ associates its real part v. Moreover, this identifies the operators i on $T^{1,0}X$ and J on $TX_{\mathbb{R}}$. Clearly $T^{1,0}X$ is generated by the $v - iJv, v \in TX_{\mathbb{R}}$.

Furthermore, in the case where $X = \mathbb{C}^n$, consider the isomorphism $T\mathbb{C}^n_{\mathbb{R}} = \mathbb{C}^n \times R^{2n}$ given by the sections $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ of the tangent bundle of $\mathbb{C}^n$, where $z_k = x_k + iy_k$ and z_k are complex linear coordinates on $\mathbb{C}^n$. The induced complex structure operator J on $T\mathbb{C}^n_{\mathbb{R}}$ sends $\frac{\partial}{\partial x_j}$ to $\frac{\partial}{\partial y_j}$. Thus, the tangent vectors of type $(1, 0)$ are generated over $\mathbb{C}$ at each point by the

$$\left(\frac{\partial}{\partial x_j} - iJ\frac{\partial}{\partial y_j}\right) = 2\frac{\partial}{\partial z_j}.$$

In conclusion, we have shown the following.

Proposition 3.12. *If X is a complex manifold, then X admits an almost complex structure, and the subbundle $T^{1,0}X \subset TX_{\mathbb{R}} \otimes \mathbb{C}$ defined by J is equal, as a complex vector subbundle of $TX_{\mathbb{R}} \otimes \mathbb{C}$, to the holomorphic tangent bundle TX.*

Complex conjugation acts naturally on the complexified tangent bundle $TX_{\mathbb{C}} = TX_{\mathbb{R}} \otimes \mathbb{C}$ of a differentiable manifold X. If J is an almost complex structure on X, we have the subbundle $T^{0,1}X$ of $TX_{\mathbb{C}}$, defined as the complex conjugate of $T^{1,0}X$. We can also define it as the set of the complexified tangent vectors which are the eigenvectors of J associated to the eigenvalue $-i$. Thus, it is clear that we have a direct sum decomposition

$$TX_{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X.$$

Remark 3.13. When X is an almost complex manifold, the vector bundle $T^{1,0}X$ does not a priori have the structure of a holomorphic bundle. In what follows, if X is a complex manifold, a section of TX will be taken to mean a holomorphic section of TX, while a section of $T^{1,0}X$ will be a differentiable section.

If $\psi : X \to Y$ is a holomorphic map between two complex manifolds, we define a morphism of holomorphic vector bundles $\psi_* : TX \to TY$ in the obvious way. In holomorphic local charts which trivialise TX and TY, the matrix of ψ_* is given by the holomorphic Jacobian matrix $\frac{\partial \psi_k}{\partial z_j}$ of ψ. This morphism can in fact be identified with the morphism of real vector bundles $\psi_{*,\mathbb{R}} : TX_{\mathbb{R}} \to TY_{\mathbb{R}}$ via the identifications of real bundles $TX = TX_{\mathbb{R}}, TY = TY_{\mathbb{R}}$ given by the real part Re. As a morphism of complex bundles, ψ_* can be deduced from $\psi_{*,\mathbb{R}}$ by noting that $\psi_{*,\mathbb{R}}$ is compatible with the almost complex structures of X and Y, since ψ is holomorphic, and thus induces a $\mathbb{C}$-linear morphism $\psi_*^{1,0} : T^{1,0}X \to T^{1,0}Y$.

3.13.1. **The Newlander-Nirenberg theorem.** Note first that the bracket of vector fields over a differentiable manifold X extends by $\mathbb{C}$-linearity to the complexified vector fields, i.e. to the differentiable sections of $TX_{\mathbb{C}}$. Now, let (X, J) be an almost complex manifold. As mentioned above, the almost complex structure operator J splits the bundle $TX_{\mathbb{C}}$ into elements of type $(1, 0)$, eigenvectors associated to the eigenvalue i of J, and elements of type $(0, 1)$, eigenvectors associated to the eigenvalue $-i$ of J. The bundle $T^{1,0}X$ is the complex conjugate of the bundle $T^{0,1}X$. The following theorem gives an exact description of the integrable almost complex structures.

Theorem 3.14. *(Newlander-Nirenberg) The almost complex structure J is integrable if and only if we have*

$$[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X.$$

Remark 3.15. By passing to the conjugate, this is equivalent to the condition that the bracket of two vector fields of type $(1, 0)$ is of type $(1, 0)$.

This theorem is a difficult theorem in analysis, for it implies, in particular, that the manifold X which was assumed to be only differentiable actually admits the structure of a real analytic manifold.

3.16. The operators ∂ and $\bar{\partial}$.

3.16.1. **Definition.** Let (X, J) be an almost complex manifold; the decomposition at the tangent bundle level $TX_{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X$ induces a dual decomposition

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}. \tag{3.1}$$

When X is a complex manifold, the bundle $\Omega_X^{1,0}$ of complex differential forms of type $(1, 0)$, i.e. $\mathbb{C}$-linear forms, is generated in holomorphic local coordinates $z_1, \ldots, z_n$ by the dz_i, i.e. a form a of type $(1, 0)$ can be written locally as $\alpha = \sum_i \alpha_i dz_i$, where the a_i, are C^∞ functions if α is C^∞. Since $d(dz_i) = 0$, it follows that $d\alpha = \sum d\alpha_i \wedge dz_i$. Furthermore, the decomposition 3.1 also induces the decomposition of the complex k-forms into forms of type (p, q), for $p + q = k$:

$$\Lambda^k \Omega_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q}, \tag{3.2}$$

where the bundle $\Omega_X^{p,q}$ is equal to $\Lambda^p\Omega_X^{1,0} \oplus \Lambda^q\Omega_X^{0,1}$. With this definition, formula (3.2) shows that if the almost complex structure is integrable and α is a differential form of type $(1, 0)$, then $d\alpha$ is a section of $\Omega_X^{2,0} \oplus \Omega_X^{1,1}$. In fact, using the formula

$$\chi(\alpha(\eta)) - \eta(\alpha(\chi)) = d\alpha(\chi, \eta) + \alpha([\chi, \eta]),$$

where α is a 1-form and χ, η are vector fields, we easily see that this property is equivalent to the integrability condition of theorem 3.14, and thus to the integrability of the almost complex structure.

More generally, the bundle $\Omega_X^{p,q}$ admits as generators in holomorphic local coordinates $z_1, \dots, z_n$, in the differential forms

$$dz_I \wedge d\bar{z}_J = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where I, J are sets of ordered indices $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_q \leq n$. Note that these forms are closed, i.e. annihilated by the exterior differential operator d. A form α of type (p, q) can thus be written locally as $\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$. It follows that

$$d\alpha = \sum_{I,J} d\alpha_{I,J} \wedge dz_I \wedge d\bar{z}_J$$

is the sum of a form of type $(p, q + 1)$ and a form of type $(p + 1, q)$.

Definition 3.17. For a C^∞ differential form a of type (p, q) on a complex manifold X, we define $\partial\alpha$ to be the component of type $(p, q + 1)$ of $d\alpha$. Similarly, we define $\bar{\partial}\alpha$ to be the component of type $(p + 1, q)$ of $d\alpha$.

For $(p, q) = (0, 0)$, a form of type (p, q) is a function f; $\bar{\partial}f$ is then the $\mathbb{C}$-antilinear part of df, and thus it vanishes if and only if f is holomorphic. By definition, we have

$$df = \sum_i \frac{\partial f}{\partial z_i} dz_i + \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i,$$

and thus

$$\bar{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

As mentioned above, a differential k-form α decomposes uniquely into components $\alpha^{p,q}$ of type $(p, q), p + q = k$. We then set

$$\bar{\partial}\alpha = \sum_{p,q} \bar{\partial}\alpha^{p,q}, \quad \partial\alpha = \sum_{p,q} \partial\alpha^{p,q}.$$

The following lemmas describe the essential properties of the operators ∂ and $\bar{\partial}$.

Lemma 3.18. *The operator $\bar{\partial}$ satisfies Leibniz' rule*

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \bar{\partial}\beta.$$

Similarly, the operator ∂ satisfies Leibniz' rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial\beta.$$

Proof. The second assertion follows from the first, since by definition of the operators ∂ and $\bar{\partial}$, we have the relation $\partial\bar{\alpha} = \bar{\partial}\bar{\alpha}$. As for the first relation, it suffices to prove it for α of type (p,q) and β of type (p',q'). We then obtain it immediately in this case, by taking the component of type $(p+p', q+q'+1)$ of $d(\alpha\wedge\beta)$. □

Lemma 3.19. *We have the following relations between the operators ∂ and $\bar{\partial}$.*

$$\bar{\partial}^2 = 0,\ \partial^2 = 0,\ \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proof. This follows from the formulas $d\circ d = 0$, $d = \partial + \bar{\partial}$. Indeed, these relations imply that $\partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = d^2 = 0$. Now, if α is a form of type (p,q) then $\partial^2\alpha$ is of type $(p+2,q)$, $(\partial\bar{\partial}+\bar{\partial}\partial)\alpha$ is of type $(p+1,q+1)$ and $\bar{\partial}^2\alpha$ is of type $(p,q+2)$. Thus, $d^2\alpha = 0$ implies that $\partial^2\alpha = (\partial\bar{\partial}+\bar{\partial}\partial)\alpha = \bar{\partial}^2\alpha = 0$. □

3.19.1. **Local exactness.** The Poincare lemma shows the local exactness of the operator d (see Bott and Tu [1]):

Lemma 3.20. *Let α be a closed differential form of strictly positive degree on a differentiable manifold. Then, locally, there exists a differential form β such that $\alpha = d\beta$. We say that α is locally exact.*

Now consider a complex manifold X. Let $\alpha = d\beta$ be a form of type (p,q) which is $\bar{\partial}$-exact. Then we have $\bar{\partial}\alpha = 0$ by lemma 3.19. The following proposition is a partial converse which is the analogue of the Poincare lemma for the operator $\bar{\partial}$.

Proposition 3.21. *Let α be a differential form of type (p,q) with $q>0$. If $\bar{\partial}\alpha = 0$, then there locally exists on X a differential form β of type $(p,q-1)$ such that $\alpha = \bar{\partial}\beta$.*

Proof. We first reduce to the case where $p=0$ by the following argument. Locally, we can write in holomorphic coordinates $z_1,\ldots,z_n$

$$\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J,$$

where the sets of indices I are of cardinal p and the sets of indices J are of cardinal q. Then

$$\bar{\partial}\alpha = \sum_{I,J} \bar{\partial}\alpha_{I,J}\wedge dz_I \wedge d\bar{z}_J$$

by lemma 3.18. It follows that if $\bar{\partial}\alpha = 0$, for every I of cardinal p the form α_I of type $(0,q)$ defined by

$$\alpha_I = \sum_J d\bar{z}_J$$

is $\bar{\partial}$-closed. If the proposition is proved for forms of type $(0,q)$, then locally we have $\alpha_I = \bar{\partial}\beta_I$, and

$$\alpha = (-1)^p\bar{\partial}\left(\sum_I dz_I\wedge\beta_I\right).$$

It remains to show the proposition for forms of type $(0,q)$. Such a form can be written $\alpha = \sum_J \alpha_J d\bar{z}_J$. We do the proof by induction on the largest integer k such that there exists J with $k \in J$ and $\alpha_J \neq 0$. Necessarily $k \geq q$. If $k = q$, we have $a = f d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$. The condition $\bar{\partial}\alpha = 0$ is then equivalent to the fact that the function f is holomorphic in the variables $z_l, l > q$. We then apply Cauchy's theorem : note that its proof also gives the following. □

Proposition 3.22. *Let $f(z_1, \ldots, z_n)$ be a differential function which is holomorphic in the variables $z_l, l > q$. Then there locally exists a differential function g, holomorphic in the variables $z_l, l > q$, such that $\frac{\partial g}{\partial \bar{z}_q} = f$.*

3.22.1. **Dolbeault complex of a holomorphic bundle.** Let E be a holomorphic vector bundle of rank over a complex manifold X. Let $\mathcal{E}^{0,q}(E)$ denote the space of C^∞ sections of the bundle $\Omega_X^{0,q} \otimes_{\mathbb{C}} E$. In a holomorphic trivialisation of E, $\phi_U : E_{|U} = U \times \mathbb{C}^n$, such a section can be written $(\alpha_1, \ldots, \alpha_k)$, where the α_i, are C^∞ forms of type $(0, q)$ on U. We then set $\bar{\partial}_U \alpha = (\bar{\partial}\alpha_1, \ldots, \bar{\partial}\alpha_k)$; it is a section of $\Omega_U^{0,q+1} \otimes_{\mathbb{C}} E$. We will show that this local definition in fact gives a form $\bar{\partial}\alpha \in \mathcal{E}^{0,q+1}(E)$.

Lemma 3.23. *Let V be an open subset of X and $\phi_V : E_{|V} = V \times \mathbb{C}^k$ a holomorphic trivialisation of E over V. Then for $\alpha \in \mathcal{E}^{0,q}(E)$, we have*

$$\bar{\partial}_U \alpha \mid_{U\cap V} = \bar{\partial}_V \alpha \mid_{U \cap V} .$$

Proof. Let M_{UV} be the transition matrix, with holomorphic coefficients, which enables us to pass from the trivialisation ϕ_U to the trivialisation ϕ_V. Then, by definition, if α_U is a section of E over U, $\alpha_U = (\alpha_{1,U}, \ldots, \alpha_{k,U})$ in the trivialisation ϕ_U, and α_V is a section of E over V, $\alpha_V = (\alpha_{1,V}, \ldots, \alpha_{k,V})$in the trivialisation ϕ_V, the sections α_U and α_V coincide on $U \cap V$ if and only if

$$(\alpha_{1,V}, \ldots, \alpha_{k,V})^T = M_{UV}(\alpha_{1,U}, \ldots, \alpha_{k,U})^T.$$

We can of course replace the functions α_i by differential forms. The form α can be written $(\alpha_{1,U}, \ldots, \alpha_{k,U})$ in the trivialisation ϕ_U and $(\alpha_{1,V}, \ldots, \alpha_{k,V})$ in the trivialisation ϕ_V, and we have, as above,

$$(\alpha_{1,V}, \ldots, \alpha_{k,V})^T = M_{UV}(\alpha_{1,U}, \ldots, \alpha_{k,U})^T.$$

To see

$$\bar{\partial}_U \alpha \mid_{U\cap V} = \bar{\partial}_V \alpha \mid_{U \cap V} .$$

by the above and the definition of $\bar{\partial}_U, \bar{\partial}_V$, it suffices to show that

$$(\bar{\partial}\alpha_{1,V}, \ldots, \bar{\partial}\alpha_{k,V})^T = M_{UV}(\bar{\partial}\alpha_{1,U}, \ldots, \bar{\partial}\alpha_{k,U})^T.$$

But this follows immediately from the Leibniz formula lemma 3.18 and the fact that the matrix M_{UV} has holomorphic coefficients. □

Lemma 3.23 enables us to define an operator

$$\bar{\partial}_E : \mathcal{E}^{0,q}(E) \to \mathcal{E}^{0,q+1}(E)$$

by the condition $\bar{\partial}_E \alpha \mid_U = \bar{\partial}_U \alpha \mid_U$. Note that the meaning of this operator on the space $\mathcal{E}^{0,0}(E)$ of C^∞ sections of E is the following.

Lemma 3.24. *The kernel*

$$\mathrm{Ker}\left(\bar\partial_E : \mathcal{E}^{0,0}(E) \to \mathcal{E}^{0,1}(E)\right)$$

contains exactly the holomorphic sections of E.

Proof. This is clear, since the holomorphic sections are those which are given by n-tuples of holomorphic functions in local holomorphic trivialisations. But by definition, $\bar\partial_E$ acts like the operator $\bar\partial$ on these n-tuples, and we know that the functions annihilated by $\bar\partial$ are exactly the holomorphic functions. □

Naturally, this operator satisfies the same local properties as the operator $\bar\partial$ on the forms.

Lemma 3.25. *The operator $\bar\partial_E$ satisfies Leibniz' rule*

$$\bar\partial_E(\alpha \wedge \beta) = \bar\partial_E \alpha \wedge \beta + (-1)^q \alpha \wedge \bar\partial_E \beta.$$

Here, α is a differential form of type $(0, q)$, and β is a differential form of type $(0, q')$ with coefficients in E, so that $\alpha \wedge \beta$is naturally a differential form of type $(0, q + q')$ with coefficients in E.

Clearly, the operator $\bar\partial_E$ also satisfies the property $\bar\partial_E^2 = 0$. Finally, the local exactness of $\bar\partial_E$ follows from that of the operator $\bar\partial$.

Proposition 3.26. *Let α be a form of type $(0, q)$ with coefficients in E, and $q > 0$. If $\bar\partial_E \alpha = 0$, then locally on X there exists a form β of type $(0, q-1)$ with coefficients in E such that $\alpha = \bar\partial_E \beta$.*

Recall that a complex (of vector spaces for example) is a family of vector spaces V_i together with morphisms $d_i : V_i \to V_{i+1}$ satisfying $d_{i+1} \circ d_i = 0$. The standard example is the *de Rham complex* of a differentiable manifold X, where $V_i = \mathcal{E}^i(X)$ is the space of differential forms of degree i, and $d_i = d$.

Definition 3.27. The complex

$$\left(\mathcal{E}^{0,*}(E), \bar\partial_E\right) = \left(\cdots \to \mathcal{E}^{0,q-1}(E) \xrightarrow{\bar\partial_E} \mathcal{E}^{0,q}(E) \xrightarrow{\bar\partial_E} \mathcal{E}^{0,q+1}(E) \to \dots\right)$$

is called the *Dolbeault complex* of E.

3.28. **Examples of complex manifolds.**

3.28.1. **Riemann surfaces.** Let us consider 2-dimensional differentiable manifolds. If we restrict ourselves to the compact oriented case, these manifolds are classified by their genus g: such a surface is diffeomorphic to the g-holed torus. Furthermore, we can always put complex structures on such surfaces X. Indeed, we first note that the Newlander-Nirenberg integrability condition is automatically satisfied by an almost complex structure on X, by the fact that the rank of the complex vector bundle $T^{0,1}X$ is equal to 1, and the bracket of vector fields is alternating. Thus, every almost complex structure is induced by a complex structure. Moreover, the existence of almost complex structures follows from the existence of Riemannian metrics on X: An almost complex structure on an oriented surface X is equivalent to a conformal structure on X, i.e. to a Riemannian metric on X defined up to multiplication by a positive function. Compact Riemann surfaces are called curves in algebraic geometry. Indeed, they are 1-dimensional complex manifolds (varieties).

3.28.2. **Complex projective space.** The complex projective space $\mathbb{CP}^n$ is the set of complex lines of $\mathbb{C}^{n+1}$, or equivalently, the quotient of $\mathbb{C}^{n+1} \setminus 0$ by the equivalence relation identifying collinear vectors on $\mathbb{C}$. The topology is the quotient topology. The complex structure is obtained as follows. For each i, consider the open subset $\tilde{U}_i$ of $\mathbb{C}^{n+1} \setminus 0$ consisting of the points z such that $z_i \neq 0$. Let U_i be the image of $\tilde{U}_i$ in $\mathbb{CP}^n$. Each point $z \in U_i$ admits a unique lifting z to $\tilde{U}_i$ which satisfies the condition $z_i = 1$. Thus, U_i is naturally homeomorphic to $\mathbb{C}^n$, which provides the holomorphic charts for $\mathbb{CP}^n$, which is covered by the U_i. It remains simply to check that the change of chart morphisms are holomorphic. But $U_i \cap U_j$ can obviously be identified with the classes of non-zero vectors $z \in \mathbb{C}^{n+l}$ such that $z_i \neq 0$ and $z_j \neq 0$. Given such a vector, the image of the representative of its class in the chart $U_i = \mathbb{C}^n$ is given by $\left(\frac{z_1}{z_i}, \ldots, 1, \ldots, \frac{z_{n+1}}{z_i}\right)$, where the 1 is in the i^{th} place, while the image of the representative of its class in the chart $U_j = \mathbb{C}^n$ is given by $\left(\frac{z_1}{z_j}, \ldots, 1, \ldots, \frac{z_{n+1}}{z_j}\right)$, where the 1 is in the j^{th} place. The transition morphism is thus given, up to the order of the coordinates, by

$$(\zeta_1, \ldots, \zeta_n) \to \left(\frac{1}{\zeta_j}, \frac{\zeta_1}{\zeta_j}, \ldots, \frac{\zeta_n}{z_j}\right) \tag{3.3}$$

on $\mathbb{C}^n \setminus \{\zeta_j = 0\}$. As (3.3) is clearly holomorphic, we have equipped $\mathbb{CP}^n$ with a complex structure.

3.28.3. **Complex tori.** Let Γ be a lattice in $\mathbb{C}^n$, i.e. a free additive subgroup generated by a basis of $\mathbb{C}^n$ over $\mathbb{R}$. The group Γ acts by translation on $\mathbb{C}^n$, and the action is proper and fixed-point-free. The quotient $\mathbb{T} = \mathbb{C}^n/\Gamma$ is compact. In fact, there exists a $\mathbb{R}$-linear automorphism of $\mathbb{C}^n = \mathbb{R}^{2n}$ sending Γ to $\mathbb{Z}^{2n}$ so that this quotient is naturally homeomorphic to $(\mathbb{R}/\mathbb{Z})^n = (\mathbb{S}^1)^n$. Clearly, $\mathbb{T}$ admits a natural differentiable structure for which the quotient map is a local diffeomorphism. We then put an almost complex structure onto $\mathbb{T}$ by taking the holomorphic charts to be the local inverses of the quotient map. As these local inverses are defined up to translation by an element of Γ, the change of chart morphisms are given by these translations, which are obviously holomorphic. Thus, $\mathbb{T}$ is equipped with a holomorphic structure.

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