TOPOLOGY OF PLANE ALGEBRAIC CURVES

1. Introduction

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2. Fundamental concepts

2.1. **Affine plane algebraic curves.** The following notation for the zeroes set of a polynomial will be sometimes used. For $f \in \mathbb{C}[x, y]$ the set

$$V(f) := \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}.$$

is called the *variety* of f.

We now introduce the fundamental notion of an affine plane curve.

Definition 2.2. A subset $C \subset \mathbb{C}^2$ is called an *affine plane algebraic curve* if there exists a polynomial $f \in \mathbb{C}[x, y]$ of degree ≥ 1 such that

$$C = V(f) = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}.$$

It is clear that any polynomial $c \cdot f^k$ with $c \in \mathbb{C}^*$ and $k \in \mathbb{N}^*$ defines the same curve as f. We will see that this is the only indeterminacy in the definition of C.

Note that if f is a divisor of g then $V(f) \subset V(g)$. Using the following lemma we can go the other way.

Lemma 2.3. *Let $f, g \in \mathbb{C}[x, y]$. If f is irreducible of degree ≥ 1 and $V(f) \subset V(g)$ then f is a divisor of g.*

We will give a proof of this lemma to illustrate both a basic technique and the important notion of resultant of two polynomials. The strategy is to reduce the assertion to the fundamental theorem of algebra.

Definition 2.4. If A is a commutative ring with unity and if $P = a_0X^m + a_1X^{m-1} + \cdots + a_m$ and $Q = b_0X^n + b_1X^{n-1} + \cdots + b_n$ are polynomials in $A[X]$ the *resultant* $R_{P,Q}$ of P and Q is defined as the determinant of the square $(m + n)$-matrix

$$\begin{pmatrix} a_0 & & \cdots & \cdots & & a_m & & \\ & \ddots & & & & & \ddots & \\ & & a_0 & & \cdots & \cdots & & a_m \\ b_0 & & \cdots & \cdots & & b_n & & \\ & \ddots & & & & & \ddots & \\ & & b_0 & & \cdots & \cdots & & b_n \end{pmatrix}$$

where the coefficients of P are arranged on the first n rows and of Q on the last m rows. Note that $R_{P,Q} \in A$.

The usefulness of the resultant is highlighted by the following fact.

Theorem 2.5. *Suppose A is a unique factorization domain and $a_0, b_0 \neq 0$. Then the following are equivalent:*

(i) *P and Q have a commnon factor of degree ≥ 1 in $A[X]$.*

(ii) $R_{P,Q} = 0$ *in* A.

We now return to the proof of Lemma 2.3.

Proof. Write f and g as polynomials in $\mathbb{C}[x][y]$:

$$f = a_0 y^m + a_1 y^{m-1} + \cdots + a_m,\ g = b_0 y^n + b_1 y^{n-1} + \cdots + b_n,$$

where $a_i, b_j \in \mathbb{C}[x]$ and $a_0, b_0 \not\equiv 0$. We may assume $m \geq 1$.

Now intersect the sets $V(f)$ and $V(g)$ with the vertical lines $x = t$ for all $t \in \mathbb{C}$. Note that $n \geq 1$ as well: otherwise choose t_0 such that $a_0(t_0), b_0(t_0) \neq 0$, and then $x = t_0$ would intersect $V(f)$ but not $V(g)$.

Consider the resultant $R = R_{f,g}$ in $A = \mathbb{C}[x]$. Since f is irreducible, if we show that $R \equiv 0$ then by Theorem 2.5 we are done. But R is itself a polynomial in $\mathbb{C}[x]$ so it is enough to show that $R(x) = 0$ for infinitely many $x \in \mathbb{C}$. It is clear that $a_0(t), b_0(t) \neq 0$ for almost all t. For such a t the line $x = t$ will intersect both $V(f)$ and $V(g)$. In other words, the polynomials $P = f(t, y)$ and $Q = g(t, y)$ in $\mathbb{C}[y]$ have a nonconstant common factor $y - \lambda$ for some $\lambda \in \mathbb{C}$. We conclude by Theorem 2.5 that $R(t) = R_{P,Q} = 0$. □

Remark 2.6. By using this method of projecting a curve on a line we can easily see that an algebraic curve $C \subset \mathbb{C}^2$ contains infinitely many points.

We discuss now the decomposition of algebraic curves into components. This is based on the the fact that polynomial ring $\mathbb{C}[x, y]$ is a UFD.

Definition 2.7. An algebraic curve $C \subset \mathbb{C}^2$ is called *reducible* if $C = C_1 \cup C_2$ for some curves C_1 and C_2 such that $C_1 \neq C_2$. Otherwise we say that C is *irreducible*.

The following is an immediate consequence of Lemma 2.3.

Lemma 2.8. *A curve* $C = V(f) \subset \mathbb{C}^2$ *is irreducible if and only if there exists* $k \in \mathbb{N}^*$ *and an irreducible* $g \in \mathbb{C}[x, y]$ *such that* $f = g^k$.

This lemma and the prime factorization of a defining polynomial implies the following.

Theorem 2.9. *Any algebraic curve* $C \subset \mathbb{C}^2$ *is the union* $C = C_1 \cup \cdots \cup C_r$ *of irreducible curves. The* $C_i, 1 \leq i \leq r$ *are called the* components *of* C*, and they are unique up to a permutation.*

An important property of irreducible curves is the following.

Theorem 2.10. *An irreducible algebraic curve* $C \subset \mathbb{C}^2$ *is connected as a topological space.*

This theorem will be proved later.

Remark 2.11. Notice that some of the previous statements are false if $\mathbb{C}$ is replaced by $\mathbb{R}$.

If $C = V(f)$ and $f = f_1^{k_1} \cdot \dots \cdot f_r^{k_r}$ is a prime factorization then any any other polynomial g such that $C = V(g)$ will be of the form $cf_1^{l_1} \cdot \dots \cdot f_r^{l_r}$ where $c \in \mathbb{C}^*$ and $l_i \in \mathbb{N}^*$. We may call $f_1 \cdot \ldots f_r$ the *minimal defining polynomial* of C. A curve $V(f) \subset \mathbb{C}^2$ with f minimal is called *reduced*. For a curve $C = V(f)$ given by $f = f_1^{k_1} \cdot \dots \cdot f_r^{k_r}$ we may regard k_i as the multiplicity of the component $C_i = V(f_i)$.

We introduce next an important invariant of an algebraic curve, its degree.

Definition 2.12. If $C = V(f) \subset \mathbb{C}^2$ with f minimal then the *degree of the curve* C, denoted by $\deg C$, is the degree of f.

The geometric meaning of the degree can be seen by intersecting the curve with lines. Let L be a line in $\mathbb{C}^2$ given by a parametrization $t \to (\alpha(t), \beta(t))$ where $\alpha(t), \beta(t)$ are linear. If $C = V(f)$ we obtain a single variable polynomial $h(t) = f(\alpha(t), \beta(t))$, whose zeroes correspond to the points in $C \cap L$. Since $h \equiv 0$ is equivalent to $L \subset C$, the inequality $\deg h \leq \deg f$ implies that $\#C \cap L \leq \deg C$.

This bound is almost always attained. If $0 \leq k \leq d = \deg f$, then $f = f_0 + f_1 + \cdots + f_d$, where $f_k = \sum_{k_1+k_2=k} a_{k_1,k_2} x^{k_1} y^{k_2}$ is the *homogeneous part* of degree k of f. If $\alpha(t) = a_1 t + a_2, \beta(t) = b_1 t + b_2$, then the coefficient of t in h is given by $f_d(a_1, b_1)$. Because $f_d \not\equiv 0$, f_d can vanish for at most d distinct slopes $a_1 : b_1$ of L. For all the remaining slopes, $\deg h = \deg f = d$. Thus there are two obstructions to attaining the bound $\#C \cap L = \deg C$:

(a) The line L may have an exceptional slope.
(b) The polynomial h may have multiple zeroes.

The second problem can be resolved by counting points of intersection with their multiplicities, and the first by also considering points of intersection at infinity.

2.13. **Projective plane algebraic curves.** The complex projective plane $\mathbb{P}^2$ is the set of all lines through the origin in $\mathbb{C}^3$. If $x = (x_0, x_1, x_2) \in \mathbb{C}^3 \setminus \{0\}$ then $[x_0 : x_1 : x_2] = \mathbb{C} \cdot (x_0, x_1, x_2)$ denotes the line through x. In these *homogeneous coordinates*, we have that $[x_0 : x_1 : x_2] = [y_0 : y_1 : y_2]$ if and only if $(x_0, x_1, x_2) = \lambda \cdot (y_0, y_1, y_2)$ for some $\lambda \in \mathbb{C}^*$.

An embedding of the affine plane is given by $\mathbb{C}^2 \to \mathbb{P}^2$,$(x_1, x_2) \to (1, x_1, x_2)$. The *line at infinity* of $\mathbb{C}^2$ is then $\mathbb{P}^2 \setminus \mathbb{C}^2 = \{x_0 = 0\}$. Instead of this line, any other line $L \subset \mathbb{P}^2$ can play the role of the line at infinity of the affine plane $\mathbb{P}^2 \setminus L$.

We can now give the definition of a projective plane curve.

Definition 2.14. A subset $\bar{C} \subset \mathbb{P}^2$ is called a *projective plane algebraic curve* if there exists a homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2]$ of degree ≥ 1 such that

$$\bar{C} = V(F) = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid F(x_0, x_1, x_2) = 0\}.$$

We extend now an affine curve C to a projective curve $\bar{C}$. If $f \in \mathbb{C}[x,y]$ the *homogenization* of f is $F \in \mathbb{C}[x_0,x_1,x_2]$ given by $F(x_0,x_1,x_2) = x_0^d \cdot f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$, where $d = \deg f$. Note that $f(x,y) = F(1,x,y)$. Clearly, if $f = f_0 + f_1 + \cdots + f_d$ then $F(x_0,x_1,x_2) = x_0^d \cdot f_0 + \cdots + x_0 \cdot f_{d-1} + f_d$.

If $C = V(f)$ is an affine curve and F is the homogenization of f, then $\bar{C} = V(F) \subset \mathbb{P}^2$ is called the *projective closure* of C. Clearly $C = \bar{C} \cap \mathbb{C}^2$.

Example 2.15. The cubic C with $f = x^3 - y^2$ has projective closure $\bar{C}$ given by $F = x_1^3 - x_0^2 x_2$. Then $\bar{C}$ has an inflection at $[1:0:0]$ and a cusp at $[0:0:1]$. This last point lies in the affine part $x_2 = 1$ with coordinates (x_0, x_1).

Remark 2.16. The projective closure $\bar{C}$ of C coincides with the topological closure of C in $\mathbb{P}^2$.

The notions introduced above for affine curves have analogues for projective curves: irreducibility, decomposition into components, degree, etc.

First, note that $f \in \mathbb{C}[x,y]$ is irreducible if and only if its homogenization $F \in \mathbb{C}[x_0,x_1,x_2]$ is irreducible. Then, the *degree* of the projective plane algebraic curve C is the degree of a minimal defining polynomial F in $\mathbb{C}[x_0,x_1,x_2]$. The *irreducible components* of C are then defined by the irreducible factors of a minimal F.

Remark 2.17. The group $\mathrm{PGL}_2(\mathbb{C})$ of projective transformations acts on $\mathbb{P}^2$. Since the homogeneous coordinates are transformed linearly, we can easily see that the degree and the irreducibility property are independent of projective transformations. Such numbers or properties are called *(projective) invariants.*

2.18. **Intersection numbers and Bézout's theorem.** The most important result of the elementary curve theory is Bézout's theorem, which gives the number of points of intersection of two algebraic curves.

We begin by treating the special case of the intersection of a curve and a line. Let $C = V(F) \subset \mathbb{P}^2$ be a curve of degree $d \geq 1$. For simplicity choose coordinates so that the line is given by $x_2 = 0$. The points in $C \cap L$ correspond to the zeroes of $G(t_0,t_1) = F(t_0,t_1,0)$. Write $F(x_0,x_1,x_2) = F_0 x_2^d + F_1 x_2^{d-1} + \cdots + F_d$, where $F_k \not\equiv 0$ are homogeneous of degree k in x_0, x_1. Then $G = F_d$.

If $F_d = 0$, then F is divisible by x_2, that is $L \subset C$. Otherwise $\deg G = d$, and by the fundamental theorem of algebra we have a factorization

$$G = (b_1 t_0 - a_1 t_1)^{k_1} \dots (b_m t_0 - a_m t_1)^{k_r},$$

where $[a_i : b_i] \in \mathbb{P}^1$ are uniquely determined, distinct points, and $k_i \in \mathbb{N}^*$ are independent on the choice of coordinates.

Thus we can define the *intersection multiplicity* of C and L to be

$$\mathrm{Int}_p(C,L) := k,$$

where $k = k_i$ for point $p = [a_i : b_i : 0]$ and $k = 0$ for all other p in $\mathbb{P}^2$. Since $k_1 + \cdots + k_m = d$, we obtain:

Proposition 2.19. *If $C \subset \mathbb{P}^2$ is a curve of degree $d \geq 1$ and L a line not contained in C, then the total number of points of intersection of C and L, counted with multiplicities, is d. For almost all lines $\#C \cap L = d$.*

Example 2.20. Let $C = V(x_0x_2^2 - x_1^3)$ be a cuspidal cubic and L be the line through $p = [1 : 0 : 0]$ with slope in the affine plane determined by $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$. A parametrization of L is given by $(t_0, t_1) \to (t_0, \lambda_0 t_1, \lambda_1 t_1) = \alpha(t)$. Then $F(\alpha(t)) = G(t) = t_1^2(\lambda_1^2 t_0 - \lambda_0^3 t_1)$. The factor t_1^2 describes the intersection point p, and the second a point $q = (\lambda_0^3 : \lambda_0\lambda_1^2 : \lambda_1^3)$. If the line is horizontal then $[\lambda_0 : \lambda_1] = [1 : 0]$, so $p = q$ and $\mathrm{Int}_p(C, L) = 3$. Otherwise $p \neq q$ and $\mathrm{Int}_p(C, L) = 2, \mathrm{Int}_p(C, L) = 1$. For $[\lambda_0 : \lambda_1] = [0 : 1]$, $q = [0 : 0 : 1]$ is an inflection point at infinity, but L is not the inflectional tangent. The latter is the line at infinity $L_\infty = V(x_0)$, and $\mathrm{Int}_q(C, L_\infty) = 3$.

Now we determine the number of intersection points of two algebraic curves $C_1 = V(F_1)$ and $C_2 = V(F_2)$ in $\mathbb{P}^2$.

Theorem 2.21. *If $C_1, C_2 \subset \mathbb{P}^2$ are algebraic curves with no common component, then the number of intersection points satisfies the inequality*

$$\#C_1 \cap C_2 \leq \deg C_1 \cdot \deg C_2.$$

Proof. We may assume that C_1, C_2 do not pass through $q = [0 : 0 : 1]$. For a point $x = [x_0 : x_1 : 0]$, let L_x denote the line through x and q. Expand in terms of x_2:

$$F_1 = a_0 x_2^m + a_1 x_2^{m-1} + \cdots + a_m,$$

$$F_2 = b_0 x_2^n + b_1 x_2^{n-1} + \cdots + b_n,$$

where $a_i, b_j \in \mathbb{C}[x_0, x_1]$ and $\deg a_i = i, \deg b_j = j$ if $a_i, b_j \not\equiv 0$. Since $q \notin C_1$ and $q \notin C_2$, we have $a_0, b_0 \not\equiv 0$. We take the resultant $R = R_{F_1,F_2}$. Then $R \in \mathbb{C}[x_0, x_1]$ is a homogeneous polynomial of degree mn. In the special case when $C_2 = V(x_2)$ is a line $R = \pm a_m$.

If C_1 and C_2 are algebraic curves with have no common component, then $R \not\equiv 0$ by Theorem 2.5. Moreover, $R(x_0, x_1) = 0$ if and only if C_1 and C_2 have an intersection point on L_x. For a fixed x there are only finitely many intersection points because otherwise L_x would be a common component of C_1 and C_2. Hence $C_1 \cap C_2$ is finite.

Now we count the intersection points. There are only finitely many lines connecting them. If coordinates are chosen so that q lies on none of these lines, then there is at most one point of intersection on each L_x. In other words, altogether there are at most as many as the zeroes of the resultant R. □

To turn the inequality above into an equality, we need to count the intersection points with their multiplicities. The hard part is to define the intersection multiplicity.

Definition 2.22. Let $C_1 = V(F_1)$ and $C_2 = V(F_2)$ be algebraic curves in $\mathbb{P}^2$ that have no common component. Moreover, suppose that they do not pass through the point $q = [0 : 0 : 1]$, and on each line through q there is at most one point of intersection of C_1 and C_2. Let $R \in \mathbb{C}[x_0, x_1]$ be the resultant of F_1 and F_2. If $p = [p_0 : p_1 : p_2] \in C_1 \cap C_2$ and $r = [p_0 : p_1]$ then

$$\mathrm{Int}_p(C_1, C_2) := \mathrm{mult}_r(R),$$

that is, the *intersection multiplicity* of C_1 and C_2 is the multiplicity of the zero of R at r.

Combining this definition with the arguments in the proof of Theorem 2.21 immediately gives the following theorem, which was discovered in 1765.

Theorem 2.23. *For $C_1, C_2 \subset \mathbb{P}^2$ algebraic curves with no common component,*

$$\sum_{p \in C_1 \cap C_2} \mathrm{Int}_p(C_1, C_2) = \deg C_1 \cdot \deg C_2.$$

We give a very simple example.

Example 2.24. If $C_1 = V(x_0 x_1^2 - x_2^3)$ and $C_2 = V(x_0 x_1^2 + x_2^3)$, then $R = 8x_0^3 x_1^6$. The two cubics intersect at $q = [0 : 1 : 0]$ with multiplicity 3 and at $p = [1 : 0 : 0]$ with multiplicity 6.

In fact the intersection multiplicity is a *local* invariant: it is determined by the behaviour of the two curves in an arbitrarly small neighborhood of the point of intersection.

2.25. **Tangents and singularities.** The local properties of a curve can be studied in an affine part of the projective plane.

Definition 2.26. Let $C = V(f) \subset \mathbb{C}^2$ be an algebraic curve with $f \in \mathbb{C}[x, y]$ minimal. We say that C is *smooth* at a point p if

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right) \neq (0, 0).$$

If p is not smooth, we say it is *singular*.

If C is smooth at p, the line

$$T_p C = \left\{(x, y) \in \mathbb{C}^2 \mid \frac{\partial f}{\partial x}(p) \cdot x + \frac{\partial f}{\partial y}(p) \cdot y = c\right\}$$

is called the *tangent line* to C at p. Here c is chosen so that $p \in T_p C$.

The set $\operatorname{Sing} C$ of the singular points of C is called the *singular locus* of C.

Proposition 2.27. *For an algebraic curve $C \subset \mathbb{C}^2$ the set $\operatorname{Sing} C$ is finite.*

Proof. If $C = V(f)$ set $C_x = V(\frac{\partial f}{\partial x})$ and $C_y = V(\frac{\partial f}{\partial y})$. Then $\operatorname{Sing} C = C \cap C_x \cap C_y$. We may assume that $\deg C = \deg f \geq 2$ since a line is smooth everywhere. This implies that $\deg \frac{\partial f}{\partial x} \geq 1$ say. Then C_x is an algebraic curve, and $\operatorname{Sing} C \subset C \cap C_x$. Thus it suffices to show that $C \cap C_x$ is finite. This follows from Bézout's theorem (for divisors) once we show C and C_x have no component in common.

Suppose that f and $\frac{\partial f}{\partial x}$ do have a common prime factor g. Then $f = g \cdot h$ and

$$\frac{\partial f}{\partial x} = g \cdot g_1 = h\frac{\partial g}{\partial x} + g\frac{\partial h}{\partial x}.$$

Hence g is also a divisor of $h \cdot \partial g/\partial x$. If $\partial g/\partial x \not\equiv 0$ then g is a divisor of h, so g^2 is a divisor of f. But this contradicts the minimality of f.

If $\partial g/\partial x \equiv 0$ then $g = y - a$. But the coordinates can be chosen from the outset so that C contains no lines parallel to the axes. □

If $\deg C = d$, it follows that C has at most $d(d-1)$ singular points. This estimate will be improved later.

To get the first measure of how bad a singularity is, we consider higher derivates of the defining polynomial.

Definition 2.28. Let $C = V(f) \subset \mathbb{C}^2$ with $f \in \mathbb{C}[x, y]$ minimal. The *multiplicity* of C at p is the order of the polynomial f at p:

$$\operatorname{mult}_p(C) := \operatorname{ord}_p(f),$$

where by definition $\operatorname{ord}_p(f) := \min\{k \mid f_k \not\equiv 0\}$ with f_k the homogeneous part of degree k of the Taylor expansion of f about p.

It is clear from the definition that:

(a) $0 \leq \operatorname{mult}_p(C) \leq \deg C$;
(b) $p \in C$ iff $\operatorname{mult}_p(C) > 0$;
(c) C is smooth at p iff $\operatorname{mult}_p(C) = 1$;
(d) C is singular at p iff $\operatorname{mult}_p(C) > 1$.

The extreme case $\operatorname{mult}_p(C) = \deg C = d$ occurs iff $f = f_d$. This means that C consists of d lines through p.

Example 2.29. The origin has multiplicity 3 in the case of the quartic $V((x^2 + y^2)^2 + 3x^2y - y^3)$ and multiplicity 4 in the case of the sextic $V((x^2 + y^2)^3 - 4x^2y^2)$. In these examples, the multiplicity counts just the local branches, but the next is very different. The quartic $V((x^2 + y^2)^2 - x^3)$ has multiplicity 3 at the origin, but a single branch there.

It is clear that possible intersections of two curves at a point depend on the multiplicities of the curves.

Proposition 2.30. *If $C \subset \mathbb{C}^2$ is an algebraic curve and L is a line through $p \in C$, then*

$$\text{mult}_p(C) \le \text{Int}_p(C, L),$$

and the inequality is strict for at most $\text{mult}_p(C)$ *lines through p.*

Proof. Let f be a minimal polynomial of $C = V(f)$ and let $f = f_r + \cdots + f_d$, where $r = \text{ord}_O(f)$ and $d = \deg f$, be its Taylor expansion at the origin O. If the line L is parametrized by $\alpha(t) = (\lambda_1 t, \lambda_2 t)$, then $g(t) = f(\alpha(t)) = \sum_{k=r}^{d} f_k(\lambda_1, \lambda_2)t^k$. By definition $\text{Int}_p(C, L) = \text{ord}_p(g)$. This is greater than r iff $f_r(\lambda_1, \lambda_2) = 0$. □

For convenience, set $\text{Int}_p(C, L) = \infty$ if $L \subset C$. This allows us to define tangents to singular points as well.

Definition 2.31. As above, let $p \in C \cap L$. The line L is called a *tangent* to C at p if

$$\text{mult}_p(C) < \text{Int}_p(C, L).$$

We call p an *ordinary r-fold* singular point if $r = \text{mult}_p(C)$ and there are r distinct tangents at p, i.e. the homogeneous polynomial f_r has r distinct zeroes.

Example 2.32. The origin is not an ordinary point for quartic $V((x^2 + y^2)^2 - x^3)$ and for the cuspidal cubic $V(x^3 - y^2)$. In these examples there is only one tangent which must be counted three times, respectively twice.

Definition 2.33. If $p \in C$ is a smooth point and T is the tangent at p, then $k := \text{Int}_p(C, T)$ is called the *order of contact*. We have that $2 \le k \le \infty$. If $k = 2$ then T is called a *simple tangent*, otherwise is called an *inflectional tangent*. In the latter case p is called an *inflection point*.

Calculations become much more complicated when we intersect two general curves. Nevertheless one can prove the following result.

Theorem 2.34. *Let $C_1, C_2 \subset \mathbb{C}^2$ be algebraic curves with no common component. Then*

$$\text{Int}_p(C_1, C_2) \ge \text{mult}_p(C_1) \cdot \text{mult}_p(C_2)$$

for $p \in C_1 \cap C_2$. Equality holds iff C_1 and C_2 do not have a common tangent at p.

Example 2.35. If $C = V(x_2^3 + x_0 x_1^2)$ and $C_n = V(x_2^n + x_0^{n-1} x_1)$, then $\text{Int}_p(C_1, C_2) = 3$ independent of n. The common tangent to C and C_n at $p = [1 : 0 : 0]$ is $T = V(x_1)$.

In order to apply the global statement of Bézout's theorem we have to projectivize.

Proposition 2.36. *Let $C = V(F) \subset \mathbb{P}^2$, where F is minimal, and let $p \in C$. Then*

(a) *C is smooth at p iff*
$$\left(\frac{\partial F}{\partial x_0}(p), \frac{\partial F}{\partial x_1}(p), \frac{\partial F}{\partial x_2}(p)\right) \neq (0,0,0);$$

(b) *If C is smooth at p, the projective line*
$$T_pC = \left\{\frac{\partial f}{\partial x_0}(p)\cdot x_0 + \frac{\partial f}{\partial x_1}(p)\cdot x_1 + \frac{\partial f}{\partial x_2}(p)\cdot x_2 = 0\right\}$$
is called the tangent line *to C at p.*

Proof. It follows from Euler's formula for F, where $C = V(F)$.
$$x_0\frac{\partial f}{\partial x_0} + x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} = (\deg F)\cdot F.$$
□

Corollary 2.37. *For an algebraic curve $C \subset \mathbb{P}^2$ the set $\operatorname{Sing} C$ is finite.*

Example 2.38. The Fermat curve $C = V(x_0^d + x_1^d - x_2^d)$ is irreducible and smooth for every $d \geq 1$.

2.39. **Polars and Hessian curves.** It is natural to ask the following questions for an algebraic curve $C \subset \mathbb{P}^2$: 1) How can we find tangents T to C passing through a prescribed point q? 2) How can we find the inflection points of C? In order to answer these questions is useful to introduce polar and Hessian curves.

Definition 2.40. Let $C = V(F)$ with F minimal of degree $d \geq 2$. Let $q = [q_0 : q_1 : q_2]$ be an arbitrary point in $\mathbb{P}^2$ and set $D_qF = q_0\frac{\partial F}{\partial x_0} + q_1\frac{\partial F}{\partial x_1} + q_2\frac{\partial F}{\partial x_2}$. If $\deg D_qF \geq 1$ then the curve $P_qC := V(D_qF)$ is called the *polar* of C with respect to the pole q.

Example 2.41. a) $F = x_1x_2$. Then $D_qF = q_1x_2 + q_2x_1$ and P_qC is a line: undefined for $q = [1:0:0]$, included in C for $q = [q_0 : 1 : 0]$. b) C smooth quadric. Then P_qC is a line. c) $C = \{x_2^3 + x_0x_1^2 = 0\}$. Then $D_qF = q_0x_1^2 + 2q_1x_0x_1 + 3q_2x_2^2$. C has a cuspidal tangent $T = V(x_1)$ at $p = [1:0:0]$. If $q \in T$ then P_qC consists of two lines through p and $\operatorname{Int}_p(C \cap P_q) = 4$. If $q_1 \neq 0, q_2 = 0$ then $P_qC = T \cup L$, where L is a line through $[0:0:1]$, and $\operatorname{Int}_p(C \cap P_q) = 3$. If $q_1 \neq 0, q_2 \neq 0$ then P_qC is a smooth quadric with tangent T at p, and $\operatorname{Int}_p(C \cap P_q) = 3$.

The polar curves enjoy a lot of useful propreties.

Proposition 2.42. *Let $C = V(F)$ and $d = \deg F$. We then have:*

(a) *P_qC is independent on the choice of coordinates in $\mathbb{P}^2$.*
(b) *$D_qF \equiv 0$ if and only if C is the union of d lines through q.*
(c) *If $D_qF \not\equiv 0$ then $\deg D_q = d - 1$.*

(d) *C and P_qC have a common component if and only if C contains a line through q.*
(e) *If $p \in \operatorname{Sing} C$ then $p \in P_qC$.*
(f) *Intersection $C \cap P_qC$ consists of the tangency points of tangents to C passing through q, together with $\operatorname{Sing} C$.*
(g) *Suppose C has a simple tangent T at p (i.e. $\operatorname{Int}_p(T, C) = 2$). If $q \in T$ is distinct from p, then C and P_qC intersect transversely at p, that is $\operatorname{Int}_p(C, P_qC) = 1$.*

Definition 2.43. For $F \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous of degree $d \geq 2$, the 3×3 matrix $H_F = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)$ is called the Hessian of F. If $C = V(F)$ with F minimal, and $\deg(\det H_F) \geq 1$ then $H(C) = V(\det H_F)$ is called the *Hessian curve* of C.

Example 2.44. a) $F = x_0^2 + x_1^2 + x_2^2$. Then $\det H = 8$ and $H(C) = \emptyset$. b) $F = x_1x_2(x_1 - x_2)$. Then $\det H = 0$ and $H(C) = \mathbb{P}^2$. c) $F = x_0x_1x_2$. Then $\det H = 2F$ and $H(C) = C$. d) $F = x_0(x_0^2 + x_1^2 + x_2^2)$. Then $C = L \cup C_1$, $\det H = 8x_0(3x_0^2 - x_1^2 - x_2^2)$, and $H(C) = L \cup C_2$, where C_1, C_2 are distinct quadrics and C and $H(C)$ have the line $L = V(x_0)$ in common.

It is useful to have alternate formulae for $\det H_F$:

Lemma 2.45. *With notations $F_i = \frac{\partial F}{\partial x_i}, F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$ we have*

$$\det H_F = \frac{d-1}{x_0} \det \begin{pmatrix} F_0 & F_1 & F_2 \\ F_{01} & F_{11} & F_{21} \\ F_{02} & F_{12} & F_{22} \end{pmatrix} = \frac{d-1}{x_0^2} \det \begin{pmatrix} dF & F_1 & F_2 \\ (d-1)F_1 & F_{11} & F_{21} \\ (d-1)F_2 & F_{12} & F_{22} \end{pmatrix}$$

Proof. It follows from Euler's formula applied to F_i, where 0^{th} row is multiplied by x_0, while for $j = 1, 2$ the j^{th} row is multiplied by x_j and added to it. Similar procedure on columns gives the second equality. □

It is readily seen the following

Proposition 2.46. (a) *$H(C)$ is independent on coordinates.*
(b) $\deg(\det H_F) = 3(d-2)$ *if* $\det H_F \not\equiv 0$.
(c) $\operatorname{Sing} C \subset H(C)$.

The crucial property of Hessian curves is their relationship with inflection points.

Theorem 2.47. *Let $C = V(F) \in \mathbb{P}^2$ containing no line. Then:*
(a) $\det H_F \not\equiv 0$.
(b) *$p \in C \setminus \operatorname{Sing} C$ is an inflection point if and only if $p \in H(C)$.*
(c) *C and $H(C)$ have no common component.*
(d) *If $p \in C$ is a simple inflection point then* $\operatorname{Int}_p(C, H(C)) = 1$.

In order to prove Theorem 2.47 we need the following lemma, which is an easy consequence of Taylor's expansion.

Lemma 2.48. *Let $C = V(f) \in \mathbb{C}^2$ smooth at $p = (0,0)$ with $T = V(x_2)$. If $k = \mathrm{Int}_p(C,T) < \infty$ then $f(x_1,x_2) = x_1^k g(x_1) + x_2 h(x_1,x_2)$, where $g(0) \neq 0, h(0,0) \neq 0$.*

Proof. Theorem 2.47. Let $p = [1:0:0] \in C$ smooth point with tangent $T = V(x_2)$. By Lemma 2.48 we can write $F(1,x_1,x_2)$ in the form $f = x_1^k g + x_2 h$, where $x_1^k g = a_2 x_1^2 + a_3 x_1^3 + \dots$, $h = b + b_1 x_1 + b_2 x_2 + \dots$. Here $2 \leq k < \infty, b \neq 0$ and $a_2 = 0$ if and only if $k \geq 3$, i.e. p is inflection point. Using Lemma 2.45 we find

$$\det H_F(p) = (d-1)^2 \det \begin{pmatrix} 0 & 0 & b \\ 0 & 2a_2 & b_1 \\ b & b_1 & 2b_2 \end{pmatrix} = -2(d-1)^2 b^2 a_2.$$

This gives (b). To prove (a) is enough to show that there exists a smooth point which is not an inflection point. That follows from the implicit function theorem.

To prove (c) we have to track down lowest order terms in the Taylor expansion of $\det H_F$ about p. We obtain: $\det H_F = x_1^{k-2}\tilde{G}(x_0,x_1) + x_2 \tilde{H}(x_0,x_1,x_2)$, where $\tilde{G}(1,0) \neq 0$. This implies (c) because it follows from the inequality common component at a smooth point.

For $k = 3$, $H(C)$ is smooth at p, with tangent distinct from T. This gives (d). □

The situation when C and $H(C)$ have a component in common is also clear:

Proposition 2.49. *A curve $C \subset \mathbb{P}^2$ is included in its Hessian $H(C)$ if and only if C is a union of lines. More precisely: (a) If C contains a line L, then $L \subset H(C)$. (b) If D is a common irreducible component of C and $H(C)$, then D is a line.*

2.50. **Applications of Bézout's theorem.** It is natural to ask the following questions for a curve $C \subset \mathbb{P}^2$ of degree d: How many singular points C can have? How many tangents to C pass through a given point q? How many inflection points C can have?

In order to estimate the number of singularities we use a construction of algebraic curves through prescribed points. Foe example, exactly one line passes through two points, exactly one quadric through five non-collinear points.

Algebraic plane curves of degree d are parametrized by the projective space $\mathbb{P}^N$, where $N = \binom{d+2}{2} - 1 = \frac{d(d+3)}{2}$. Every point in $\mathbb{P}^2$ through which the curve C is to pass determines a hyperplane in $\mathbb{P}^N$. The intersection of N hyperplanes contains at least one point in $\mathbb{P}^N$. Points $p_1, \dots, p_N$ are said to be in *general position* (with respect to curves of degree d) if the hyperplanes they determine in $\mathbb{P}^N$ intersect in exactly one point. We thus have the following lemma.

Lemma 2.51. *Through $\frac{d(d+3)}{2}$ points in $\mathbb{P}^2$ there passes at least one algebraic curve of degree $\leq d$.*

In the plane d lines can intersect in $d(d-1)/2$ points, thus a degree d curve can have $d(d-1)/2$ singularities. There is in fact a better bound for irreducible curves.

Theorem 2.52. *An irreducible algebraic curve $C \subset \mathbb{P}^2$ of degree d has at most $(d-1)(d-2)/2$ singular points.*

Proof. We may assume $d \geq 3$. Suppose there were $s;1$ singularities on C, where $s = (d-1)(d-2)/2$. We adjoin $d-3$ more points to $\operatorname{Sing} C$, so altogether there are $s+1+d-3 = (d+1)(d-2)/2$ points. Now there is a curve D of degree $m \leq d-2$ that passes through all these points. For each singular point p of C we have $\operatorname{Int}_p(C,D) \geq 2$ and for each of the additional $d-3$ points $\operatorname{Int}_p(C,D) \geq 1$, so

$$\sum_{p \in C \cap D} \operatorname{Int}_p(C,D) \geq 2(s+1)+d-3 = d(d-2)+1.$$

Now C is irreducible and $\deg D < d$, so C cannot be a component of D. Bézout's theorem gives

$$\sum_{p \in C \cap D} \operatorname{Int}_p(C,D) = d \cdot m \leq d(d-2),$$

contradicting the previous inequality. □

For a more accurate count, we can assign weights to the singularities and show that

$$\sum_{p \in \operatorname{Sing} C} \operatorname{mult}_p(C)(\operatorname{mult}_p(C)-1) =\leq (d-1)(d-2)$$

By induction on the number of components we have the following.

Corollary 2.53. *An arbitrary algebraic curve $C \subset \mathbb{P}^2$ of degree d has at most $d(d-1)/2$ singular points.*

Remark 2.54. It is very natural to ask now whether, for each n there exist irreducible curves of degree d that have the maximal number of singularities. Severi showed that there even exist irreducible curves in $\mathbb{P}^2$ with exactly n ordinary points for $0 \leq n \leq (d-1)(d-2)/2$ and no other singularities.

Remark 2.55. If we apply Bézout's theorem to a curve containing no lines C and to its polar P_qC, we can easily conclude from Proposition 2.42 that there are at most $d(d-1)$ tangents to C through q. We shall see later that this number is reduced by bitangents, inflection points and singularities of C.

Remark 2.56. If we apply Bézout's theorem to a curve containing no lines C and to its Hessian $H(C)$, we can easily conclude from Theorem 2.47 that there are at most $3d(d-2)$ inflection points on C. The maximum number of points is attained only if C is smooth and all its inflection points are simple.

Example 2.57. $F = x_0^3+x_1^3+x_2^3$. Then the Fermat cubic $C = V(F)$ is smooth and all its 9 inflection points are simple. We have $\det H_F = 6^3 x_0 x_1 x_2$ so $H(C)$ splits into 3 lines. On $x_0 = 0$ there are 3 inflection points: $[0 : 1 : -1], [0 : \zeta : -1], [0 : \zeta^2 : -1]$, where ζ is a primitive cube root of unity. There are also 3 inflection points on each of the lines $x_1 = 0$ ($[1 : 0 : -1]$, etc.) and $x_2 = 0$ ($[1 : -1 : 0]$, etc.). The 3 real inflection points lie on the line $x_0 + x_1 + x_2 = 0$. In general, every smooth cubic has exactly 3 real inflection points which always lie on a line.

3. Puiseux Series

In this section we will show how to solve the polynomial equation $f(x, y) = 0$, for y as a series in fractional powers of x, in the neigborhood of the origin $O = (0, 0)$. It makes no difference if we allow f to be a formal power series. The proof goes back to Newton.

Let $f(x, y) = \sum a_{r,s}x^r y^s$. Consider a real plane $\mathbb{R}^2$ with coordinates (r, s) and mark those points for which the coefficient $a_{r,s}$ is non-zero. The *Newton diagram* of f is the convex hull of the regions above and to the right of the marked points. Its boundary is made up of straight line segments. The union of those segments which do not lie on the coordinate axes is the *Newton polygon* N_f of f.

We say that $f \in \mathbb{C}[[x, y]]$ is *general of order* m in y if $f(0, y) = y^m \cdot h(y)$ such that $h(0) \neq 0$.

Theorem 3.1. *Any equation $f(x, y) = 0$, where f is a polynomial with $f(O) = 0$, or more generally $f \in \mathbb{C}[[x, y]]$ with zero constant term, and general in y of order $m \geq 1$, admits at least one solution in formal power series of the form:*

$$x = t^n, \quad y = \phi(t) = a_1 t + a_2 t^2 + \dots,$$

for some $n \in \mathbb{N}$.

Proof. We try writing $y = c_0 x^\alpha +$ terms of higher order, for some $\alpha \in \mathbb{Q}$, and then substitute for y in $f(x, y)$. Each term $a_{r,s}x^r y^s$ contributes $a_{r,s}c_0^s x^{r+s\alpha}$. When we add these, the terms of least order $r + s\alpha$ in x will have to cancel.

In order for that to happen we need to choose a line $r + s\alpha = e$ lying along an edge of N_f. Write the endpoints of this edge as (r_0, s_0) and $(r_0 + ka, s_0 - kb)$, where a, b and k are positive and a and b are coprime, so that the integral points on the edge are the $(r_0 + la, s_0 - lb)$ with $0 \leq l \leq k$. When the substitution is made, no powers of x lower than x^e appear, and the coefficient of x^e is

$$\sum_{l=0}^{k} a_{r_0+la, s_0-lb} c_0^{s_0-lb}.$$

If we set $\phi(T) = \sum_{l=0}^{k} a_{r_0+la,s_0-lb} T^{k-l}$, then the coefficient of x^e may be written as $c_0^{s_0-kb}\phi(c_0^b)$.

We start now the constructive part of the proof. First we check whether f is divisible by x: if so, then one solution is parametrised by $(x, y) = (0, t)$. From now on we will assume that f is not divisible by x. It follows that N_f intersects the s-axis in a point determined by the order of $f(0, y)$, which we denote by m_0.

Now choose an edge of N_f, and a solution T_0 of the equation $\phi(T) = 0$, and set $c_0 = T_0^{1/b}$. Since ϕ has a non-zero constant term, T_0 and c_0 cannot be zero. We take $y = c_0 x^{a/b}$ as a first approximation to a solution of $f(x, y) = 0$.

Set $x = x_1^b, y = x_1^a(c_0 + y_1)$, and substitute in $f(x, y) = 0$. By the above choices, $f(x_1^b, x_1^a(c_0 + y_1))$ is divisible by $x_1^{br_0+as_0}$. Divide by this to get $f_1(x_1, y_1)$, say. Now repeat the above procedure, replacing f, x, y by f_1, x_1, y_1. Write m_1 for the order of $f_1(0, y_1)$. Note that $f_1(0, y_1)$ is in fact given by $(c_0 + y_1)^{s_0-kb}\phi((c_0 + y_1)^b)$. Since ϕ has degree k, we deduce that $m_1 \leq kb \leq m_0$.

Iterating this procedure yields after r steps an expression of the form

$$Y_r := x^{\frac{a_0}{b_0}}(c_0 + x^{\frac{a_1}{b_0 b_1}}(c_1 + x^{\frac{a_2}{b_0 b_1 b_2}}(c_2 + \ldots (c_r + x^{\frac{a_r}{b_0 b_1 \cdots b_r}}) \ldots))),$$

which multiplies out to an expression in incresing fractional powers of x.

Observe that Y_r differs from Y_{r-1} only in coefficients of $x^{\frac{a_r}{b_0 b_1 \cdots b_r}}$ and higher terms. Also, if the lowest non-vanishing coefficient in $f(x, Y_{r-1})$ is the coefficient of $x^{l_{r-1}}$, then since the procedure amounts to killing this coefficient, we have $l_r > l_{r-1}$.

It remains to prove the following claim: The denominators $b_0 b_1 \ldots b_r$ are bounded, or equivalently, that $b_r = 1$ for all large enough r. To that end we will show that

Lemma 3.2. *If $b_r > 1$ then $m_r > m_{r+1}$.*

The claim will follow now from the previous lemma, since m_r is a monotone decreasing sequence of positive integers.

Then all the y_r are power series in $x_d = x^{1/d}$ for some fixed d and since they agree up to increasingly high powers of x_d they converge to a power series y_∞. The numbers dl_r are also integers, but strictly increasing and so they got to ∞. Thus $f(x, y_\infty) = 0$ and we have an exact solution. □

Example 3.3. Equation $f = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = 0$ leads to N_f consisting of the segment connecting $(0, 4)$ to $(6, 0)$. Substituting $x = x_1^2, y = x_1^3(1 + y_1)$ we obtain $f_1 = y_1^4 + 4y_1^3 + 4y_1^2 - 4y_1x_1 - 4x_1 - x_1^2 = 0$. Then N_{f_1} is the segment $(0, 2), (1, 0)$. The substitution $x_1 = x_2^2, y_1 = x_2(1 + y_2)$ gives $f_2 = y_2 = 0$. Thus $y_2 = 0$ and $y_1 = x_2$ and a solution writes $x = x_2, y = x_2^6 + x_2^7$.

Theorem 3.1 only gives us a formal solution to $f(x, y) = 0$. In fact if f is a convergent power series in $\mathbb{C}\{x, y\}$ then ϕ is also a convergent power series in $\mathbb{C}\{t\}$. The proof of this fact uses the properties of the ring $\mathbb{C}\{x, y\}$. Most importantly that $\mathbb{C}\{x, y\}$ is an UFD.

First of all note that $\mathbb{C}\{x, y\}$ is a subring of the ring of formal power series $\mathbb{C}[[x, y]]$. If $f \not\equiv 0$ in $\mathbb{C}[[x, y]]$ then the *order* of f is $\operatorname{ord} f = \min\{d \mid f_d \not\equiv 0\}$, where f_d is the degree d homogeneous part of f; if $f \equiv 0$ then $\operatorname{ord} f = \infty$. Now f is a unit in $\mathbb{C}[[x, y]]$ if and only if $\operatorname{ord} f = 0$. Furthermore, if f is a unit in $\mathbb{C}\{x, y\}$, then f is also a unit in $\mathbb{C}[[x, y]]$. In particular, f is a unit in $\mathbb{C}\{x, y\}$ if and only if $f(O) \neq 0$.

Definition 3.4. An element $f \in \mathbb{C}\{x\}[y]$ of the form $f = y^k + a_1(x)y^{k-1} + \cdots + a_k(x)$ such that $a_1(0) = \cdots = a_k(0) = 0$ is called *Weierstrass polynomial.*

Theorem 3.5 (Weierstrass preparation theorem)**.** *Let $g \in \mathbb{C}\{x, y\}$ be general in y of order k. Then there is exactly one representation $g = u \cdot P$ with u a unit in $\mathbb{C}\{x, y\}$ and P is a Weierstrass polynomial in $\mathbb{C}\{x\}[y]$ of degree k.*

Thus, in some neighborhood of the origin, the zeroes of g are the same as those of P. Note that $\operatorname{ord} g \leq k$.

An important special case of the preparation theorem occurs when f has order $k = 1$ in y. In fact this is nothing but the implicit function theorem in the context of convergent power series. Thus the Weierstrass preparation theorem can be viewed as a generalization of the implicit function theorem.

Theorem 3.6 (Implicit function theorem)**.** *Let $\in \mathbb{C}\{x, y\}$ such that $f(0) = 0$ and $\frac{\partial f}{\partial y}(0) \neq 0$. Then there is exactly one series $\phi \in \mathbb{C}\{x\}$ such that $\phi(0) = 0$ and $f(x, \phi(x)) = 0$.*

To prove the preparation theorem, one uses the following important result.

Theorem 3.7 (Weierstrass division theorem)**.** *Let $f, g \in \mathbb{C}\{x, y\}$ and let g be general in y of order k. Then there exist $q \in \mathbb{C}\{x, y\}$ and $r \in \mathbb{C}\{x\}[y]$ such that the degree of R in y is less than k and $f = q \cdot g + r$.*

The preparation theorem is then used to prove that $\mathbb{C}\{x, y\}$ is a unique factorization domain which is moreover Noetherian.

Definition 3.8. Let $f \in \mathbb{C}\{x, y\}$ minimal and $p \in C = V(f)$. The *local branches* of C at p are the germs of the sets $V(f_1), \ldots, V(f_r)$, where, after a change of coordinates that brings p to the origin, f factors out in $\mathbb{C}\{x, y\}$ into the prime factors $f_1, \ldots, f_r$.

The Puiseux series of Theorem 3.1 gives a parametrization of the special form $t \to (t^n, \phi(t))$. We will show that any parametrization can be brought to this form after a transformation of the parameter t.

Corollary 3.9. *Let (ψ_1, ψ_2) be a formal parametrization of f with $\operatorname{ord}(\psi_1)) = k$ finite. Then there exists a series $\beta \in \mathbb{C}[[t]]$ such that $\operatorname{ord}(\beta) = 1$ and $\psi_1(\beta(t)) = t^k$. Thus a new formal parametrization of f is given by $(t^k, \phi(t))$, where $\phi(t) = \psi_1(\beta(t))$.*

We now state the geometric form of Puiseux's theorem.

Theorem 3.10. *Let $f(x, y) = y^k + a_1(x)y^{k-1} + \cdots + a_k(x) \in \mathbb{C}\{x\}[y]$ be an irreducible Weierstrass polynomial, where $k \geq 1$. Let ρ be chosen so that:*

(a) *$a_1, \ldots, a_k$ converge in $U = \{x \in \mathbb{C} \mid |x| < \rho\}$,*

(b) *the discriminant $D_f(x) \not\equiv 0$ in $U^* = U \setminus \{0\}$.*

Furthermore let $V = \{t \in \mathbb{C} \mid |t| < \rho^{1/k}\}$ and $C = (x, y) \in U \times \mathbb{C} \mid f(x, y) = 0$.

Then there exists $\phi \in \mathbb{C}\{t\}$ convergent in V such that:

(i) $f(t^k, \psi(t)) = 0$ *for all* $t \in V$;
(ii) $\Phi : V \to C, t \to (t^k, \psi(t))$ *is a bijection.*

The theorem is a consequence of the following lemmas. Denote by π the projection $U \times \mathbb{C} \to U$ on the second factor, and let $p : V \to U$ be the composition of $\pi\Phi$.

Lemma 3.11. *Let* $C^* = C \setminus \{(0,0)\} \subset U^* \times \mathbb{C}$*. Then the restriction* $\pi^* : C^* \to U^*$ *of* π *is a covering map.*

Lemma 3.12. C^* *is connected.*

Lemma 3.13. *Let* $V^* = V \setminus \{0\}$*. Then there exists a homeomorphism* $\Phi^* : V^* \to C^*$ *such that the restriction* p^* *is the composition* $\pi^*\Phi^*$*.*

Lemma 3.14. *The continuos map* $\Phi^* : V^* \to C^*$ *is holomorphic and setting* $\Phi(0) = (0,0)$ *gives a holomorphic extension* $\Phi : V \to C$ *of the form* $t \to (t^k, \phi(t))$*.*

The mappings p^* and π^* are both k-fold coverings of U^*. If we extend to U then the origin becomes a branching point of order k.

Proposition 3.15. *Let* $\Phi(t) = (t^k, \phi(t))$ *be a convergent solution and let* $\zeta = exp(2\pi i/k) \in \mathbb{C}$ *be a primitive root of unity of order* k*. For* $1 \le \nu \le k$ *let* $\Phi_\nu = \phi(\zeta^\nu t)$ *and* $\Phi_\nu(t) = (t^k, \phi_\nu(t)$*. Then* $\Phi_1, \dots, \Phi_k$ *are distinct parametrizations of* C*.*

Geometrically the maps Φ_ν differ from each other by permutations of the sheets of the covering $V^* \to C^*$, where the roots of unity act as covering transformations.

The parametrizations ϕ_ν can be used to extend the factorization $f(x,y) = (y - c_1) \cdot \dots \cdot (y - c_k)$ with $c_i \in \mathbb{C}$, which exist for every $x \in U^*$, to all of U.

Corollary 3.16. *Let* $f \in \mathbb{C}\{x\}[y]$ *be an irreducible Weierstrass polynomial of degree* $k \ge 1$ *and* $t \to (t^k, \phi(t))$ *a parametrization as in Theorem 3.10. Then in* $\mathbb{C}\{t\}[y]$ *we have*

$$f(t^k, y) = (y - \phi_1(t)) \cdot \dots \cdot (y - \phi_k(t)).$$

We can now prove the convergency of the formal Puiseux parametrization in the case when f is convergent.

Proof. Let $f \in \mathbb{C}\{x, y\}$ be the given series general in y and $\phi \in \mathbb{C}[[s]]$ be obtained by the formal construction. By the Weierstrass Preparation Theorem $f = uP_1 \cdots P_r$ for some unit u and irreducible Weierstrass polynomial $P_1, \dots, P_r$. Then since $u(s^n, \phi(s)) \ne 0$ we get $P_j(s^n, \phi(s)) = 0$ for some j. Hence we may assume that f is an irreducible Weierstrass polynomial of degree $m \ge 1$ and ϕ a formal solution.

To compare ϕ with the factorization of f from Corollary 3.16 we have to match the two parameters: $x = t^m = s^n$. In the formal construction there is no problem to achieve that by increasing the denominator n. We then may assume that $n = ml$

and hence $t = s^l$. Thus $f(s^n, y) = (y - \phi_1(s^l)) \cdot \cdots \cdot (y - \phi_m(s^l))$ in $\mathbb{C}\{s\}[y]$. Since ϕ is also a zero of $f(s^n, y)$ we must then have in the integral domain $\mathbb{C}[[s]]$ that $\phi(s) = \phi_\nu(s^l)$ for some ν. Hence ϕ itself is convergent. □

The previous proof implies the following result which relates the order m with the integer n from the statement of Theorem 3.1.

Corollary 3.17. *If $f \in \mathbb{C}\{x, y\}$ is an irreducible series general in y of order m then there exists $\phi \in \mathbb{C}\{t\}$ such that $f(t^m, \phi(t)) = 0$ in $\mathbb{C}\{t\}$.*

To be able to compute with Puiseux series that have arbitrary denominators in the exponent, we need an extension of the ring $\mathbb{C}[[x]]$: For every $n \geq 1$, regard the monomorphism $\mathbb{C}[[x]] \to \mathbb{C}[[t]]$, $x \to t^n$ as a ring extension $\mathbb{C}[[x]] \subset \mathbb{C}[[t]] = \mathbb{C}[[x^{\frac{1}{n}}]]$. Now form $\mathbb{C}[[x^*]] := \Cup_{n=1}^{\infty} \mathbb{C}[[x^{\frac{1}{n}}]]$. This is an integral domain containing all formal Puiseux series. For a fixed $\phi \in \mathbb{C}[[x^*]]$, there is an $n \in \mathbb{N}$ such that $\phi \in \mathbb{C}[[x^{\frac{1}{n}}]]$. Hence

$$\phi = \sum_{m=0}^{\infty} a_m x^{\frac{m}{n}}, \text{ where } a_m \in \mathbb{C},$$

and we can define the rational number

$$\operatorname{ord} \phi := \min \left\{ \frac{m}{n} \mid a_m \neq 0 \right\} \geq 0$$

to be the *order* of ϕ.

The importance of the ring $\mathbb{C}[[x^*]]$ can be seen in the following lemma.

Lemma 3.18. *Every normalized polynomial in $\mathbb{C}\{x\}[y]$ splits into linear factors in $\mathbb{C}[[x^*]]$.*

This fact is the key step in proving that the quotient field of $\mathbb{C}[[x^*]]$ is algebraically closed.

From Lemma 3.18 and Corollary 3.16 it follows that

Corollary 3.19. *If $f \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial of degree k, there is a factorization $f = (y - \phi_1) \cdot \cdots \cdot (y - \phi_k)$, where $\phi_i \in \mathbb{C}[[x^*]]$ and $\operatorname{ord} \phi_i > 0$ for $1 \leq i \leq k$. If f is irreducible then all ϕ have the same order.*

3.20. **Equation from a parametrisation.** We have seen above that a parametrisation determines a unique branch; conversely each branch has an essentially unique good parametrisation.

Lemma 3.21. *Given a good parametrisation for a branch B, we can write down an irreducible equation such that a point (near enough to O) satisfies the equation if and only if it is given by the parametrisation.*

Proof. Let the parametrisation be $y = \sum_1^\infty a_r t^r$, where $x = t^m$. Collect all terms in the expansion where r lies in a single congruence class modulo m, setting $r = mq+s$:

$$y = \sum_{s=0}^{m-1} t^s \left(\sum_{q=0}^{\infty} a_{mq+s} t^{mq} \right),$$

and define $\lambda_s(x) = \sum_{q=0}^\infty a_{mq+s} x^q$. These are convergent power series.

We may now regard the equations:

$$t^a y = \sum_{s=0}^{m-a-1} t^{a+s} \lambda_s(x) + \sum_{s=m-a}^{m-1} t^{a+s-m} x \lambda_s(x),$$

for $0 \le a < m$, as a system of linear equations in unkowns t^a with coefficients in $\mathbb{C}\{x, y\}$. Since the values t^a provide non-zero solutions to thse equations the determinant $D(x, y)$ of the system vanishes, and we can take $D = 0$ as the equation for the branch. □

3.22. **The Newton polygon.** We will explore here the connection between the geometric properties of the Newton polygon N_f and the algebraic properties of f.

In general N_f consists of finitely many segments with negative rational slopes. But it can also degenerate to a single point, for instance when f is a unit.

The most basic result is the following connection between slope and order.

Proposition 3.23. *Let $f \in \mathbb{C}[[x, y]]$ and $\phi \in \mathbb{C}[[x^*]]$, with $\rho := \operatorname{ord} \phi > 0$ and $f(x, \phi) = 0$. Then N_f contains a segment with slope $-1/\rho$.*

Thus the orders of all possible Puiseux parametrizations of f can be read off from N_f. For example, if the curve germ $V(f)$ has a smooth branch then N_f must have a segment with slope $-1/m$ or $-n$, where $m, n \in \mathbb{N}$. If the tangent to the branch is not one of the coordinate axes, then the slope of the segment is -1.

For simplicity assume that $f \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial of degree $k > 0$ that does not have y as a divisor. We factor f as in Lemma 3.18, and arrange the roots of f in $\mathbb{C}[[x^*]]$ according to their orders:

$$f = f_1 \cdot \dots \cdot f_l, f_j = (y - \phi_{j,1}) \cdot \dots \cdot (y - \phi_{j,k_j}), 1 \le j \le l, \tag{3.1}$$

where $\rho_j = \operatorname{ord} \phi_{j,i}$ for $1 \le i \le k_j$, and $\rho_1 > \rho_2 > \dots > \rho_l > 0$.

Then $k = k_1 + \dots + k_l$. If g is an irreducible factor of f then g divides exactly one f_j because all the roots of g have the same order by Corollary 3.19. Thus the factorization in (3.1) is cruder in general than the factorization of f into irreducible factors in $\mathbb{C}\{x, y\}$. Nevertheless the numbers k_j and ρ_j can be read off from the Newton polygon.

Theorem 3.24. *Let $f \in \mathbb{C}\{x\}[y]$ be a Weierstrass polynomial that does not have y as a divisor and factored as in (3.1). Then the Newton polygon N_f consists of segments $S_1, \dots, S_l$ of slopes $-1/\rho_j$ and heights k_j.*

Corollary 3.25. *The Newton polygon of an irreducible Weierstrass polynomial consists of a single segment.*

Hence reducibility can often be seen immediately, but not irreducibility.

Corollary 3.26. *Let $f \in \mathbb{C}\{x\}[y]$ be a Weierstrass polynomial and let $-1/\rho$ be a slope of a segment in N_f. Then there exists a convergent $\phi \in \mathbb{C}[[x^*]]$ such that $f(x, \phi(x)) = 0$ and $\operatorname{ord} \phi = \rho$.*

Thus the germ $C = V(f)$ has at least as many branches as N_f has segments.

Corollary 3.27. *If N_f of a Weierstrass polynomial f has a segment of height 1, then the germ $V(f)$ has a smooth branch.*

3.28. **Intersection multiplicities of germs.** We give here another definition of intersection multiplicity through parametrizations.

Definition 3.29. Let C_1 and C_2 be germs of curves, let $C_1 = V(f)$ with minimal $f \in \mathbb{C}\{x, y\}$, and let C_2 be irreducible with good parametrization $\Phi = (\phi_1, \phi_2)$. Then the intersection multiplicity is defined by

$$\operatorname{Int}(C_1, C_2) := \operatorname{ord} f(\phi_1(t), \phi_2(t)).$$

The definition is independent of a linear change of coordinates, the equation of f and the good parametrization of C_2.

By extending linearly, we can define $\operatorname{Int}(C_1, C_2)$ for arbitrary germ curves.

It is natural to compare the intersection multiplicity of algebraic curves with the intersection multiplicity of germs.

Let $C = V(F)$ and $D = V(G)$ two algebraic curves in $\mathbb{P}^2$, and let $p \in C \cap D$. Consider the germs C_p and D_p at p. The expected result turns out to be true.

Theorem 3.30. *With the notation above,*

$$\operatorname{Int}_p(C, D) = \operatorname{Int}(C_p, D_p).$$

Through this equality Bézout's theorem becomes indeed a local-global principle.

4. Resolution of singularities

The main topic of this section is that a curve singularity may be resolved by succesive blowings up. We develop some numerical invariants of a single branch, and we introduce infinitely near points and the dual graph of the resolution.

4.1. **Blowing up.**

Definition 4.2. Let p be a point on a smooth algebraic surface S. The *blowing up of S with centre p* is a new surface T and a map $\sigma : T \to S$ such that $\sigma^{-1}(p) = E$ is a curve, the restriction $\sigma : T \setminus E \to S \setminus \{p\}$ is an isomorphism, and the points on E correspond to different directions in S at p.

We first carry on the construction for $S = \mathbb{C}^2$ with coordinates (x, y) and $p = O$.

Let $\mathbb{P}^1$ be the projective line with coordinates $[u_0 : u_1]$. Define $T = T_1$ to be the subspace of points in $\mathbb{C}^2 \times \mathbb{P}^1$ satisfying the equation $xu_1 = yu_0$. The projection to $\mathbb{C}^2$ defines a map $\pi : T \to \mathbb{C}^2$. Any point $(x, y) \neq O$ determines a unique $[u_0 : u_1] = [x : y]$, hence a unique point $\pi^{-1}(x, y)$, while corresponding to O we have the entire line $\mathbb{P}^1$, so that $\pi^{-1}(O)$ is a curve E isomorphic to $\mathbb{P}^1$, and called the *exceptional curve* of the blow up.

For a general surface S we introduce coordinates (x, y) in a neighbourhood of a smooth point p. These define a biholomorphism between a neighbourhood U of p in S and a neighbourhood V of O in $\mathbb{C}^2$. Now the blowing up T is obtained by gluing $\pi^{-1}(V)$ to $S \setminus \{p\}$ using the equivalence of $U \setminus \{p\}$ and $\pi^{-1}(V \setminus O)$ via $V \setminus O$.

We will need to blow up with centre a point on $T = T_1$ to get a new surface T_2, then blow up with centre a point on T_2, and so on. However, all calculations will be performed by taking local coordinates in all these surfaces. The existence of these coordinates shows that the blow up of $\mathbb{P}^2$ is another nonsingular surface.

Recall that $\mathbb{P}^1$ is the union of two affine charts $U_i = \{u_i \neq 0\}$, for $i = 0, 1$, where u_1/u_0 and, respectively, u_0/u_1 can be taken as coordinate. On the part of T where $u_0 \neq 0$ write Y for u_1/u_0, and so the equation $xu_1 = yu_0$ simplifies to $y = xY$, showing that this part of T can be identified with $\mathbb{C}^2$ having coordinates (x, Y). Similarly, we can identify the part of T where $u_1 \neq 0$ with $\mathbb{C}^2$ having coordinates (X, y), where $X = u_0/u_1$. For calculations purposes, we simply introduce $Y = y/x$, or $X = x/y$, as a new coordinate.

Note in particular that the preimage E of O is isomorphic to $\mathbb{P}^1$, and in the first chart is given by $x = 0$ (with coordinate Y) and in the second by $y = 0$ (with coordinate X).

The blow up construction, although expressed in local coordinates, is independent on them.

Lemma 4.3. *The result of blowing up a surface S with centre a point p is intrinsically well-defined.*

Let $\sigma : T \to S$ be the blowing up with centre $p \in S$, and $E = \sigma^{-1}(p)$ the exceptional curve. A curve C in S not passing through p, corresponds to the unique curve $\sigma^{-1}(C)$ in T. If C is a curve through p, then $\sigma^{-1}(C)$ is called its *total transform*. This contains the exceptional curve E. The closure of $\sigma^{-1}(C) \setminus E$ is called the *strict transform* of C.

Suppose B is a branch of the germ at O of a holomorphic curve in $\mathbb{C}^2$. If $x = 0$ is not tangent to B, we have a good parametrization $x = t^m, y = \sum_{r=m}^{\infty} t^r$. Then a parametrization of the strict transform of B under the blowing up of $\mathbb{C}^2$ with centre O can be obtained by setting $Y = \sum_{r=m}^{\infty} t^{r-m}$. This intersects E at the point given by $t = 0$, where $Y = a_m$. Note that this intersection is a single point, and it corresponds to the tangent to the branch B. A neighbourhood of this point is contained in the chart with coordinates (x, Y).

4.4. **Resolution of singularities.** We first consider the case of a single branch. Write C for a branch at $O_0 = O$ of a holomorphic plane curve in $T_0 = \mathbb{C}^2$. Blowing up with centre O_0 produces a smooth surface T_1, an exceptional curve E_0 on it, and a strict transform $C^{(1)}$ meeting E_0 at a unique point O_1. Now blow up T_1 with centre O_1.

Inductively, suppose we have constructed a surface T_i containing curves E_j for $0 \leq j < i$ and a curve $C^{(i)}$ meeting E_{i-1} at a unique point O_i. Then blowing up T_i with centre O_i gives a new smooth surface T_{i+1} and a map $\pi_i : T_{i+1} \to T_i$. We write E_j again for the strict transform of E_j if $j < i$, E_i for the exceptional curve of π_i, $C^{(i+1)}$ for the strict transform of the curve (branch) $C^{(i)}$. As before, this meets E_i in a unique point O_{i+1}.

We now show that this process eventually yields a smooth curve. This process is known as *resolving* the singularity by blowing up. If $C^{(N)}$ is smooth, the projection $\pi : T_N \to T_0$ is a *resolution* of C.

Theorem 4.5. *There exist an integer N such that $C^{(N)}$ is smooth (and hence $C^{(n)}$ is smooth for all $n > N$).*

For a proof of the existence of a resolution we will resort to Puiseux parametrizations. Let B be a branch of a holomorphic germ at O such that $x = 0$ is not tangent to B, and $x = t^m, y = \sum_{r=m}^{\infty} t^r$ a good parametrization. This means that the values of r for which $a_r \neq 0$, together with m, have highest common factor 1. Note that $a_m = 0$ holds if $y = 0$ is tangent to B at O.

Definition 4.6. Define β_1 to be the exponent of the first term in the power series which is not a power of t^m, and e_1 to be the highest common factor of m and β_1. And, inductively,

$$\beta_{i+1} = \min\{k \mid a_k \neq 0,\, e_i \not| k\},\, e_{i+1} = \gcd(e_i, \beta_{i+1}),$$

continuing until we reach g with $e_g = 1$, which exists since the parametrization was chosen good. Thus β_j is the least exponent appearing in the series which does not belong to the additive group generated by m and the preceding β_j.

We shall call the sequence of numbers

$$(m\,;\,\beta_1, \ldots, \beta_g)$$

the *Puiseux characteristic* of the branch B; the β_i are called the Puiseux characteristic exponents. We also define $\beta_0 = e_0 = m$.

Example 4.7. The Puiseux characteristic of the parametrization $(x, y) = (t^4, t^6 + t^7)$ is $(4\,;\,6, 7)$.

We shall see that the characteristic is not only idependent of the choice of coordinates, but also contains deep information about the branch.

Proof. Theorem 4.5. We induct on the multiplicity m. For the curve C we have the parametrization

$$x = t^m,\; y = \sum_{r=m}^{\infty} a_r t^r.$$

More precisely, we may rewrite $y = b_1 t^m + b_2 t^{2m} + \cdots + b_q t^{qm} + c t^{\beta_1} + \ldots$, where we set $q := \left[\frac{\beta_1}{m}\right]$.

Blowing up once, we get $C^{(1)}$, parametrized by

$$x = t^m,\; Y = b_1 + b_2 t^m + \cdots + b_q t^{(q-1)m} + c t^{\beta_1 - m} + \ldots.$$

If $q \geq 2$, we shift the origin $y_1 := Y - b_1$, again have multiplicity m, and continue. After q blowups we find (after shifting if necessary) that the expansion of y_q starts with $c t^{\beta_1 - qm}$, so $C^{(q)}$ has multiplicity $\beta_1 - qm < m$.

Since, if $m > 1$, we can blow up to reduce the multiplicity m, the result follows by induction on m. □

Example 4.8. Let C be the curve $y^8 = x^{11}$.

For the first blow up set $(x, y) = (x_1, x_1 y_1)$ and substitute in f to get $x_1^8(y_1^8 - x_1^3) = 0$, the equation of the total transform. The first factor represents E_0, counted 8 times (as C has multiplicity 8 at O); the second factor f_1 is the equation of the strict transform. The singular point of $C^{(1)}$ does not lie in the chart given by substitution $(x, y) = (x_1' y_1', y_1')$.

For the second blow up we make the $(x_1, y_1) = (x_2 y_2, y_2)$. This produces a total transform of $C^{(1)}$ consisting of $E_1 = \{y_2 = 0\}$, counted 3 times, and the strict transform $C^{(2)} = \{y_2^5 - x_2^3 = 0\}$. The strict transform of E_0 is given by $x_2 = 0$.

For the third blow up we set $(x_2, y_2) = (x_3 y_3, y_3)$. In this chart, the strict transform of E_0 is given by $x_3 = 0$; the one of E_1 does not meet the domain of this chart; the exceptional curve E_2 is given by $y_3 = 0$; and $C^{(3)} = \{y_3^2 - x_3^3 = 0\}$.

For the fourth blow up we set $(x_3, y_3) = (x_4, x_4y_4)$; in this chart E_0, E_1 do not appear; E_2 is given by $y_3 = 0$ and E_3 by $x_4 = 0$; finally the strict transform $C^{(4)}$ of C is the smooth curve $y_4^2 - x_4 = 0$.

4.9. **Geometry of the resolution.** An exceptional curve is a smooth curve isomorphic to $\mathbb{P}^1$ and that is unaltered by the replacement by the strict transform. However information may be derived from the pattern of intersections of the curves E_j in the surfaces T_i constructed before.

We start with a simple lemma.

Lemma 4.10. *Let C be a smooth curve on a smooth surface S, and p a point on C. Blow up S with centre p to get surface T, an exceptional curve E, and the strict transform C' of C. Then C' is smooth, it meets E in a single point, and the intersection is transverse. Moreover, C' is isomorphic to C.*

Proof. Suppose C is given by $x = t, y = \sum_{r\geq 1} a_r t^r$ in suitable coordinates. Set $y = xY$ in the blow-up. In coordinates (x, Y) we have $E = \{x = 0\}$ and $C' = \{Y = \sum_{r\geq 1} a_r t^{r-1}\}$. Then C' is smooth and intersects E transversely at $Y = a_1$. Projection from C' to C is bijective and since x is a good parameter for both the projection is an isomorphism. □

In paticular we may take $S = T_1$ and $C = E_0$. The lemma then tells us that E_0 and E_1 are smooth and intersect transversely in a single point in T_2.

The above lemma can be generalized to the following

Lemma 4.11. *For any curve C, the intersection multiplicity of the strict transform of C with the exceptional curve E of the blow up is equal to the multiplicity of C.*

Proof. It is enough to consider the single branch case since both expressions are addiive. Choosing suitable coordinates we may take a good parametrisation $x = t^m, y = \sum_{r\geq m} a_r t^r$. Blowing up as above we find $C^{(1)}$ given by $x = t^m, Y = \sum_{r\geq m} a_r t^{r-1}$. Substituting this parametrisation in th equation $x = 0$ of E gives t^m, so the intersection number is equal to the multiplicity m of C. □

We return to the sequence of blowings up.

Proposition 4.12. *The exceptional curve E_i in T_{i+1} intersects E_{i-1} and at most one curve E_j with $j < i - 1$. These intersections are transverse, and no three of the curves E_i pass through a common point.*

Proof. Induction on i: result for $i = 1$ is already established. Inductively, assume it holds for $E_i \subset T_{i+1}$. Then T_{i+2} is the blow-up of T_{i+1} with centre a point O_{i+1} on E_i.

If O_{i+1} lies on $E_j, j < i - 1$ or on E_{i-1}, as well as on E_i then by Lemma 4.10 the curves meet transversely there, so have different tangents. These directions correspond to distinct points on E_{i+1}. By Lemma 4.10 again, E_{i+1} meets each of E_j and E_i transversely, at distinct points. Since E_{i+1} meets E_j in T_{i+2} if and only if $O_{i+1} \in E_j$ in T_{i+1}, the result follows. □

Definition 4.13. A collection of curves in a smooth surface is said to have *normal crossings* if each curve is smooth, no three meet in a point, and any intersection of two of them is transverse.

We have seen above that the configuration of exceptional curves always has normal crossings.

Definition 4.14. Given a singular point p of a curve C in a smooth surface S, a *good resolution* is a map $\pi : T \to S$ such that, if $E = \pi^{-1}(p)$, then π gives an isomorphism $T \setminus E \to S \setminus \{p\}$ and the collection $\pi^{-1}(C)$ of curves has normal crossings.

Theorem 4.15. *Any plane curve singularity has a good resolution.*

Proof. Let curve C have branches $B_1, \ldots, B_k$. By Theorem 4.5 we can find a sequence of blowings up such that the strict transform of B_1 by $\pi_{(1)} : T_1 \to T_0 = \mathbb{C}^2$ is smooth. Then the strict transform of B_2 by $\pi_{(1)}$ meets the exceptional locus of $\pi_{(1)}$ in a single point. We apply Theorem 4.5 again to obtain $\pi_{(2)} : T_2 \to T_1$ which resolves the singularities of B_2; by Lemma 4.10 the strict transform by $\pi_{(1)}\pi_{(2)}$ is still non-singular. Repeating the argument it follows by induction on k that we can find a composite π' of blowings up in which each B_j has a non-singular strict transform.

Suppose two components, say B, B', of $\pi'^{-1}(O)$ have non-transverse intersection $(B \cdot B')_p = s$ at a point p. Now blow up at p. We may take local coordinates (x, y) in which $B' = \{y = 0\}$ and $B = \{y = f(x)\}$ with f of order s. Blow up $(x, y) = (x_1, x_1 y_1)$ produces a similar situation with f replaced by $f(x_1)/x_1$ of order $s - 1$. Since no intersection numbers are increased by blow-up and the new exceptional curve is transverse to all other components (by Lemma 4.10), we may iterate to reduce all intersection numbers to 1.

Finally, if there is still a point where three or more curves meet then since any two of them are transverse, blowing up the point will separate them all. They will meet the new exceptional curve transversely in distinct points. Thus blowing up each such point leads to a good resolution of C. □

Example 4.16. We return to $y^8 = x^{11}$. Recall that after the fourth blow up, E_0, E_1 do not appear, E_2 is given by $y_3 = 0$ and E_3 by $x_4 = 0$; finally the strict transform $C^{(4)}$ of C is the smooth curve $y_4^2 - x_4 = 0$. This curve touches E_3, so we must continue blowing up.

After the fifth blow up $(x_4, y_4) = (x_5 y_5, y_5)$, we have E_3 and E_4 given by $x_5 = 0$ and by $y_5 = 0$ and $C^{(5)}$ by $x_5 = y_5$. Any two of these are transverse, but all three go through a single point, so we must blow up once more. This produces a final exceptional curve E_5 which meets each of E_3, E_4 and $C^{(6)}$ transversely, all at different points, so that at last we have a good resolution.

The procedure we have described is very simple: whenever there is a singular point or one where the normal crossing condition fails, choose one such point, and blow it up. A resolution obtained in this way is called *minimal.* The order in which we blow points up does not affect the result, since blowing up one of two points does not change what happens in a neighbourhood of the other.

Remark 4.17. For any resolution, the strict transform of C is a smooth curve $\tilde{C}$ and we have a projection $\pi : \tilde{C} \to C$ which is unique up to isomorphim. This is called the *normalisation* of C. This is not very interesting when studying germs: if C is single branch then π can be regarded as a parametrisation; if C has r branches there are r smooth points on $\tilde{C}$ and we get a parametrisation for each branch.

4.18. **Infinitely near points.** We return to the case of a single branch B and consider a good resolution of its singularity.

Definition 4.19. A point of the curve $E_0 \in T_1$ is said to be an *infinitely near point* of the first order to $O \in \mathbb{C}^2$; a point of $E_{r-1} \in T_r$ is an infinitely near point of r^{th} order to O.

The above iterative process produces one infinitely near point O_r on B of each order r. However, not all infinitely near points have the same nature. The basic geometry relation between infinitely near points is that of proximity.

Definition 4.20. Given infinitely near points O_i and O_j of orders i, respectively j with $j < i$ so that O_i determines a sequence of surfaces T_s with $s \leq i$ and maps $\pi_s : T_{s+1} \to T_s$ for $s < i$. Then O_i is *proximate* to O_j if E_j has a strict transform, also denoted E_j, in T_i and O_i lies on E_j so that E_i intersects E_j in T_{i+1}. Note that O_i is always proximate to O_{i-1}.

Along with proximity of infinitely near points we consider their multiplicities.

Definition 4.21. Denote by $m_i(B)$ the multiplicity of O_i of the strict transform $B^{(i)}$ of B in the T_i. The sequence $\{m_i(B)\}_{i \geq 0}$ is called the *multiplicity sequence* of the resolution of B.

The following are the basic properties of the proximity relation.

Proposition 4.22.
(i) *For each i there is at most one value $j < i - 1$ such that O_i is proximate to O_j.*
(ii) *If O_i is proximate to O_j and $j < k < i$ then O_k proximate to O_j.*
(iii) *The multiplicity $m_j(B)$ is equal to the sum of all $m_i(B)$ such that O_i is proximate to O_j.*

Remark 4.23. One can show that any sequence of proximity relations satisfying (i) and (ii) of Proposition 4.22 corresponds to some branch.

It follows from Proposition 4.22 that O_{i+2} is proximate to O_i if and only if $m_i(B) > m_{i+1}(B)$. The deduction of proximity relations from the sequence of multiplicities is now very simple.

Corollary 4.24. *For two irreducible curve germs with sequences* $\{O_i\}$,$\{O'_i\}$ *of infinitely near points the following are equivalent:*

(i) *we have the same proximity relations in the two sequences.*
(ii) *the sequence of multiplicities is the same for both.*

Proof. Suppose (i) holds. Note that all $m_i(B) = 1$ for i sufficiently large. If the multiplicities coincide for $i > j$, it follows from our hypothesis and Proposition 4.22 (iii) that they also coincide for $i = j$; by induction they coincide for all i.

Conversely if the multiplicities are the same, then for each j choose i such that $m_j(B) = \sum_{k=j+1}^{i} m_k(B)$. Then O_k is proximate to O_j if and only if $j < k \leq i$ and correspondingly for the O'_i. □

A convenient way to present the data of proximity relations is as follows.

Definition 4.25. We define the *proximity matrix* $P(B)$ of the branch B to have entries $P_{i,j}$ given by

$$P_{i,j} := \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } O_j \text{ is proximate to } O_i, \text{ so } i < j \\ 0 & \text{otherwise} \end{cases}$$

Here $0 \leq i, j \leq N$, where N is taken large enough to correspond to a good resolution. Since this matrix is uppertriangular, it has determinant 1 and an inverse $Q(B)$, also unitriangular, with integer entries.

Proximity relations can be graphically represented by writing the points O_i in sequence, and connecting O_i to O_j by an arc whenever O_j is proximate to O_i. Rather than write the symbols O_i it is more informative to write the multiplicities m_i.

Example 4.26. Again the curve $y^8 = x^{11}$. We had the sequence of multiplicities $8, 3, 3, 3, 2, 1, 1, \ldots$; each of O_1, O_2, O_3 is proximate to O_0; only O_2 is proximate to O_1; O_3, O_4 are proximate to O_2; O_4, O_5 are proximate to O_3; and only O_{i+1} is proximate to O_i if $i > 3$.

Lemma 4.27. *The multiplicity* $m_r(B)$ *defined above is the* (r, N) *entry in the inverse proximity matrix* $Q(B)$.

To relate the sequence of multiplicities $m_r(B)$ to the Puiseux characteristic, we need to know the effect of blowing up on the Puiseux characteristic. The invariance of the Puiseux characteristic under coordinate change is assumed.

Theorem 4.28. *Suppose given an irreducible curve whose Puiseux characteristic is $(m; \beta_1, \dots, \beta_g)$. Then the Puiseux characteristic of the curve obtained by blowing up is given by*

$$\begin{array}{ll} (m; \beta_1 - m, \dots, \beta_g - m) & \text{if } \beta_1 > 2m \\ (\beta_1 - m; m, \beta_2 - \beta_1 + m, \dots, \beta_g - \beta_1 + m) & \text{if } \beta_1 < 2m,\ (\beta_1 - m) \not| m \\ (\beta_1 - m; \beta_2 - \beta_1 + m, \dots, \beta_g - \beta_1 + m) & \text{if } (\beta_1 - m) \mid m \end{array}$$

Theorem 4.29. *The Puiseux characteristic of a branch B determines the sequence of multiplicities $m_i(B)$, and conversely.*

Proof. It follows at once from Theorem 4.28 that from the Puiseux characteristic of B we can calculate those of the branches obtained by succesive blowings up, and hence their multiplicities $m_i(B)$.

To see the converse, first observe that the three cases of the theorem are distinguished by the multiplicities $m_0(B)$ and $m_1(B)$: in the first case, $m_1(B) = m_0(B)$; in the second case $m_1(B)$ does not divide $m_0(B)$, and in the third, $m_1(B)$ is a proper divisor of $m_0(B)$.

We induct on the number of values of i for which $m_i(B) > 1$. If there are none, the curve is smooth. If there is just one, we must have $\beta_1 - m = 1$, so the Puiseux characteristic is $(m; m + 1)$, with $m = m_0(B)$. In general we may suppose that the sequence of multiplicities determines the Puiseux characteristic of the blown up curve, which is $(m; \beta_1, \dots, \beta_g)$. It follows that:

If $m_1(B) = m_0(B)$, the Puiseux characteristic of the original curve is $(m_1; \beta_1 + m_1, \dots, \beta_g + m_1)$;

If $m_1(B)$ does not divide $m_0(B)$, the Puiseux characteristic of B is $(m_0; \beta_1 + m_1, \dots, \beta_g + m_1)$;

If $m_1(B)$ is a proper divisor of $m_0(B)$, the Puiseux characteristic of B is $(m_0; m_0 + m_1, \beta_1 + m_1, \dots, \beta_g + m_1)$. □

Given a curve C with several branches, we may take a good resolution of it. If the resolution is minimal, each exceptional curve arises in the resolution of at least one component of C, and its relation to its predecessors is thus the same as before. However, instead of all the points being arranged in a single sequence, the sequences for the different components overlap, so the result takes the form of a tree. To each point of the tree there is attached a multiplicity, which is the sum of those arising from the different branches of C. It thus follows that Proposition 4.22 (iii) extends to the case of several branches.

4.30. **The dual graph.** A geometric way to present resolution data of a curve C is by the dual graph. For us a graph will consist of a set of vertices $\mathcal{V}$, and a set of edges $\mathcal{E}$, each one incident to a pair of vertices. Our graphs will have the additional property that no edge joins a vertex to itself, and there is at most one edge joining two vertices. A connected graph with no cycles is called a tree.

Let C have branches B_j, and let $\pi : T_N \to T_0$ be a good resolution (nearly always minimal). Consider the exceptional curves $E_i (0 \leq i \leq N)$ and the strict transforms $B_j^{(N)}$ in T_N.

Definition 4.31. The *dual graph* $\Gamma(C)$ is defined to be the abstract graph with vertices v_i corresponding to the curves E_i, and with an edge joining v_i to v_j if and only if the curves E_i, E_j intersect.

We can build $\Gamma(C)$ one step at a time. Suppose we already have a graph for the curves in T_{i-1}. If O_i is proximate only to O_{i-1} we adjoin a new vertex v_i and a new edge $v_i v_{i-1}$. If O_i is proximate to both O_{i-1} and O_j with $j < i-1$, then E_{i-1} and E_j intersect in T_i, so there is already an edge $v_{i-1}v_j$. We replace it by two edges $v_{i-1}v_i$ and $v_i v_j$ (one may think of this as subdividing $v_{i-1}v_j$ at a new vertex v_i). We see inductively that $\Gamma(C)$ is a tree.

Definition 4.32. The *augmented dual tree* $\Gamma^+(C)$ is defined to be the abstract graph with vertices corresponding to the curves E_i (vertices v_i) and $B_j^{(N)}$ (vertices w_j), and with an edge joining two vertices if and only if the curves intersect.

Notice that each $B_j^{(N)}$ has valence 1 in this graph: this holds at each stage of the inductive process of building up the tree.

Example 4.33. We return to curve with parametrisation $x = t^4, y = t^6 + t^7$, and hence Puiseux characteristic $(4; 6, 7)$. Blowing up to get a good resolution gives multiplicities $4, 2, 2, 1, 1$. Thus O_2 is proximate to O_0 and O_4 to O_2, as well as O_{i+1} to O_i for each i. The dual graph has edges Γ^+ has edges $v_0v_2, v_1v_2, v_2v_4, v_3v_4$ and v_4w.

Example 4.34. Let C be the union of $B_1 : y^2 = x^3$ and $B_2 : y^3 = x^4$. The first blow up $y = x_1y_1$ gives $y_1^2 = x_1$ and $y_1^3 = x_1$; the second blow up $x_1 = x_2y_2$ gives $y_2 = x_2$ and $y_2^2 = x_2$, which both go through the origin but are no longer tangent to each other; thus B_1 and B_2 have no further infinitely near points in common. The diagram of infinitely near points is thus: O_1 $(m_1 = 2)$, O_2 $(m_2 = 2)$, and $O_3^{(2)}$ $(m_3^{(2)} = 1)$ are proximate to O_0 (of multiplicity $m_0 = 5$); O_2 is only proximate to O_1; $O_3^{(1)}$ $(m_3^{(1)} = 1)$ and $O_3^{(2)}$ are both proximate to O_2.

The dual graph has edges $v_0v_3, v_1v_2, v_2v_3, w_1v_2, w_2v_3$.

For a single branch B we can analyse all our invariants in terms of the Puiseux characteristic. We now investigate how $\Gamma^+(B)$ is built up. We begin with the Newton polygon.

By a previous result, the Newton polygon of a single branch consists of a single edge, say from $(ad, 0)$ to $(0, ad + bd)$ with a and b coprime. The first group of blowings up corresponds to the steps in the Euclidean algorithm for finding the greatest common divisor d of ad and bd.

Suppose the steps in this algorithm are as follows:

$$\begin{aligned} a &= bq_1 + r_1 && (0 < r_1 < b) \\ b &= r_1 q_2 + r_2 && (0 < r_2 < q_1) \\ &\dots \\ r_{p-1} &= r_p q_{p+1}; \end{aligned}$$

so $r_p = d$; we will write $s_k = \sum_{i=1}^{k} q_i$.

Then O_0 has multiplicity $m = ad$; the next q_1 points have multiplicity bd and are proximate to O_0, as is the next, with multiplicity $r_1 d$. The next $q_2 - 1$ points also have multiplicity $r_1 d$, and are proximate to O_{q_1}. In general, O_{s_k} has multiplicity $r_{k-1}d$; the next q_{k+1} points are proximate to it, each with multiplicity $r_k d$, as is $O_{s_{k+1}+1}$, with multiplicity $r_{k+1}d$; but the following point is proximate only to $O_{s_{k+1}}$.

The coordinate transformations corresponding to these blowups are all of the standard type $(x, y) = (x', x'y')$ or $(x'y', y')$, with the new centre at the origin in each case except for O_{s_p+1}. We have a sequence of transformations of one type, followed by a sequence of those of the other type; the change-over corresponding to a proximity relation.

It is easy now to follow what happens to the dual graph. As far as v_{s_p}, the graph may be considered as a sequence of points on a line, but these lie in the order:

$$\{0, s_1 + 1, \dots, s_1 + q_2 = s_2, \dots, s_2 + q_3 = s_3, \dots, s_2 + 1, q_1 = s_1, \dots, 2, 1\}$$

We emphasise that, starting from the left, we have the first group, then the third; the odd groups preceding the even ones which conclude with the fourth group, then the second. The point v_{s_p} is somewhere in the middle; v_{s_p+1} is joined to it by a segment leaving the above line; then we have a sequence of segments end-to-end until we again start obtaining proximate points.

We can construct the whle of $\Gamma^+(B)$ inductively, using the effect of blowing up on the Puiseux characteristic. The general shape of the result is as follows.

Lemma 4.35. *If B is an irreducible curve with Puiseux characteristic $(m; \beta_1, \dots, \beta_g)$, then $\Gamma^+(B)$ consists of a single chain of edges from the initial vertex v_0 to the vertex w, with g side branches, each of a single chain, attached at distinct vertices of the original chain.*

Proof. We have described above the sequence of blowings up required to reduce the length of the Puiseux characteristic. We also found that the dual graph up to this point consists of a chain from v_0 to v_{s_p}, a side chain attached at v_{s_p}, and that any later vertices of the graph will be attached at v_{s_p}. We can now repeat the procedure; the result follows by induction on g. □

Remark 4.36. For a curve C with several branches, $\Gamma^+(C)$ is in some sense the union of the trees $\Gamma^+(B_j)$ corresponding to the different branches, overlapping as appropriate. Although the sequence of proximity relations (or of multiplicities) determines $\Gamma^+(C)$, the abstract structure of the graph does not suffice to allow us to reconstruct his data, as is clear from the above discussion, where many different sequences yield straight line graphs. The graph will thus always be used in conjunction with additional data.

We can label each vertex v_r by the integer r corresponding to its position in the sequence of blowups of infinitely near points O_r. This already suffices to determine the proximity relations and hence, the sequence of multiplicities.

Lemma 4.37. *For a single branch $B$the graph $\Gamma^+(B)$, labelled by the order in which vertices are generated, determines the proximity relations.*

Remark 4.38. Other conventions that encode the Puiseux characteristic are used in the literature.

The *characteristic Puiseux pairs* $(m_i, n_i), 1 \leq i \leq g$ are defined by $m_i = \beta_i/e_i$, $n_i = e_{i-1}/e_i$, so that we can recover e_i as $m/(n_1 n_2 \dots n_i)$ and β_i as $e_i m_i$.

The *Zariski characteristic pairs* $(p_i, q_i), 1 \leq i \leq g$ are defined by $q_i = n_i$, $p_1 = m_1$ and $p_i = m_i - n_i m_{i-1}$ for $1 < i \leq g$. These arise by direct application of the algorithm of Section 3, and occur in the normal forms

$$y = x^{\frac{q_1}{p_1}} \left(\alpha_1 + x^{\frac{q_2}{p_1 p_2}} (a_2 + \dots) \right),$$

for a parametrisation and

$$\dots((y^{p_1} - A_1 x^{q_1})^{p_2} - A_2 x^{q_2}) \dots = 0$$

for an equation.

4.39. **The semigroup of a branch.** The set of intersection numbers of B with other curves in the plane forms a semigroup $S(B)$. We begin by determining the structure of this semigroup in terms of the Puiseux characteristic of the branch, and establish an important duality property. We then establish he equivalence of several invariants of the branch: the Puiseux characteristic, the semigroup and the sequence of multiplicities. Two branches with the same invariants are called equisingular. A corresponding result for curves with several branches is also discussed.

In this section a semigroup S will be a subset of $\mathbb{Z}$ closed under addition and containing 0. If S contains positive and negative elements, it is a subgroup of $\mathbb{Z}$, hence it is of the form $d\mathbb{Z}$, for some $d \in \mathbb{N}$. From now on we suppose that $S \subseteq \mathbb{N}$, and that $S \neq \{0\}$. Write $d(S)$ for the highest common factor of all elements of S. S contains all but finitely many positive multiples of $d(S)$. Write $N(S)$ for the largest multiple of $d(S)$ not in S. We call S a dual semigroup if, for r divisible by $d(S)$, we have $r \in S$ if and only if $N(S) - r \notin S$. For example $S = d\mathbb{N}$ is a dual semigroup, with $N(S) = -d$.

Definition 4.40. For any curve C, with defining equation $f(x, y) = 0$, the quotient ring $\mathbb{C}\{x, y\}/(f)$ is defined to be the *local ring* $\mathcal{O}_C$ of C. For a single branch B, we may use a parametrisation γ to identify $\mathcal{O}_B$ with the image of $\gamma^* : \mathcal{O} = \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$, a subring of $\mathbb{C}\{t\}$.

We define the *semigroup of the branch* $S(B)$ by

$$S(B) := \{\operatorname{ord} \phi \mid \phi \in \mathcal{O}_B\}.$$

We also define the *double point number* $\delta(B)$ to be the cardinality of the set of gaps $\delta(B) = \#\{r \geq 0 \mid r \notin S(B)\}$.

Clearly $S(B)$ is a semigroup since $\operatorname{ord}(\phi_1\phi_2) = \operatorname{ord} \phi_1 + \operatorname{ord} \phi_2$. It is immediate that $S(B)$, and hence also $\delta(B)$, is unaltered by a change of holomorphic coordinates at $O \in \mathbb{C}^2$. It is also unaffected by a change of good parametrisation for B.

We will investigate these invariants in terms of $(m; \beta_1, \ldots, \beta_g)$, the Puiseux characteristic of B. We introduce a sequence of numbers that will play an important role in the sequel. Set:

$$\bar{\beta}_i := \frac{1}{m}\left[m\beta_1 + e_1(\beta_2 - \beta_1) + \cdots + e_{i-1}(\beta_i - \beta_{i-1})\right] \qquad (1 \leq i \leq g).$$

Lemma 4.41. (i) *The set $S(B)$ consists of the intersection numbers $(B \cdot C)_O$ of B with germs C at O not having B as a component.*
(ii) *For each $q \geq 0, \beta_q \in S(B)$.*
(iii) *For any branch B, $\delta(B)$ is finite.*

We can give a basic structural result for $\mathcal{O}_B$.

Proposition 4.42. (i) *For N sufficiently larg, any element of $\mathbb{C}\{t\}$ of order $> N$ belongs to $\mathcal{O}_B$.*
(ii) *The powers t^r with $r \notin S(B)$ form a basis of $\mathbb{C}\{t\}/\mathcal{O}_B$. Hence we have $\dim_{\mathbb{C}} \mathbb{C}\{t\}/\mathcal{O}_B = \delta(B)$.*

We will write $N(S(B))$ for the greatest integer not in $S(B)$.

Theorem 4.43. (i) *$S(B)$ is generated by $\bar{\beta}_0, \ldots, \bar{\beta}_g$.*
(ii) *This is a minimal set of generators: $\bar{\beta}_q$ is the least element of $S(B)$ not divisible by e_{q-1}.*

(iii) *$S(B)$ is a dual semigroup.*

Write S_q for the semigroup generated by $\bar\beta_0, \dots, \bar\beta_q$. The proof depends on the following lemma, proved by induction on q.

Lemma 4.44. (i) $d(S_q) = e_q$.
(ii) $N(S_q) = -\bar\beta_0 + \bar\beta_{q+1} - \beta_{q+1}$.
(iii) $N(S_q) = N(S_{q-1}) + (\frac{e_{q-1}}{e_q} - 1)\bar\beta_q$.
(iv) *S_q is a dual semigroup.*

Corollary 4.45. *We have $2\delta(B) = 1 + N(S(B))$.*

Example 4.46. For a curve with Puiseux characteristic $(4; 6, 7)$, the semigroup is generated by $4, 6$ and 13; the first elements are $0, 4, 6, 8, 10, 12, 13, 14, 16, 17, 18$; and $r \in S$ iff $15 - r \notin S$.

Now consider the number of gaps in $S(B)$. Using Lemma 4.44, since $N(S_0) = -m$, we obtain

$$N(S(B)) = -m + \sum_{q=1}^{g} \bar\beta_q \left(\frac{e_{q-1}}{e_q} - 1 \right).$$

We can rewrite this number in several ways:

$$N(S(B)) = -m + \bar\beta_{g-1} - \beta_g = \sum_{q=1}^{g} (e_{q-1} - e_q)(\beta_q - 1) - 1.$$

We record a summary of the equivalence of several relations between branches.

Proposition 4.47. *Any one of the following sets of data determines the others:*

- *the Puiseux characteristic $(m; \beta_1, \dots, \beta_g)$,*
- *the sequence $\{\bar\beta_0, \dots, \bar\beta_g\}$,*
- *the semigroup $S(B)$,*
- *the sequence of multiplicities $\{m_i(B)\}$,*
- *the proximity relations between infinitely near points,*
- *the proximity matrix.*

In particular, all of the above are indepedent of any choices of coordinates or parametrisation.

Definition 4.48. Two branches with the same Puiseux characteristic, and hence sharing the other properties listed, are said to be *equisingular*. Be aware that this does not imply that they are equivalent up to change of coordinates.

For the case of curves with several branches, there is a corresponding notion.

Proposition 4.49. *For the plane curves C and C', the following are equivalent.*

(a) *There is a bijection* $B_i \leftrightarrow B'_i$ *between branches of* C *and* C' *such that, for each* i*,* B_i *and* B'_i *are equisingular and for all* i, j *the intersection numbers coincide* $B_i \cdot B_j = B'_i \cdot B'_j$.

(b) *There is an isomorphism between the trees of infinitely near points in good resolutions of* C *and* C' *preserving proximity relations.*

Definition 4.50. We say that two curves are *equisingular* if the equivalent conditions of Proposition 4.49 hold.

Remark 4.51. Results stating that two curves are equisingular iff the dual trees of minimal good resolutions are isomorphic, may also be formulated, but some care is needed in the formulations to ensure that sufficient information is attached to the tree. The tree of infinitely near points, with multiplicities attached does not suffice.

4.52. **Intersections and infinitely near points.** The intersection numbers of two branches can also be expressed in the terminology of infinitely near points. The key to this is the following

Lemma 4.53. *Let the branches* B *and* B' *respectively have multiplicities* m *and* m'*; suppose that blowing up the origin gives branches* B_1 *and* B'_1*. Then if* B *and* B' *have distinct tangents at* O*, their intersection number is* mm'*. If they have a common tangent,* B_1 *and* B'_1 *are both centered at the point* P *corresponding to it, and* $(B \cdot B')_O = mm' + (B_1 \cdot B'_1)_P$.

Given two branches B and B', blowing up yields branches B_1 and B'_1, say. If B and B' have the same tangent, B_1 and B'_1 have the same centre, and we can blow up again, obtaining B_2 and B'_2. We obtain a sequence of branches B_i and B'_i, with multiplicities m_i and m'_i, say. Apllying the above lemma inductively yields

$$(B \cdot B')_O = \sum_{i=0}^{q-1} m_i m'_i + (B_q \cdot B'_q).$$

The next results follows at once.

Lemma 4.54. *If* B *and* B' *are distinct branches, then after finitely many blowings up, they will have different centres. Their intersection number is given by the sum*

$$(B \cdot B')_O = \sum_i m_i m'_i.$$

of products of multiplicities over those infinitely near points that they have in common.

5. Topology of the singularity link

For each germ curve C defined at $O \in \mathbb{C}^2$ we can define a link in the 3-sphere $\mathbb{S}^3$. This consists of a disjoint union of embedded circles, one for each branch of C at O. This link is defined uniquely up to isotopy.

Take coordinates (x, y) in $\mathbb{C}^2$ and write D_ϵ for the disc $|x|^2+|y|^2 \leq \epsilon^2$ with center O and radius ϵ, and S_ϵ for its boundary sphere $|x|^2 + |y|^2 = \epsilon^2$. We may suppose ϵ small enough so that C is defined in the neighbourhood D_ϵ of O. We will describe the intersection $C \cap D_\epsilon$: the first point is that it is essentially independent of ϵ provided that ϵ is small enough.

Lemma 5.1. *For ϵ sufficiently small, $K = C \cap S_\epsilon$ is a 1-manifold smoothly embedded in S_ϵ, and there is a homeomorphism of the pair $(D_\epsilon, C \cap D_\epsilon)$ to the cone on $(S_\epsilon, C \cap S_\epsilon)$, which may be chosen compatible with the natural projection on $[0, \epsilon]$.*

Proof. For suitable ϵ a vector field ξ on $D_\epsilon \setminus \{O\}$ is constructed such that: (i) at all points, ξ has inner product with the radius vector equal to 1, and (ii) at points of C, ξ is tangent to C.

After constructing ξ we integrate the vector field $-\xi$. Since the inner product of ξ with the radius vector is 1, if $r = \sqrt{|x|^2 + |y|^2}$, then $\xi(r) = 1$ so that if we integrate on the compact set $\epsilon_1 \leq r \leq \epsilon$, the integral $G : S_\epsilon \times \mathbb{R} \to D_\epsilon$ will be defined for all $t \leq \epsilon - \epsilon_1$ and at $t = \epsilon - \epsilon_1$ it will take values on the sphere S_{ϵ_1}. As $t \to \epsilon$ these converge uniformly to the origin, so G gives a continuos map $S_\epsilon \times [0, \epsilon] \to D_\epsilon$. Each integral curve meets each concentric sphere S_{ϵ_1} in just one point, and there is just one integral curve through ech point on each such sphere, so our map is bijective except that $S_\epsilon \times \epsilon$ is mapped to the origin. Hence it induces a homeomorphism of the cone on S_ϵ onto D_ϵ. Finally, since ξ is tangent to C, each integral curve that meets C stays within C so that our homeomorphism does indeed take the cone on $S_\epsilon \cap C$ onto $D_\epsilon \cap C$. □

The final clause of the lemma means that the points with $|x|^2 + |y|^2 = \delta^2$ are mapped to the points on the cone of height δ, so that the pairs $(S_\epsilon, C \cap S_\epsilon)$ and $(S_\delta, C \cap S_\delta)$ are homeomorphic: this is what we mean by 'essentially independent of ϵ'.

To analyse $C \cap S_\epsilon$, consider a branch B of C. We may suppose B tangent to $y = 0$ and take a parametrisation $x = t^m, y = \sum_{r>m} a_r t^r$. The vector field, restricted to B, defines a vector field in the t plane. Since for t small the radius vector is nearly in the plane $y = 0$, we see that $|t|$ increases along integral curves of the vector field. Thus $B \cap S_\epsilon$ corresponds to a curve in the t plane encircling 0, and is homeomorphic to a circle.

Using diffeomorphims $\mathbb{S}^1 \to B \cap S_\epsilon$ and the central projection $S_\epsilon \to \mathbb{S}^3$ we obtain a knot; using all the components of $C \cap S_\epsilon$ we have a link. We have shown that up to diffeomorphism these depend only on C.

In order to obtain explicit parametrisations for the knots K, we introduce a small modification to our construction of K. Consider the case when C consists of a single branch, or more generally of several branches all with the same tangent at O, say $y = 0$. Any such branch has a Puiseux parametrization $x = t^m, y = \sum_{r>m} a_r t^r$. Since $t^{-m}y$ tends to 0 with t, we can choose $\epsilon > 0$ such that $|x| < \epsilon$ implies $|y| < c|x|$ for some small constant $c > 0$. We may replace S_ϵ by a nearby manifold Σ_ϵ which coincides with $|x| = \epsilon$ in the region where $|y| < c|x|$. This has the advantage that taking $|t| = \epsilon^{1/m}$ or setting $t = \epsilon^{1/m} e^{2\pi i\theta}$ gives an explicit parametrization for the knot $C \cap \Sigma_\epsilon$. Moreover the change does not affect the end result.

Lemma 5.2. *For ϵ small enough, the knots $(S_\epsilon, C \cap S_\epsilon)$ and $(\Sigma_\epsilon, C \cap \Sigma_\epsilon)$ are homeomorphic.*

We can use the the method of proof of Lemma 5.1: the only difference is that we replace the radius vector (x, y) by the vector field $(x, 0)$.

Definition 5.3. For any two embeddings of $\mathbb{S}^1$ in $\mathbb{S}^3$ with disjoint images K_1 and K_2, the *linking number* is defined as follows. Span K_2 by an oriented surface F meeting K_1 transversely and count the number of intersection points of K_1 with F, with appropriate signs. The result is an integer, denoted by $\mathrm{Lk}(K_1, K_2)$. This does not depend on the choice of F, since if F' is another such surface, the union $F \cup -F'$ is a closed surface in $\mathbb{S}^3$, which necessarily has zero intersection number with the closed curve K_1.

We may also span both K_1 and K_2 by transverse surfaces in the disc D^4, and we can again count their signed intersections. This definition implies that the $\mathrm{Lk}(K_1, K_2) = \mathrm{Lk}(K_2, K_1)$, since the intersection number is symmetric.

This understood, we then have

Lemma 5.4. *Given two branches B, B' at O, their intersection number at O is equal to the linking number $\mathrm{Lk}(K, K')$ of the corresponding knots K and K'.*

5.5. **The geometry of the link.** In this section, we give an explicit geometrical model for the links we have just defined, and show that the isotopy class of the link depends only on the equisingularity class of C. The model resembles one of a solar system in that it consists of circles with their centers on other circles, all revolving together.

We begin with an example: $y = x^{3/2} + x^{7/4}$. First look at the simpler curve $y = x^{3/2}$. For the corresponding knot in Σ_ϵ we have a parametrisation $(x, y) = (\epsilon e^{2i\theta}, \epsilon^{3/2} e^{3i\theta})$. Each value of x determines 2 values of y, which lie on the circle with center 0 and (small) radius $\epsilon^{3/2}$; as x moves around the circle $|x| = \epsilon$ once, the values of y move around this circle 3/2 times, getting interchanged in the process.

For the given curve, the obvious parametrisation leads to the knot $(x, y) = (\epsilon e^{4i\theta}, \epsilon^{3/2} e^{6i\theta} + \epsilon^{7/4} e^{7i\theta})$. For each value of x we have 4 values of y. Since $\epsilon^{7/4}$ is small compared to $\epsilon^{3/2}$, it is natural to think of these as being near the previous points: indeed, we may draw circles of radius $\epsilon^{7/4}$ centered at the points corresponding to the previous curve; then we have two points on each of these circles, and as x moves around the circle once, as well as our 'first order points' moving around the circle $|y| = \epsilon^{3/2}$, these points move around these auxilliary circles 7/4 times.

The picture of a hierarchy of circles, each with center on a circle at the preceding level, and with radii each small compared to earlier ones, and all rotating at appropriate rates may be appropriately called a *carousel*. It is in fact qualitatively correct for any branch B, as we discuss next.

Take a Puiseux parametrisation $x = t^m, y = \sum_{r=k}^{\infty} a_r t^r$ of branch B with Puiseux characteristic $(m; \beta_1, \ldots, \beta_g)$. We define a sequence of branches $B_n, n \geq k$ by the parametrisations (not necessarily good) $x = t^m, y = \sum_{r=k}^{n} a_r t^r$ and consider what happens to the knot $K_n = B_n \cap \Sigma_\epsilon$, as we increase n.

If $n < \beta_1$, y_n is a polynomial in x, so we have a unique value of y_n for each value of x. Thus K_n is unknotted.

If $n = \beta_1$, we describe a circle of radius $|a_n|\epsilon^{\beta_1/m}$ with center y_{n-1}. The values of y_n give m/e_1 points equally spaced around this circle, and as x moves around the circle $|x| = \epsilon$, they rotate uniformly; this rotation is superimposed on the motion of the center y_{n-1} of the circle. Each of these points correspond to e_1 values of x.

For $\beta_1 < n < \beta_2$ we have a similar situation to the case $n < \beta_1$: the further terms form a polynomial in $t^{e_1} = x^{e_1/m}$, so we still have m/e_1 points and the effect, since the later terms are much smaller than the radius of the above circle (for ϵ small enough) is to move these points by distances which are small compared to their distances apart.

More precisely, these knots K_n are all isotopic to the knot K_{β_1}. Indeed, multiplying the other coefficients by s provides an isotopy.

$$x = \epsilon e^{ie_1\theta}, y = \sum_{r=k}^{\beta_1} a_r e^{ir_1\theta/m} \epsilon^{r/m} + \sum_{r=\beta_1+1}^{n} s a_r e^{ir_1\theta/m} \epsilon^{r/m}.$$

The general pattern is now clear: as n increases by 1, in general K_n is changed by an isotopy, but when n attains a value b_q each point splits into e_{q-1}/e_q points lying on a small circle surrounding it. We proceed to a more formal treatement.

Proposition 5.6. *Two branches with the same Puiseux characteristic determine isotopic knots in Σ_ϵ for ϵ small enough.*

Proof. Consider a branch B and notations as above. The knot K is given by the parametrisation $x = \epsilon e^{im\theta}$, $y = \sum_{r=k}^{\infty} a_r \epsilon^{r/m} e^{ir\theta}$.

The following deformation, for $0 \leq s \leq 1$, defines an isotopy of K, provided ϵ is small enough:

$$x = \epsilon e^{im\theta}, y = \sum_{r=k}^{\infty} s_r a_r \epsilon^{r/m} e^{ir\theta},$$

where $s_r = 1$ if $r = \beta_q$ for some q and $s_r = s$ otherwise.

This shows that K is isotopic to the knot corresponding to the branch B' with parametrisation $x = t^m, y = \sum_{q=1}^{g} a_{\beta_q} t^{\beta_q}$. We will reduce the coefficients to 1 by a further isotopy. This will complete the proof. Write $a_{\beta_q} = e^{l_q}$, and consider for $0 \leq s \leq 1$ the deformation

$$x = t^m, y = \sum_{q=1}^{g} e^{sl_q} t^{\beta_q}.$$

□

Using the same methods we can obtain an important extension of this result.

Proposition 5.7. *Let C and C' be two equisingular curve germs. Then the associated links K and K' are isotopic in $\mathbb{S}^3$.*

The proofs of the previous two propositions lead to the following

Corollary 5.8. *Any curve C is equisingular to a curve C' whose branches have parametrisations with real coefficients; thus the singularity links of C and C' are isotopic.*

Remark 5.9. The carousel description fits into one of the standard approaches to knot theory: the theory of braids. A *braid* is given by a set of m points in a disc D, which may be taken as $\{y \in \mathbb{C} \mid |y| < \delta\}$, varying continuously as strictly monotonic functions of θ with $0 \leq \theta \leq 2\pi$, and having the same points at both ends $\theta = 0$ and $\theta = 2\pi$ (though perhaps in a different order). This is converted to a link by identifying the two ends, thereby giving one or more closed curves in $D \times \mathbb{S}^1$, and embedding this in turn into $\mathbb{S}^3$ in a standard way.

Our next aim is to prove the converse of Proposition 5.7, but for this new tools are needed.

5.10. **Cable knots.** The carousel presented above gave a picture of the possible values of y for a given x; we must integrate this to give an effective description of the knot. That can be achieved by means of the knot construction known as cabling. In order to apply the properties of this construction we need to evaluate the relevant parameters in terms of the Puiseux characteristic of the branch.

Definition 5.11. For any knot K given by an embedding $f : \mathbb{S}^1 \to \mathbb{S}^3$, we can find neighbourhoods given embeddings $F : \mathbb{S}^1 \times D^2 \to \mathbb{S}^3$ such that $F(x, 0) = f(x)$ for each $x \in \mathbb{S}^1$. Such an embedding, which is called *tubular neighbourhood* of the knot K, is almost unique up to isotopy. Any simple closed curve traced on $\mathbb{S}^1 \times \mathbb{S}^1$ is mapped by F to the boundary of the neighbourhood, and so defines by composition a further knot. Such a knot is said to be a *cabled knot* about K.

Since by Proposition 5.6 we may choose any branch with the given Puiseux characteristic to get an equivalent knot, we may take our knot to be parametrised by $x = e^{im\theta}, y = \sum_{q=1}^{g} \epsilon_q e^{i\beta_q\theta}$, where $\epsilon_0 = 1$ and each ϵ_q is very small compared to its predecessor (suffice to take $\epsilon_q = 10^{-\beta_q}$). With this choice the unit circle $|x| = 1$ is sufficiently small in the sense of the proposition.

Recall that the first term leads to a small circle $|y| = \epsilon_1$. In the 3-sphere Σ_1, this sweeps out a torus $|x| = 1, |y| = \epsilon_1$, which is the boundary of the solid torus $|x| = 1, |y| \leq \epsilon_1$ which is a neighbourhood of the circle $y = 0$. Similarly the next term leads us to consider the solid torus which is the union of the discs $x = e^{im\theta}, |y - \epsilon_1 e^{i\beta_1\theta}| \leq \epsilon_2$, etc.

Let B_{q+1} be the branch defined by $x = t^m, y = \sum_{1\leq r<\beta_{q+1}} a_r t^r$, so meets Σ_1 in the knot parametrised by $x = e^{im\theta}, y = \sum_{s=1}^{q} \epsilon_s e^{i\beta_s\theta}$. We now denote this knot K_q (it is the K_{β_q} of the preceding section). It is immediate that for each q, K^q is a cable knot about K^{q-1}; thus in particular K^1 is a cable knot about the trivial knot $K^0 (y = 0)$.

We need some standard facts about the topology of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$, which may be considered as the boundary of the solid torus $M = \mathbb{S}^1 \times D^2$; write $i : T \to M$ for inclusion. Consider on $\mathbb{S}^1$ the orientation given by positive sense of rotation, that is by the parametrisation $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$. This defines a fundamental homology class in $H^1(\mathbb{S}^1)$. Parametrise M by $(e^{i\theta}, \rho e^{i\theta})$, where $0 \leq \rho \leq 1$.

The group $H_1(T)$ is free abelian with generators u, v the classes determined by the embeddings of $\mathbb{S}^1$ in T as $\mathbb{S}^1 \times 1$, respectively $1 \times \mathbb{S}^1$. The group $H_1(M)$ is infinite cyclic. The natural map $i_* : H_1(T) \to H_1(M)$ sends u to a generator and v to zero. The class v is called a *meridian* of T: up to sign, is characterised as generating $\ker i_*$. Intersection numbers on T are easily determined:

$$u \cdot u = v \cdot v = 0,\ u \cdot v = 1,\ v \cdot u = -1,$$

and for arbitrary classes $(au + bv) \cdot (a'u + b'v) = ab' - a'b$.

If a and b are coprime, $(e^{i\theta}, \rho e^{i\theta}), 0 \leq \theta \leq 2\pi$ parametrises a simple closed curve on T with homology class $au + bv$.

Lemma 5.12. *If the torus T is embedded in $\mathbb{S}^3$ with image disjoint from the knot K, taking linking numbers with K defines a homomorphism $\phi : H_1(T) \to \mathbb{Z}$. If*

$j : M \to \mathbb{S}^3$ is an embedding of the solid torus, K is the knot $j(\mathbb{S}^1 \times 0)$, and v is a meridian, then $\phi(v) = 1$.

With the notation of the lemma, if $\phi(u) = s$ then $\phi(u - sv) = 0$. The class $u - sv$, or a representative curve on $j(T)$, is called a *longitude* of the torus in $\mathbb{S}^3$. This is characterised by having linking number zero with the knot K.

Definition 5.13. Fix u to be a longitude. A knot K' which is cabled bout K is the image by j of a simple closed curve on T. If this curve has class $mu + nv$, then m is called the *winding number* of K' about K. The pair (m, n) will be called the *cabling data*. The class $mu + nv$ determines the curve up to isotopy on T.

Theorem 5.14. *For each q, the knot K^q is the cable about K^{q-1} with cabling data $(\frac{e_{q-1}}{e_q}, \frac{\bar{\beta}_q}{e_q})$. Thus the singularity knot K of a branch with Puiseux characteristic $(m; \beta_1, \dots, \beta_g)$ is the iterated torus knot about the unknot with cabling data $\{(\frac{m}{e_1}, \frac{\bar{\beta}_1}{e_1}), \dots, (\frac{e_{g-1}}{e_g}, \frac{\bar{\beta}_g}{e_g})\}$.*

5.15. **The Alexander polynomial.** In order to recover numerical information from topology, we require some topological invariant for knots. It turns out that the Alexander polynomial is convenient for our purposes. The key result we need is the behaviour of the Alexander polynomial under the cabling construction. This leads easily to the calculation of the polynomial for our knots, and the conclusion that the equisingularity class of the branch is determined by this polynomial, and hence by the isotopy type of the knot.

Let K be a link in $\mathbb{S}^3$ of r components and $G = \pi_1(X, *)$ the fundamental group of its complement $X = \mathbb{S}^3 \setminus K$, where $*$ is the base point. If G is made into an abelian group G^{ab} by factoring out its commutator subgroup $G' = [G, G]$ we obtain the homology group $G^{\text{ab}} = G/G' = H_1(X) = \mathbb{Z}^r$.

We next consider the abelianization of G', namely $G'^{\text{ab}} = G'/G''$, viewed as a module over the group ring $\mathbb{Z}G^{\text{ab}}$, where G/G' acts on G'/G'' by conjugation. We may identify G' with the fundamental group of the universal abelian cover $p : X^{\text{ab}} \to X$ of the link complement. Then $B = G'/G'' = H_1(X^{\text{ab}})$ is known as the Alexander invariant. It is more convenient to consider a closely related module, $A = H_1(X^{\text{ab}}, p^{-1}(*))$, known as the Alexander module. If we choose a basis $t_1, \dots, t_r$ for $\mathbb{Z}^r$, we may identify $\mathbb{Z}G^{\text{ab}}$ with the ring of Laurent polynomials $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. This ring is equipped with an augmentation homomorphism $\epsilon : \Lambda \to \mathbb{Z}$ defined by $\epsilon(t_i) = 1$, whose kernel is the augmentation ideal $I = (t_1 - 1, \dots, t_r - 1)$. Then the Λ-modules above fit into an exact sequence: $0 \to A \to B \to I \to 0$.

A finite presentation for the Alexander module A as Λ-module can be obtained via free calculus from the group G. Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ be a finite presentation of the link group. As usual, $r_1, \dots, r_n$ are words in the free group $\mathbb{F}$ generated by $x_1, \dots, x_n$. For every $1 \le i \le n$ a derivation $\frac{\partial}{\partial x_i}$ of the group ring $\mathbb{Z}\mathbb{F}$

is defined as follows:

$$\frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i}\epsilon(u) + u\frac{\partial v}{\partial x_i}, \quad \frac{\partial x_j}{\partial x_i} = \delta_{i,j}.$$

Let $\alpha : G \to G^{\mathrm{ab}}$ be the abelianization and $\phi : \mathbb{F} \to G$ the presentation homomorphism of G. Then the Alexander matrix $M = (m_{i,j})$ of G is the abelianized Jacobian matrix of the relators:

$$m_{i,j} = \alpha\phi\left(\frac{\partial r_i}{\partial x_j}\right).$$

The matrix M is a presentation matrix for the Λ-module A. More precisely, we have the exact sequence of Λ-modules: $\Lambda^n \xrightarrow{M} \Lambda^n \to A \to 0$.

The matrix M is a square matrix of determinant zero. Let E be the Fitting ideal in Λ generated by the $(n-1)\times(n-1)$ minors of M. Then if $r > 1$ we have $E = I \cdot (\Delta_K)$ the product of the augmentation ideal I and a principal ideal, and if $r = 1$ we have $E = (\Delta_K)$ a principal ideal. The Laurent polynomial $\Delta_K = \Delta_K(t_1, \ldots, t_r)$ is called the *Alexander polynomial* of the link K. The element Δ_K is unique up to a multiplication by a unit of Λ, i.e. an element of the form $\pm t_1^{a_1} \cdots t_r^{a_r}$.

An example which is crucial for us is the case of torus knots.

Example 5.16. Consider the standard torus $T = \{(z,w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$. Then for any mutually coprime integers m, n we have a knot $K_{m,n}$ parametrized by $z = e^{im\theta}, w = e^{in\theta}$, known as the torus knot of type (m,n). The fundamental group may be presented as $\langle x, y \mid x^m = y^n \rangle$. The projection to $\mathbb{Z} = \langle t \rangle$ takes x to t^n and y to t^m. A free calculus computation yields the Alexander polynomial

$$\Delta_{K_{m,n}}(t) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}.$$

This example is a key ingredient in the following.

Theorem 5.17. *Suppose the knot K' is a (m,n) cable about K. Then*

$$\Delta_{K'}(t) = \Delta_K(t^m)\Delta_{K_{m,n}}(t).$$

We return to our singularity knot K. We recall that we have a sequence K^0 (unknot), $K^1, \ldots, K^g = K$ and for each q, K^q is the $(\frac{e_{q-1}}{e_q}, \frac{\bar\beta_q}{e_q})$ cable about K^{q-1}. We may thus use the above theorem to calculate Δ_K inductively.

Proposition 5.18.

$$\Delta_{K^q}(t) = \frac{(t-1)}{\left(t^{\frac{m}{e_q}}-1\right)} \prod_{r=1}^{q} \frac{\left(t^{\frac{e_{r-1}\bar\beta_q}{e_r e_q}}-1\right)}{\left(t^{\frac{\bar\beta_r}{e_q}}-1\right)}.$$

The degree of $\Delta_{K^q}(t)$ is easily to be $(e_{q-1}\bar\beta_q - \beta_q - m + e_q)/e_q$. In particular, since $e_g = 1$ we have

Corollary 5.19.

$$\Delta_K(t) = \frac{(t-1)}{(t^m-1)}\prod_{q=1}^{g}\frac{\left(t^{\frac{e_{q-1}\bar\beta_q}{e_q}}-1\right)}{\left(t^{\bar\beta_q}-1\right)}.$$

and its degree is $(e_{g-1}\bar\beta_g - \beta_g - m + 1) = N(S(B)) + 1$.

We are now ready to apply these calculations to obtain our main conclusions.

Theorem 5.20. *The Alexander polynomial* $\Delta_K(t)$ *and hence the fundamental group* $G = \pi_1(\mathbb{S}^3 \setminus K)$ *determines the Puiseux characteristic of the branch.*

Proof. We first observe that there are no non-trivial relations between the binomials $t^k - 1$, so a product of such polynomials is uniquely determined. Secondly note observe that in the expression of $\Delta_K(t)$ there can be no further cancellation. For if, for example, $\frac{e_{q-1}\bar\beta_q}{e_q} = \bar\beta_r$, it would follow that $\bar\beta_r$ was a divisor of $\bar\beta_q$, which is a contradiction. Thus the denominator terms are just the $(t^{\bar\beta_q} - 1), 0 \leq q \leq g$. But the set $\{\bar\beta_q, 0 \leq q \leq g\}$ determines the Puiseux characteristic. □

Remark 5.21. Although our expression for $\Delta_k(t)$ involves denominators, these in fact cancel out. Explicitely we do this using cyclotomic polynomials. For an integer d, let Φ_d be the cyclotomic polynomial whose roots are the d th primitive roots of unity. Then we have $t^n - 1 = \prod_{d|n}\Phi_d(t)$. If these factorisations are inserted in $\Delta_k(t)$ then we will easily obtain it in polynomial form.

Example 5.22. If Puiseux characteristic is $\beta = (4;6,7)$ and $e = (4;2,1)$ then $\bar\beta = (4;6,13)$. After cancelations we obtain $\Delta_K(t) = \Phi_{26}(t)\Phi_1 2(t)$ of degree 16.

We put together now the full circle of equivalences between the different notions of equisingularity.

Theorem 5.23. *The following conditions on a pair of branches* B, B' *are equivalent:*

(i) B *and* B' *are equisingular.*
(ii) *The corresponding knots* K *and* K' *have the same Alexander polynomial.*
(iii) *The fundamental groups* $\pi_1(\mathbb{S}^3 \setminus K)$ *and* $\pi_1(\mathbb{S}^3 \setminus K')$ *are isomorphic.*
(iv) *The knots* K *and* K' *are isotopic.*
(v) *The pairs* $(D^4_\epsilon, D^4_\epsilon \cap B)$ *and* $(D^4_\epsilon, D^4_\epsilon \cap B')$ *are topologically equivalent for small enough* ϵ.

The picture for curves with several branches is more complicated. Two types of iteration occur in the geometry of a singularity link of several components. If $L = L_1 \cup \cdots \cup L_r$ is an oriented link of r components, we assume the iteration is occuring on the last component L_r. Suppose K is a (m, n)-cable knot about L_r. We then have a new link L' obtain as follows. Type (i) iteration $L' = L_1 \cup \cdots \cup L_{r-1} \cup K$, where L_r is replaced by K, and type (ii) iteration $L' = L_1 \cup \cdots \cup L_r \cup K$, where K is added to obtain a link of $r + 1$ components.

Proposition 5.24. *Links of plane curve singularities (also known as algebraic links) are iterated torus links obtained from the unknot via a finite sequence of cabling operations of types (i) and (ii).*

The multivariable Alexander polynomial behaves nicely with respect to these cabling operations.

Theorem 5.25. *If L is a link of $r \geq 2$ components and L' is obtained from L by a type (i) (m, n)-cabling then*

$$\Delta_{L'}(t_1, \ldots, t_r) = \Delta_L(t_1, \ldots, t_{r-1}, t_r^m)\Delta_{L_{m,n}}(t_r, t),$$

where $L_{m,n}$ is the 2-component link formed by torus knot $K_{m,n}$ and the unknotted meridian curve on the boundary torus containing $K_{m,n}$, and $t = t_1^{l_{1,r}} \ldots t_{r-1}^{l_{r-1,r}}$.

If L is a knot and L' is obtained from L by a type (ii) (m, n)-cabling then

$$\Delta_{L'}(t_1, t_2) = \Delta_L(t_1 t_2^m)\Delta_{L_{m,n}}(t_1, t_2),$$

where $L_{m,n}$ is the 2-component link formed by torus knot $K_{m,n}$ and the unknotted core of the torus containing $K_{m,n}$.

If L is a link of $r \geq 2$ components and L' is obtained from L by a type (ii) (m, n)-cabling then

$$\Delta_{L'}(t_1, \ldots, t_{r+1}) = \Delta_L(t_1, \ldots, t_{r-1}, t_r^m t_{r+1})\Delta_{L_{m,n}}(t_r, t_{r+1}, t),$$

where $L_{m,n}$ is the 3-component link formed by torus knot $K_{m,n}$, the unknotted meridian curve and the unknotted core of the torus torus containing $K_{m,n}$, and $t = t_1^{l_{1,r}} \ldots t_{r-1}^{l_{r-1,r}}$.

To state the equisingularity result for curves with several branches, we need to introduce the *peripheral classes* of a link $L \subset \mathbb{S}^3$ as the collection of conjugacy classes in $\pi_1(\mathbb{S}^3 \setminus L)$ consisting of the classes of meridians in tori bounding tubular neighbourhoods of the components of L.

Theorem 5.26. *Suppose C and C' are plane curve singularities defining links L, L'. Then the following are equivalent:*

(i) *C and C' are equisingular.*

(ii) *The corresponding links L and L' have the same multivariable Alexander polynomial.*
(iii) *There is an isomorphism $\pi_1(\mathbb{S}^3 \setminus K) \to \pi_1(\mathbb{S}^3 \setminus K')$ taking the peripheral classes of L to those of L'.*
(iv) *The links L and L' are isotopic.*
(v) *The pairs $(D^4_\epsilon, D^4_\epsilon \cap C)$ and $(D^4_\epsilon, D^4_\epsilon \cap C')$ are topologically equivalent for small enough ϵ.*

6. The Milnor fibration

A fibration is a kind of twisted product. More precisely, a map $\pi : E \to B$ is the projection of a *fibration* with *fiber* F if each point $b \in B$ has a neighbourhood U such that there is a homeomorphism ϕ of $\pi^{-1}(U)$ onto $F \times U$ whose second component is the restriction of π.

We will be interested in fibrations over the circle. If $B = \mathbb{S}^1$, we may take U to be either the upper or lower semicircle: U^+, U^- say. We thus have homeomorphisms $\phi^\pm$ as above. Each gives a homeomorphism $h^\pm$ of $\pi^{-1}(-1)$ onto F. If we replace ϕ^- by $((h^+ \circ (h^-)^{-1}) \times 1) \circ \phi^-$, the two homeomorphisms of $\pi^{-1}(-1)$ onto F agree. The final picture is thus as follows. We have the product of F by an interval $[0, 2\pi]$, and will identify, for each $x \in F$, the point $(x, 2\pi)$ with $(h(x), 0)$ for a suitable homeomorphism h. The resulting space is called the mapping torus of h; we map it to $\mathbb{S}^1$ by taking (x, θ) to $e^{i\theta}$. The map h is called the *monodromy* of the fibration. It is unique up to isotopy.

For each curve singularity there are two fibrations, equivalent to each other, describing how the defining function f behaves in a neighbourhood.

Consider the germ at O of an equation $f(x, y) = 0$ such that f has no repeated factor. Then O is an isolated point of $\{(x, y) \mid \partial f/\partial x = \partial f/\partial y = 0\}$.

Consider the discs $B_\epsilon = \{(x, y) \mid |x|^2 + |y|^2 \leq \epsilon^2\}$ in $\mathbb{C}^2$, with boundary sphere S_ϵ and $D_\eta = \{z \mid |z| \leq \eta\}$ in the complex plane, with boundary S_η.

Following Milnor we define two closely related fibrations.

Theorem 6.1. *If ϵ is small enough, we can find η_0 such that for $\eta < \eta_0$, the map*

$$f_1 : B_\epsilon \cap f^{-1}(D_\eta \setminus \{0\}) \to D_\eta \setminus \{0\},$$

defined by the restriction of f, is the projection of a smooth fibration.

Write $K = f^{-1}(0) \cap S_\epsilon$ for the singularity link, with tubular neighbourhood $N(K) = f^{-1}(D_\eta) \cap S_\epsilon$, which has boundary $\partial N(K) = f^{-1}(S_\eta) \cap S_\epsilon$. This boundary divides the sphere S_ϵ into two parts: one is $N(K)$, the closure of the other one we denote by $E(K)$, the closed complement of the link.

Theorem 6.2. *The map f_2 defined by $f/|f|$ from $M(K)$ to $\mathbb{S}^1$ is also a fibration, equivalent to the restriction f_1 of f to $B_\epsilon \cap f^{-1}(S_\eta)$.*

The fibration of Theorem 6.1 fits well into general constructions; the fibration f_2 is crucial to a more detailed study of the link K. We refer to either fibrations as the *Milnor fibration* of the curve $f = 0$, and to the fiber F of these fibrations as the *Milnor fiber*.

The function $f/|f|$ is also defined on $S_\epsilon \setminus K$. We may identify $N(K)$ with a product $K \times D_\eta$ in such a way that the projection on the second factor is given by f; hence the closure of the preimage of a point θ by $f/|f|$ is a smooth surface F_θ°

whose closure is a compact smooth surface F_θ with boundary K. Thus F_θ consists of a fiber of the Milnor fibration, extended by attaching a copy of $\partial F \times [0, 2\pi]$ to the boundary, so that the new boundary is singularity link K. The construction of the monodromy extends to these surfaces, and by continuity it follows that the monodromy reduces to identity on the boundary K.

We now give the basic facts about the Milnor fiber F.

Proposition 6.3. *The Milnor fiber F is a compact, connected, oriented surface with r boundary components, where the curve C has r branches.*

This result gives a complete description of the topology of the fiber F up to saying what the genus is.

Definition 6.4. The rank of the first Betti numbers of F is known as the *Milnor number* of the singularity, and denoted μ.

This is a very important invariant, a key measure of the complexity of the singularity. Observe that μ determines the genus g of F by the formula $g = (\mu - r + 1)/2$.

In the case when C consists of a single branch, we can identify μ with an invariant we have already encountered.

First observe that the complement $\mathbb{S}^3 \setminus K$ of the knot has the same homotopy type of the complement M of an open tube surrounding K. Now M is fibered over $\mathbb{S}^1$ with fiber F. The homotopy sequence of the fibration yields an exact sequence of fundamental groups:

$$1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\mathbb{S}^1) \to 1.$$

We can identify $\pi_1(M)$ with knot group $G = \pi_1(\mathbb{S}^3 \setminus K)$. The group $\pi_1(\mathbb{S}^1$ is infinite cyclic, and can be identified with $G^{\text{ab}} = H_1(\mathbb{S}^3 \setminus K)$. It follows from the exact sequence that the kernel G' of the projection $G \to G^{\text{ab}}$ is isomorphic with $\pi_1(F)$. Making this kernel abelian gives an isomorphism with $H_1(F)$, which is a free abelian group of rank μ. On the other hand, the Alexander polynomial can be understood as the characteristic polynomial of the map from G'/G'' to itself induced by the monodromy. But the degree of this polynomial coincides with the rank of the abelian group.

Proposition 6.5. *If C has just one branch, then*

$$\mu = \deg \Delta_K(t) = e_{g-1}\bar{\beta}_g - \beta_g - m + 1.$$

Thus $\mu(C) = N(S(C)) + 1 = 2\delta(C)$.

The simplest case of several branches is the following: the singularity at a transverse intersection of two smooth curves has Milnor number $\mu = 1$. More generally, we have

Theorem 6.6. *The Milnor of a union of two curves is given by* $\mu(C \cup C') = \mu(C) + \mu(C') + 2C \cdot C' - 1$. *Hence if* C *has irreducible components* $B_1, \ldots, B_r$, *we have*

$$\mu(C) = \sum_i \mu(B_i) + 2\sum_{i<j}(B_i \cdot B_j) - r + 1.$$

Using this, we can prove that a curve singularity has $\mu = 1$ if and only if it is the union of two smooth branches meeting transversely.

We give next a few more formulae for the Milnor number.

Theorem 6.7. *The value of* μ *is equal to the local intersection number of the polar curves* $C_x = \{\partial f/\partial x = 0\}$ *and* $C_y = \{\partial f/\partial y = 0\}$.

Example 6.8. We apply the theorem to compute the Milnor number of $f = x^a + y^b$. Then C_x consists of the y-axis with multiplicity $a - 1$, and C_y of the x-axis with multiplicity $b - 1$, hence $\mu(f) = (a - 1)(b - 1)$.

Using Theorem 6.7, we may analyse the effect on μ of blowing-up, and hence obtain a formula for μ in terms of the tree of infinitely near points.

Theorem 6.9. *Suppose* C *has a simple tangent and multiplicity* m. *Then the Milnor number of the strict transform is equal to*

$$\mu(C) - m(m - 1).$$

Theorem 6.10. *The Milnor number of a curve-germ* C *is given by*

$$\mu = \sum_P m_P(m_P - 1) - r + 1,$$

where r *is the number of branches of* C *and the sum runs over infinitely near points* P *of multiplicity* m_P *of* C.

Remark 6.11. The *double point number* $\delta(C)$ of a curve singularity is sometimes used in place of μ. We have $\mu(B) = 2\delta(B)$ for a single branch. In general we can define $\delta := (\mu + r - 1)/2$, so by Theorem 6.6, $\delta(C \cup C') = \delta(C) + \delta(C') + 2C \cdot C'$. Then Theorem 6.10 may be written as $\delta = \sum_P \binom{m_P}{2}$, where runs over infinitely near points.

Remark 6.12. The importance of the Milnor number as an invariant is illustrated by the fact that if $\{f_t(x, y) \mid 0 \leq t \leq 1\}$ is a family of functions, continuous in t, with $\mu(f_t)$ constant, then f_0 and f_1 are equisingular.

Finally, we have a purely algebraic expression for the Milnor number.

Theorem 6.13. *For a function* $f(x, y)$ *having an isolated singularity at the origin* $(0, 0)$ *we can define* $\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{J(f)}$, *where* $J(f)$ *denotes the Jacobian ideal* $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

7. Fundamental group and braid monodromy

In this section we will outline a method for producing presentations for the fundamental group of the complement to an algebraic curve $C \subset \mathbb{P}^2$ of degree d. For simplicity, we assume the line at infinity L_∞ to be transverse to C. We choose a point q on L_∞, but not on C, and coordinates such that $q = [0 : 1 : 0]$ and $L_\infty = \{x_0 = 0\}$. We set affine coordinates $x = x_1/x_0, y = x_2/x_0$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$. We denote the affine part $C \cap \mathbb{C}^2 = \{f(x, y) = 0\}$ of C again by C.

Consider the projection $p : \mathbb{C}^2 \to \mathbb{C}$ on the first coordinate. Let $\Delta = \{x \in \mathbb{C} \mid D_y(f) = 0\}$ be the set of critical values $\{x_1, \dots, x_k\}$ of $p_{|C}$ and $B = p^{-1}(\Delta)$ its preimage. If L_x denotes the line $p^{-1}(x)$ and Y_x the set of points $L_x \cap C$, then B consists of the lines $L_{x_i}, 1 \le i \le k$. With these notations we have the following

Lemma 7.1. *The restriction $p_| : \mathbb{C}^2 \setminus (C \cup B) \to \mathbb{C} \setminus \Delta$ is a locally trivial fibration.*

Choose discs D and E such that $\Delta \subset D \subset \mathbb{C}$ and $C \cap (D \times \mathbb{C}) \subset D \times E$. Fix a base point $* = (x_0, y_0)$ for $\mathbb{C}^2 \setminus (C \cup B)$ such that $x_0 \in \partial D$ and $y_0 \in \partial E$. Then x_0 will be the base point on $\mathbb{C} \setminus \Delta$ and y_0 on the generic fiber $L_{x_0} \setminus Y_{x_0}$. The long exact sequence in homotopy of the fibration p gives:

$$1 \to \pi_1(L_{x_0} \setminus Y_{x_0}) \to \pi_1(\mathbb{C}^2 \setminus (C \cup B)) \to \pi_1(\mathbb{C} \setminus \Delta) \to 1.$$

Choose generators $\gamma_1, \dots, \gamma_d$ such that $\pi_1(L_{x_0} \setminus Y_{x_0})$ is the free group $\mathbb{F}(\gamma_1, \dots, \gamma_d)$, and $\partial E = \gamma_1 \dots \gamma_d$. Similarly for $\pi_1(\mathbb{C} \setminus \Delta) = \mathbb{F}(\alpha_1, \dots, \alpha_s)$ we have $\partial D = \alpha_1 \dots \alpha_s$. The closed paths γ_i and α_j can be chosen to form good systems of generators.

With these notations we have the following presentation

$$\pi_1(\mathbb{C}^2 \setminus (C \cup B)) = \langle \gamma_1, \dots, \gamma_d, \tilde{\alpha_1}, \dots, \tilde{\alpha_s} \mid \tilde{\alpha_i}^{-1} \gamma_j \tilde{\alpha_i} = \beta_i(\gamma_j) \rangle,$$

where $\tilde{\alpha_i}$ are lifts of α_i and $\beta_i(\gamma_j)$ are some words in $\mathbb{F}(\gamma_1, \dots, \gamma_d)$.

Since $p_|$ is a locally trivial fibration its monodromy associates to each α_j a homeomorphism h_j of the fiber $L_{x_0} \setminus Y_{x_0} = \mathbb{C} \setminus \{d \text{ points}\}$. We can choose h_j to be identity outside E. Moreover, we choose the lift α_i to lie on the line $y = y_0$. In that case h_j determines a braid β_j on d strings, and $\beta_i(\gamma_j)$ is the action of β_j on γ_j as an automorphism of $\mathbb{F}(\gamma_1, \dots, \gamma_d)$.

We need to make precise the notion of meridian in this context.

Definition 7.2. Let be C an algebraic curve on a complex manifold X, $*$ a base point and $p \notin C$ a smooth point of X. Consider an analytic disc D intersecting C transversally at $\{p\} = D \cap C$. Let α be a path joining $*$ to a point on ∂D. A closed path $\pi_1(X \setminus D, *)$ of the form $\alpha \partial D \alpha^{-1}$ (with ∂D positively oriented) will be called *meridian* of C in X.

Note that $\tilde{\alpha_i}$ as chosen above is a meridian of L_{x_i} in $\mathbb{C}^2 \setminus C$.

Proposition 7.3. *Let C be an irreducible algebraic curve in a complex manifold X. Then any two meridians of C in X are conjugate in $\pi_1(X \setminus D)$. If m is a meridian of C in X then $\pi_1(X) = \pi_1(X \setminus D)/\langle m \rangle$.*

This implies $\pi_1(\mathbb{C}^2 \setminus (C \cup B))/\langle \tilde{\alpha_1}, \ldots, \tilde{\alpha_s} \rangle = \pi_1(\mathbb{C}^2 \setminus C)$, and we have the following result:

Proposition 7.4. *The fundamental group of the complement to C in $\mathbb{C}^2$ is a finitely presented group with presentation*

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle \gamma_1, \ldots, \gamma_d \mid \gamma_j = \beta_i(\gamma_j) \rangle.$$

We obtain next a description of $\pi_1(\mathbb{P}^2 \setminus C)$. Note that $\pi_1(\mathbb{P}^2 \setminus (C \cup L_\infty) = \pi_1(\mathbb{C}^2 \setminus C)$. But a meridian of L_∞ is given by $(\gamma_1 \ldots \gamma_d)^{-1}$. As a consequence we have

Theorem 7.5. *The fundamental group of the complement to C in $\mathbb{P}^2$ is a finitely presented group with presentation*

$$\pi_1(\mathbb{P}^2 \setminus C) = \langle \gamma_1, \ldots, \gamma_d \mid \gamma_j = \beta_i(\gamma_j),\, \gamma_1 \ldots \gamma_d = 1 \rangle.$$

8. The dual curve and Plücker formulas

We start with the classical construction of projective duality.

Definition 8.1. The *dual projective space* $\mathbb{P}^{2*}$ is the space of lines in $\mathbb{P}^2$. To a point $y = [y_0 : y_1 : y_2] \in \mathbb{P}^{2*}$ there corresponds the line $V(y_0x_0 + y_1x_1 + y_2x_2) \subset \mathbb{P}^2$.

Definition 8.2. Let $C \subset \mathbb{P}^2$ be an algebraic curve. The set $C^* = \{L \in \mathbb{P}^{2*} \mid L$ tangent to C at some point $p \in C\}$ is the *dual curve* to C.

Note that by definition, the condition on L means that $\mathrm{mult}_p(C) < \mathrm{Int}_p(C, L)$. For each $p \in C$ there exist only finitely many such lines. If C is a line then C^* is a single point.

Theorem 8.3. *Let $C \subset \mathbb{P}^2$ be an algebraic curve containing no line. Then:*

(a) *C^* is an algebraic curve.*
(b) *If C is irreducible then C^* is also irreducible and* $\deg C^* \geq 2$.
(c) *$C^{**} = C$.*

Proof. Let $C = V(F)$, $d = \deg F$ and $L = V(y_0x_0 + y_1x_1 + y_2x_2)$. If $y_2 \neq 0$ we eliminate x_2 from F. Set $G(x_0, x_2) = y_2^d F(x_0, x_1, -(y_0x_0 + y_1x_1)/y_2)) = b_0x_1^d + b_1x_1^{d-1}x_0 + \cdots + b_dx_0^d$, where $b_i \in \mathbb{C}[y_0, y_1, y_2]$ are homogeneous of degree d. The zeroes of G correspond to the intersection $C \cap L$.

Let $D \in \mathbb{C}[y_0, y_1, y_2]$ be the discriminant of $g(x_1) = G(1, x_1)$. Then D is homogeneous of degree $2d^2 - d$ and $D \not\equiv 0$. Then $C' = V(D)$ is an algebraic curve.

Now, if line L determined by y is tangent to C, it has at least a double intersection point wit it, so G has a multiple zero. If it is $[0 : 1]$ then $b_0(y) = 0$, otherwise g has a multiple zero. In either case, $D(y) = 0$, so $y \in C'$. This shows that $C^* \subset C'$.

We find an equation for C^* from the factors of D: There exist lines $L_1^*, \ldots, L_k^*$ such that $C' = C^* \cup L_1^* \cup \cdots \cup L_k^*$. Thus taking out the linear factors in D we get an equation for C^*. There are two types of lineat factors: those coming from points in $C \cap V(x_0)$ and those coming from points in $\mathrm{Sing}\, C$. To prove the claim, it remains to show that C^* is the closure in $\mathbb{P}^{2*}$ of $C' \setminus (L_1^* \cup \cdots \cup L_k^*)$. But this follows from the next result, and part (a) of the theorem is proved.

Theorem 8.4. *Let $p \in C \subset \mathbb{P}^2$, and $L \subset \mathbb{P}^2$ a line. Then L is tangent to C at p if and only if there exist a sequence of smooth points $\{p_\nu\} \subset C$ convergent to p such that $L = \lim_{\nu\to\infty} T_{p_\nu}C$.*

For part (b) we need the desingularisation theorem:

Theorem 8.5. *For any irreducible algebraic curve $C \subset \mathbb{P}^2$ there exist a compact Riemann surface S and a holomorphic map $\phi : S \to C$ biholomorphic away from* $\mathrm{Sing}\, C$ *and such that for every $p \in C$ we have a bijection $\phi^{-1}(p) \leftrightarrow \{$Branches of C at $p\}$.*

We use $\phi : S \to C$ to construct a holomorphic parametrisation $\phi^* : S \to C^*$. First we do it locally.

Let $O \in S$ and t a coordinate centered at O in a small neighbourhood $U \subset S$. The restriction $\phi_{|U}$ lifts to a holomorphic map $\varphi : U \to \mathbb{C}^3 \setminus \{0\}$. The map ϕ is an immersion at t if and only if the vectors φ and $\dot{\varphi} = \frac{d\varphi}{dt}$ are linearly independent. In this case, they span the tangent $T_{\varphi(t)}C \subset \mathbb{P}^2$. The equation of this tangent is given by $a_0(t)x_0 + a_1(t)x_1 + a_2(t)x_2 = 0$, where $a_i = (-1)^i(\varphi_j\dot{\varphi}_k - \varphi_k\dot{\varphi}_j)$. Thus $\phi^*(t) = [a_0(t) : a_1(t) : a_2(t)] \in \mathbb{P}^{2*}$ if ϕ is an immersion at t. Map $U \to \mathbb{C}^3, t \to (a_0(t), a_1(t), a_2(t))$ is holomorphic. Extend it to the points where ϕ is not an immersion to get a holomorphic map $\phi^* : U \to \mathbb{P}^{2*}$. This local parametrisation is independent on coordinate and choice of lift. Thus we obtain a global parametrisation $\phi^* : S \to C^*$, which by Theorem 8.5 must satisfy $\phi^*(S) = \mathbb{C}^*$.

It is immediate that C^* is irreducible. If C^* were a line, then infinitely many tangents to C would pass through a fixed point which is impossible. This proves part (b).

For part (c) we are going to use Puiseux parametrisations to explicitely describe the passage from C to C^*.

Let U be an open neighbourhood of the origin and $\phi : U \to \mathbb{P}^2$ a holomorphic map such that $\phi(U)$ is not contained in a line. Then there exist natural numbers α, β such that, after a linear transformation of $\mathbb{P}^2$, ϕ can be written as $\phi(t) = [\phi_0(t) : \phi_1(t) : \phi_2(t)]$, where $\phi_0(t) = 1$, $\phi_1(t) = t^{1+\alpha} + \dots$, and $\phi_2(t) = t^{2+\alpha+\beta} + \dots$. The numbers α, β are known as the local numerical invariants of the parametrisation ϕ at 0. Then $C = \phi(U)$ is a piece of a curve which is defined and holomorphic aroun $O = \phi(0)$. We have that $1 + \alpha = \mathrm{mult}_p(C)$ and $2 + \alpha + \beta = \mathrm{Int}_p(C, T_pC)$. In particular, $p \in \operatorname{Sing} C$ iff $\alpha \neq 0$, and p flex of C iff $\beta \neq 0$ (and $\alpha = 0$).

Now we can use these special Puiseux coordinates to calculate $a_0 = (1+\beta)t^{2+2\alpha+\beta} + \dots$, $a_2 = -(2 + \alpha + \beta)t^{1+\alpha+\beta} + \dots$, and $a_3 = (1 + \alpha)t^{\alpha} + \dots$. This gives the local parametrisation $\phi^*_{|U} = ((1 + \beta)t^{2+\alpha+\beta} + \dots, -(2 + \alpha + \beta)t^{1+\beta} + \dots, (1 + \alpha) + \dots)$. Letting α^*, β^* be the local invariants of ϕ^*, we see that $\alpha^* = \beta$ and $\beta^* = \alpha$.

To obtain ϕ^{**} and thus C^{**} we have to repete the computations for ϕ^*. We have in particular $\phi^{**}(0) = [1 : 0 : 0] = \phi(0)$ so $C^{**} = C$ because $O \in C$ was arbitrary. This concludes part (c). $\square$

Example 8.6. $C = V(x_1^3 - x_0x_2^2)$. Consider the rational parametrisation $\phi : \mathbb{P}^1 \to C \subset \mathbb{P}^2$, $[t_0 : t_1] \to [t_0^3 : t_0t_1^2 : t_1^3]$. For $t \neq [1 : 0]$ the tangent to C at $\phi(t)$ is spanned by the $\mathbb{C}^3$ vectors $\frac{\partial\phi}{\partial t_0} = (3t_0^2, t_1^2, 0)$ and $\frac{\partial\phi}{\partial t_1} = (0, 2t_0t_1, 3t_1^2)$. An equation for the tangent is obtained by taking the cross-product of the two vectors: $(3t_1^4, -9t_0^2t_1^2, 6t_0^3t_1)$. Thus $y = [t_1^3 : -3t_0^2t_1 : 2t_0^3] \in \mathbb{P}^{2*}$ are the coordinates of the tangent for all t, including $[1 : 0]$ as $V(x_0)$ is the cuspidal tangent. This gives a rational parametrisation $\phi^* : \mathbb{P}^1 \to C^* \subset \mathbb{P}^2$. The inflection point $[0 : 0 : 1] =$

$\phi^*([1:0])$ of C^* corresponds to the cusp $[1:0:0]=\phi([1:0])$ of C. Also the cusp $[1:0:0]$ of C^* to the inflection point $[0:0:1]$ of C. Eliminating x_0 in $F=x_1^3-x_0x_2^2$ gives $G(x_1,x_2)=y_0^2(y_0x_1^3+y_1x_1x_2^2+y_2x_2^3)$ and $D=y_0^{12}(4y_1^3+27y_0y_2^2)=y_0^{12}F^*$, and so $C^*=V(F^*)$.

Example 8.7. $C=V(F)$ the nodal cubic with $F=x_1^3+x_0x_1^2-x_0x_2^2$. Then $C^*=V(F^*)$ with $F^*=4(y_1^2-y_2^2)^2-4y_0y_1(y_1^2-9y_2^2)-27y_0^2y_2^2$ is the quartic known as cardioid, having one cusp corresponding to the flex of C and one bitangent corresponding to the node.

Example 8.8. Let $C=V(F)$ with F minimal. Assume the point $p=[1:0:0]$ is singular of $\mathrm{mult}_p(C)=2$. Let $f(x_1,x_2)=F(1,x_1,x_2)$. If $f_2=c_0x_1^2+c_1x_1x_2+c_2x_2^2$ is the degree 2 part of f, we have two cases: (a) p is a double point if $c_1^2\neq 4c_0c_2$ (b) p is a cusp if $c_1^2=4c_0c_2$. A point as in case (a) is called a *simple double point* (or just a *node*) if $\mathrm{mult}_p(C)=2$ and $\mathrm{Int}_p(C,T)=3$. A point as in case (b) is called a *simple cusp* (or just a *cusp*) if $\mathrm{mult}_p(C)=2$ and $\mathrm{Int}_p(C,T)=3$. The local invariants of a simple cusp are $\alpha=1,\beta=0$ and $\alpha^*=0,\beta^*=1$. Thus simple cusps correspond to simple inflection points under duality.

We can now investigate the invariants of the dual curve C^* and their realtionship with the invariants of C.

Definition 8.9. Let $C\subset\mathbb{P}^2$ be an algebraic curve of degree d. Then the number d^* of tangents from a point $q\in\mathbb{P}^2$ to smooth points of C is called the *class* of C.

Remark 8.10. If C is irreducible and $d\geq 2$ then $d^*=\deg C^*$. The maximum number of tangents is attained for almost all $q\in\mathbb{P}^2$.

Definition 8.11. An algebraic curve $C\subset\mathbb{P}^2$ is called a *Plücker curve* if $\deg C\geq 2$ and $\mathrm{Sing}\,C$ and $\mathrm{Sing}\,C^*$ consists of at most nodes and cusps.

The following number are invariants: $\delta=\#\{\text{nodes of } C\}$, $\kappa=\#\{\text{cusps of } C\}$, $\delta^*=\#\{\text{nodes of } C^*\}$, $\kappa^*=\#\{\text{cusps of } C^*\}$.

Remark 8.12. It is easy to show that for a Plücker curve C we have that:

(a) $\delta^*=\#\{\text{bitangents of } C\}$, $\kappa^*=\#\{\text{flexes of } C\}$
(b) $\delta=\#\{\text{bitangents of } C^*\}$, $\kappa=\#\{\text{flexes of } C^*\}$

Now we establish how the invariants $\delta,\kappa,\delta,\kappa$ are related.

Theorem 8.13 (Plücker formulae)**.** *For a Plücker curve C we have:*

(1) $d^*=d(d-1)-2\delta-3\kappa$ *(the class formula).*
(2) $\kappa^*=3d(d-2)-6\delta-8\kappa$ *(the inflection formula) .*
(3) $d=d^*(d^*-1)-2\delta^*-3\kappa^*$.
(4) $\kappa=3d^*(d^*-2)-6\delta^*-8\kappa^*$.

Proof. (1) Intersect C with polar P_qC for q generic. First, there must be exactly d^* tangents through q at smooth points $p_1, \dots, p_{d^*}$ of C. We have $\mathrm{Int}_p(C, P_qC) = 1$ for $p = p_i$. Hence by Bézout we have:

$$d(d-1) = d^* + \sum_{p \in \mathrm{Sing}\, C} \mathrm{Int}_p(C, P_qC).$$

For smooth curves $d^* = d(d-1)$. It remains to show that, assuming q generic, $\mathrm{Int}_p(C, P_qC) = 2$ if p node, and $\mathrm{Int}_p(C, P_qC) = 3$ if p cusp.

(2) Intersect C with Hessian $H(C)$. By Bézout we have:

$$3d(d-2) = \kappa^* + \sum_{p \in \mathrm{Sing}\, C} \mathrm{Int}_p(C, H(C)).$$

Thus it remains to show that $\mathrm{Int}_p(C, P_qC) = 6$ if p node, and $\mathrm{Int}_p(C, P_qC) = 8$ if p cusp. □

Remark 8.14. Every irreducible quadric or cubic is a Plücker curve. But there exist irreducible quartics which are not. On the other hand, for every $d \geq 2$ there exists a Plücker curve of degree d.

Example 8.15. (a) C irreducible cubic. Then C has at most one singular point. Plücker's formulae give:

- $d^* = 6, \kappa^* = 9, \delta^* = 0$ for $\delta = \kappa = 0$ and C is smooth cubic;
- $d^* = 4, \kappa^* = 3, \delta^* = 0$ for $\delta = 1, \kappa = 0$ and C is nodal cubic;
- $d^* = 3, \kappa^* = 1, \delta^* = 0$ for $\delta = 0, \kappa = 1$ and C is cuspidal cubic.

(b) C smooth Plücker quartic with $d^* = 12, \kappa^* = 24, \delta^* = 28$. If $f_1 = x_1^2 + \frac{1}{4}x_2^2 - 1$ and $f_2 = \frac{1}{4}x_1^2 + x_2^2 - 1$ then $C_\lambda = V(f_1 f_2 - \lambda)$ is Klein's quartic. If $\lambda > 0$, then C_λ has 28 real bitangents.

9. The genus formula

In this section a Riemann surface is associated to an algebraic curve.

Definition 9.1. A *Riemann surface* S is a connected Hausdorff space, together with a complex atlas: a collection of homeomorphisms $\psi_i : U_i \to V_i, i \in I$, on open sets $V_i \subset \mathbb{C}, U_i \subset S$ such that, for all i, j the transition functions $\psi_{ij} : \psi_i^{-1}(U_i \cap U_j) \to \psi_j^{-1}(U_i \cap U_j)$ are biholomorphisms.

A map $\phi : S \to T$ between Riemann surfaces is holomorphic if when viewed on charts is holomorphic.

Remark 9.2. A Riemann surface S is a 1-dimensional complex manifold, thus also a 2-dimensional real orientable manifold. Any compact orientable 2-dimensional real manifold S is homeomorphic to $\mathbb{S}^2$ with g handles attached, where g is the genus of S. It is a classical result that any such surface can be made into a Riemann surface. The complex structure is unique for $g = 0$, but for $g = 1$ the complex structures are parametrised by one complex parameter, respectively $3g-3$ parameters if $g > 1$.

Remark 9.3. A map $\phi : S \to C$ from a Riemann surface to an algebraic curve in $\mathbb{P}^2$ is holomorhic if viewed as map from $\phi : S \to \mathbb{P}^2$.

9.4. **The desingularisation theorem.**

Theorem 9.5. *For any irreducible algebraic curve $C \subset \mathbb{P}^2$ there exist a compact Riemann surface S and a holomorphic map $\phi : S \to C$ such that:*

(i) *The map $\phi : \phi^{-1}(C \setminus \operatorname{Sing} C) \to C \setminus \operatorname{Sing} C$ is biholomorphic,*
(ii) *For every $p \in C$ we have a bijection $\phi^{-1}(p) \leftrightarrow \{$Branches of C at $p\}$. In particular $\phi^{-1}(p)$ is finite for every p.*

Definition 9.6. Let $C \subset \mathbb{P}^2$ be an irreducible algebraic curve. We define the *genus* $g(C)$ of C as the genus of the Riemann surface S from Theorem 9.5. If $g(C) = 0$ we say C is *rational*.

Remark 9.7. Every compact Riemann surface can be realized as a smooth algebraic curve $S \subset \mathbb{P}^3$. If we choose a suitable point $z \in \mathbb{P}^3 \setminus S$ as center, we obtain a projection $\pi : \mathbb{P}^3 \setminus \{z\} \to \mathbb{P}^2$ such that $\phi_{|S} : S \to C = \pi(S)$ is biholomorphic almost everywhere. THe projection can even be chosen such that C is a Plücker curve and has at most nodes as singularities.

Proof. Theorem 9.5. We construct S by patching together open sets of $\mathbb{C}$.

If $p \in C \setminus \operatorname{Sing} C$ then there exist open set $V_p \subset \mathbb{C}$, neighbourhood $W_p \subset \mathbb{P}^2$ of p, and biholomorphism $\psi_p : V_p \to C \cap W_p \subset C \setminus \operatorname{Sing} C$.

For each $q \in \operatorname{Sing} C$, choose neighbourhood $q \in W_q \subset \mathbb{P}^2$ such that $\{W_q\}_q$ are pairwise disjoint and $C \cap W_q = C_{q,1} \cup \cdots \cup C_{q,r_q}$ is the branch decomposition of C at q. In particular let $C_{q,i} \cap C_{q,j} = \{q\}$ if $i \neq j$. Further, let W_q be chosen so small that for all i, there exist Puiseux parametrisations $\psi_{q,i} : V_{q,i} \to C_{q,i}$.

Take $M = (\bigcup_{p \in C \setminus \operatorname{Sing} C} V_p) \cup (\bigcup_{q \in \operatorname{Sing} C} V_{q,1} \cup \cdots \cup V_{q,r_q})$. The maps $\{\psi_p\}$ and $\{\psi_{q,i}\}$ yield a holomorphic map $\psi : M \to C$.

Now things are glued together in M as follows. For $p, p' \in C \setminus \operatorname{Sing} C$ and $q \in \operatorname{Sing} C$ we have: $v \in V_p$ and $v' \in V_p'$ are equivalent iff $\psi_p(v) = \psi_{p'}(v') \in C \setminus \operatorname{Sing} C$, and $v \in V_p$ and $v' \in V_{q,i}'$ are equivalent iff $\psi_p(v) = \psi_{q,i}(v') \in C \setminus \operatorname{Sing} C$. There is no gluing between sets $V_{q,i}$ and $V_{q,j}$.

Let $S = M/\sim$ the quotient space of M under the equivalence relation decribed above, with quotient topology and $\phi : S \to C$ the map induced by ψ.

To verify that S is a Riemann surface is routine work. First, note that $\phi : \phi^{-1}(C \setminus \operatorname{Sing} C) \to C \setminus \operatorname{Sing} C$ is bijective by construction. Secondly, it can be shown that S is Hausdorff by a standard argument for quotient spaces. To show that S is compact one uses the compactness of S and the finiteness of $\phi^{-1}(\operatorname{Sing} C)$. The connectedness of S follows from the irreduibility of C. Finally, the uniqueness of S results from the Riemann extension theorem for holomorphic functions. □

Remark 9.8. Obvserve that it follows from the normalisation theorem that C and $C \setminus \operatorname{Sing} C$ are connected sets.

9.9. **The Riemann-Hurwitz formula and applications.** Let $f : S \to T$ be a nonconstant holomorphic map between compact Riemann surfaces. For every $p \in S$ there exist $k \geq 1$ natural number and local coordinates s around p and t around $f(p)$ such that f is described by $s \to t = s^k$.

Definition 9.10. We call $\operatorname{ord}_p(f) := k$ the *order* of f at p and $\nu_p(x) := \operatorname{ord}_p(f) - 1$ the *branching index* of f at p. We call p a *branch point* of f if $\nu_p(f) \geq 1$.

Note that f is a biholomorphism in a neighbourhood of p iff $\nu_p(f) = 0$. Thus, the image $B \subset T$ of all the branch points of f in S is finite and $f_{|S \setminus f^{-1}(B)} : S \setminus f^{-1}(B) \to T \setminus B$ is a covering map. The degree of this cover is denoted $n(f)$.

The Riemann-Hurwitz formula relates the genera of S and T with unbranching data $n(f)$ and branching data $\nu(f)$ of f.

Theorem 9.11 (Riemann-Hurwitz). *Let $f : S \to T$ be a nonconstant holomorphic map between compact Riemann surfaces. We then have:*

$$2g(S) - 2 = n(f) \cdot (2g(T) - 2) - \nu(f),$$

where $\nu(f) = \sum_{p \in S} \nu_p(f)$ is the branching index of f. In particular if $T = \mathbb{P}^1$ then $g(S) = \frac{1}{2}\nu_(f) - n(f) + 1$.

We can apply the Riemann-Hurwitz formula to compute the genus of an irreducible curve $C \subset \mathbb{P}^2$ as follows. We use maps $S \xrightarrow{\phi} C \xrightarrow{\pi} \mathbb{P}^1$, where ϕ is a desingularisation and π is a projection with center z off C. Then $f = \pi \circ \phi : S \to \mathbb{P}^1$ is a branched covering with degree $d = \deg C$. It remains to compute $n(f)$. This is especially easy when C is smooth.

Lemma 9.12. *If $C \subset \mathbb{P}^2$ is smooth of degree d then $n = d(d-1)$ for generic z.*

Proof. As S is smooth we may assume $S = C$. Let $z = [0 : 0 : 1] \notin C$. Then π is given by $C \to \mathbb{P}^1, p = [p_0 : p_1 : p_2] \to q = [p_0 : p_1]$. Now p is a branch point of π iff the line zq is a tangent at p. If z does not lie on any bitangent or inflectional tangent, then there are exactly $d^* = d(d-1)$ simple tangents from z to C. Being a simple tangent means $\nu_p(\pi) = 1$ and we are done. □

Corollary 9.13. *A smooth curve $C \subset \mathbb{P}^2$ of degree d has genus $g = \frac{1}{2}(d-1)(d-2)$.*

Proof. Set $T = \mathbb{P}^1$ in the Riemann-Hurwitz formula. □

Similarly we can prove Clebsch's formula:

Proposition 9.14. *A Plücker curve $C \subset \mathbb{P}^2$ of degree d with δ nodes and κ cusps has genus $g = \frac{1}{2}(d-1)(d-2) - \delta - \kappa$.*

Proof. Choose coordinates such that $z = [0 : 0 : 1]$ does not lie on any of the tangents to nodes or cusps and the tangents to smooth points $p_1, \dots, p_{d^*}$ pass through z. We compute $\nu_x(f)$ for $x \in S$:

(a) $\phi(x) \notin \operatorname{Sing} C \cup \{p_1, \dots, p_{d^*}\}$. Then $\nu_x(f) = 0$.
(b) $\phi(x) \in \{p_1, \dots, p_{d^*}\}$. Then $\nu_x(f) = 1$.
(c) Let $\phi(x)$ be a cusp. Since line $L = z\phi(x)$ is not the cuspidal tangent we get $\operatorname{Int}_{\phi(x)}(C, L) = 2$ so $\nu_x(f) = 1$.
(d) Let $\phi(x)$ be a node. Since the branch of C at $\phi(x)$ corresponding to x cuts the ray of projection transversely, $\nu_x(f) = 0$.
We obtain $\nu(f) = d^* + \kappa = d(d-1) - 2(\delta + \kappa)$. Substitute this into the Riemann-Hurwitz formula to obtain the genus.

□

Remark 9.15. Since a desingularisation $\phi : S \to C$ of C determines a desingularisation $\phi^* : S \to C^*$ of C^* we have that C and C^* share the genus. Thus we obtain the dual genus formula: $g = \frac{1}{2}(d^* - 1)(d^* - 2) - \delta^* - \kappa^*$.

One can generalise the previous genus formulae by introducing an analytic invariant $c_p \in 2\mathbb{N}$ associated to a point $p \in C$ such that $c_p = 0$ iff p smooth and $c_p = 2$ iff p node or cusp. Once this is done, we have the following genus formula:

Theorem 9.16 (Max Noether)**.** *The genus of an irreducible algebraic curve $C \subset \mathbb{P}^2$ is given by*

$$g = \frac{1}{2}\left[(d-1)(d-2) - \sum_{p \in \operatorname{Sing} C} c_p\right].$$

We now introduce the Gorenstein-Rosenlicht invariant c_p. We start with a geometric definition and then provide the algebraic one.

Let $C = V(f)$ be a germ of a curve at $O \in \mathbb{C}^2$ with $f \in \mathbb{C}\{x, y\}$ minimal, and $C = C_1 \cup \cdots \cup C_r$ the branch decomposition. Let L be a line through O parametrised by $t \to (at, bt), [a : b] \in \mathbb{P}^1$. Its point at infinity in $\mathbb{P}^2$ is $q = [0 : a : b] \in \bar{L}$. For germ C at q we can define the polar curve germ P_qC as $V(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y})$. The polar P_qC has no branch in common with C if L is not a branch of C.

Lemma 9.17. *Suppose C_i is not a line and q is the point at infinity of the line L. THen the integer $\operatorname{Int}(C_i, P_qC) - \operatorname{Int}(C_i, L)$ is independent of the choice of L. In particular, $\operatorname{Int}(C_i, P_qC)$ is independent of q if q is not on the tangent to C_i.*

Definition 9.18. For $1 \leq i \leq r$ let $c_i = \operatorname{Int}(C_i, P_qC) - \operatorname{Int}(C_i, L) + 1$. Then put $c := c_1 + \cdots + c_r$.

Theorem 9.19. *The number c has the following properties:*

(1) *c is invariant under linear changes of coordinates.*
(2) *if L is not tangent to any branch of C, then $c = \operatorname{Int}(C, P_qC) - \operatorname{mult}(C) + r$.*
(3) *$c \geq m(m-1)$, where $m = \operatorname{mult}(C)$, in particular $c \geq 0$ with equality iff C smooth at 0.*

The number c admits an alternate definition:

Theorem 9.20. *If $R = \mathbb{C}\{x, y\}/(f)$ and $\bar{R}$ is its integral closure then $c = 2\dim_{\mathbb{C}}(\bar{R}/R)$. In particular $c \in 2\mathbb{N}$ and c is an analytic invariant.*

The invariant $\delta = c/2$ is usually used instead of c.

Remark 9.21. If $p_1, \ldots, p_s$ is the sequence of infinitely near points of the germ (C, O) and $m_1, \ldots, m_s$ are their multiplicities, then $c = \sum_{j=1}^{s} m_j(m_j - 1)$.

Now using c we can provide a generalization of the class formula.

Theorem 9.22 (General class formula)**.** *The class of an irreducible algebraic curve $C \subset \mathbb{P}^2$ of degree $d \geq 2$ is given by*

$$d^* = d(d-1) - \sum_{p \in \operatorname{Sing} C} (c_p + m_p - r_p),$$

where $m_p = \operatorname{mult}_p(C)$ and r_p is the number of branches of C at p.

Proof. Choose coordinates in $\mathbb{P}^2$ such that through $q = [0:0:1]$ there pass exactly d^* tangents through smooth points of C, and no tangents to a singular point passes through q. By Bézout's theorem: $d(d-1) = d^* + \sum_{p \in \operatorname{Sing} C} \operatorname{Int}_p(C, P_qC)$. But $\operatorname{Int}_p(C, P_qC) = c_p + m_p - r_p$. □

Remark 9.23. Note that $c_p + m_p - r_p$ is 2 if p node and 3 if p cusp, thus we recover the class formula for Plücker curves.

We can now prove the genus formula in Theorem 9.16.

Proof. Theorem 9.16. We use the maps $S \xrightarrow{\phi} C \xrightarrow{\pi} \mathbb{P}^1$ and their composite $f : S \to \mathbb{P}^1$, where ϕ is the desingularisation and π the projection with center $[0:0:1]$. We only have to show that the branching index of f satisfies:

$$\nu(f) = d^* + \sum_{p \in \operatorname{Sing} C} (m_p - r_p). \tag{9.1}$$

For then, we get $\nu(f) = d(d-1) - \sum_{p \in \operatorname{Sing} C} c_p$ using the class formula in Theorem 9.22.

To prove (9.1) we compute the contribution of a singular point to the branching index. If $p \in \operatorname{Sing} C$ then $\phi^{-1}(p) = \{x_1, \ldots, x_{r_p}\}$ and to every x_i there corresponds a branch C_i of the germ (C, p). We can change coordinates such that $p = [1 : 0 : 0]$. Since ϕ is constructed by means of Puiseux parametrisations, we can choose a coordinate t around x_i such that the parametrisation of C_i is described by $t \to (t_{m_i}, \varphi_i(t)), m_i = \operatorname{mult}_p(C_i)$. Thus f is described around x_i by $t \to t^{m_i}$ hence $\nu_{x_i} = m_i - 1$. The total contribution of p to the branching index is $\sum_{i=1}^{r_p}(m_i - 1) = m_p - r_p$ and (9.1) follows. □

Example 9.24. $C = V(y^m - x^n), 1 \leq m < n, (m, n) = 1$. Then $\operatorname{mult}(C) = m$ and the parametrisation $t \to (t^m, t^n)$ gives $c = (m-1)(n-1)$. Thus by a suitable choice of m and n we can spread the apriori inequality $0 \leq m(m-1) \leq c$ arbitrarly far apart.

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