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Institution-independent Ultraproducts

Răzvan Diaconescu *
Institute of Mathematics "Simion Stoilow"
PO Box 1-764, Bucharest 014700, Romania
Razvan.Diaconescu@imar.ro

Abstract. We generalise the ultraproducts method from conventional model theory to an institution-independent (i.e. independent of the details of the actual logic formalised as an institution) framework based on a novel very general treatment of the semantics of some important concepts in logic, such as quantification, logical connectives, and ground atomic sentences. Unlike previous abstract model theoretic approaches to ultraproducts based on category theory, our work makes essential use of concepts central to institution theory, such as signature morphisms and model reducts. The institution-independent fundamental theorem on ultraproducts is presented in a modular manner, different combinations of its various parts giving different results in different logics or institutions. We present applications to institution-independent compactness, axiomatizability, and higher order sentences, and illustrate our concepts and results with examples from four different algebraic specification logics. In the introduction we also discuss the relevance of our institution-independent approach to the model theory of algebraic specification and computing science, but also to classical and abstract model theory.

Keywords: Institutions, model theory, algebraic specification, ultraproducts

1. Introduction

The theory of institutions [15] is a categorical abstract model theory which formalises the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them. Institutions become a common tool in the study of algebraic specification theory and can be considered its most fundamental mathematical structure. It is already an algebraic specification tradition to have an institution underlying each language or system, in which all language/system constructs and features can be rigorously explained as mathematical entities. This has been first spelt out as a programme with a sample definition

*Address for correspondence: Institute of Mathematics "Simion Stoilow", PO Box 1-764, Bucharest 014700, Romania

of specification language constructs in [32]. Most modern algebraic specification languages follow this tradition, including CASL [5], Maude [29], or CafeOBJ [11]. There is an increasing multitude of logics in use as institutions in algebraic specification and computing science. Some of them, such as first order predicate (in many variants), second order, higher order, Horn, type theoretic, equational, modal (in many variants), infinitary logics, etc., are well known or at least familiar to the ordinary logicians, while others such as behavioural or rewriting logics are known and used mostly in computing science.

The original goals of institution theory are to do as much computing science and model theory as possible, independent of what the actual logic may be [15]. This mathematical paradigm is often called 'institution-independent' computing science or model theory. While the former goal has been greatly accomplished in the algebraic specification literature, there were only very few and rather isolated attempts towards the latter [34, 35, 36, 31]. The paper [34] formulated for the first time institution-independent concepts of logical connectives, interpolation, etc. A related work is [27], however this is oriented towards proof theoretic aspects of logics rather than model theory. This situation contrasts with the feeling shared by some researchers that deep concepts and results in model theory can be reached in a significant way via institution theory. This paper can be regarded as a new step towards this goal, part of a coming series of works in institution-independent model theory.

The significance of institution-independent model theory is manifold:

- It provides model-theoretic results and analysis for various logics in a generic way. Only a limited number of model-theoretic properties are usually studied for the logics in use in computing science and algebraic specification, however it is important to have as deep as possible understanding of the model-theoretic properties of the underlying logic because the specification or software engineering properties of the logic depend intimately on the former ones ([12] is one of the works that support this argument). We sometimes notice that the failure of some specification properties of a logic is due to the rather subtle wrong definition of some details of the logic. We also notice that often the right definition of a logic can be checked through its model-theoretic properties, otherwise said good model-theoretic properties lead to good specification properties.
- It exports model-theoretic methods from classical logic to other logics. Classical first-order predicate logic has developed very rich powerful model-theoretic methods, which exported to an institution-independent framework can become available for the multitude of computing science or algebraic specification logics.
- It provides a new way of doing model theory. While the points we made above have a more application oriented significance, this point has a pure mathematics methodological significance. The institution-independent way of obtaining a model theoretic result, or just viewing a concept, leads to a deeper understanding of *why* a certain model theoretic phenomenon holds. Such top-down understanding is not suffocated by the details of the actual logic, it decompose the model-theoretic phenomenon (in various layers of abstract conditions), and provides a clear picture of its limits.

Although these points are largely valid for any form of abstract model theory, they are especially relevant for the institution-independent abstract model theory. One of the reasons for this is that up to our knowledge, the theory of institutions provides the most complete definition of abstract model theory, the only one including signature morphisms, model reducts, and even mappings (morphisms) between logical systems, as primary concepts. Also, as mentioned above, the current algebraic specification logics and an increasing number of computing science logics are formalised as institutions.

This work exports one of the most important and powerful classical model theory methods, namely the ultraproducts method [7, 21], to an institution-independent framework. This framework not only clarifies the conditions that are necessary for the development of the ultraproducts method, but also develops a simple but effective institution-independent approach to quantification and logical connectives. In this approach the concept of variable and valuation is presented in a more uniform and much simpler way than in the usual presentations of logic, without the need to distinguish between closed and open formulæ and naturally including higher order variables. We think that this very simple and general institution-independent 'internal logic' is one of the main contributions of this work, reflecting the benefits of the way of doing model theory promoted by the theory of institutions.

Since the categorical definition of the ultraproduct construction, there have been a few abstract model theoretic approaches to ultraproducts, [3] being one of the most representative. If we compare it to [3], our institution-independent approach to ultraproducts is different in many essential aspects. For example, we work with the given sentences of the institution rather than defining a semantics-oriented concept of sentence and satisfaction as in [3, 4] which leads to very complex combinatorial definitions and proofs and does not go beyond first order. Besides gains in simplicity and clarity, our approach make the applications much easier and the understanding of the ultraproducts method smoother. This is a direct consequence of the more fundamental difference of using institutions rather than simple categories as the basic framework for the ultraproducts method. By using institutions rather than categories, we are able to make use of essential model theoretic concepts such as signature morphism and model reduct and expansion (not possible in other abstract model theoretic approaches [3]), and also gets clearer and simpler proofs, in part due to being able to exploit proof theory, model theory, and the relation between them in a flexible way.

1.1. Summary and Contributions of this Work

In the preliminary section, besides briefly reviewing some terminology, concepts, and notations about filters, categories, and institutions, we introduce the novel institution concept of *representable signature morphism* and explore some of its basic properties. Representable signature morphisms can be regarded as an abstract institution-independent formulation of the concept of first-order signature entities (such as variables or constants).

The next section is devoted to an institution-independent study of logical connectives, quantification (in both existential and universal form), and of basic sentences, which are the simplest sentences matching the model theoretic structure of the institution. We show that in the applications, all sentences can be obtained by iteration of some of the logical connectives and some quantification over the basic sentences. This decomposition of the satisfaction relation between models and sentences into satisfaction of basic sentences, of logical connectives, and of quantification, is one of the contributions of this section. While the institution-independent concept of logical connectives is obvious and the concept of basic sentence is based upon a simple form of satisfaction via injectivity in the sense of [4], the key contribution of this section lies in our approach to quantification. In the applications, the latter includes naturally both first-order and higher-order forms of quantification.

The main section of the paper starts by recalling the categorical definition of reduced products and ultraproducts, then studies the interaction between reduced products and model reducts. The latter plays a crucial role for dealing with quantifiers in our institution-independent approach to the fundamental theorem on ultraproducts.

By following the structure of the internal logic introduced in the previous section, our formulation of the institution-independent fundamental theorem on ultraproducts deconstructs this main result on ultraproducts into parts having individual significance. Depending of the actual institution, these parts can be combined in various different ways and can also be used independently for obtaining weaker preservation properties but for a larger class of sentences. This presentation of the main result has also the benefit of enabling a clear perspective on the semantic limits of an actual logic or institution.

The final section is devoted to some applications, such as institution-independent compactness, elementary axiomatizability, or higher-order quantification. The applications are meant only to illustrate in a rather limited way the institution-independent ultraproducts method, the emphasis of this paper being on the fundamentals. Wider applications is topic for further research based on this work.

The concepts introduced and the results obtained are illustrated with examples from four different institutions: first-order predicate logic (with equality), rewriting logic, partial algebra, and hidden algebra for behavioural logic. All these four logics are very briefly presented in the Appendix, mainly for setting up some notation and terminology. The reader is required to have some familiarity with them or else to study the corresponding literature. Although the examples from these actual institutions serve also as application ground for the results of this paper, they are mainly used for helping the understanding of the concepts introduced by this work.

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2. Preliminaries

2.1. Filters and Ultrafilters

In this section we recall the basic concepts and definitions about filters and ultrafilters, restricted to the case of subsets of a set, partially ordered by inclusion. Let I be a nonempty set. We denote by 2^I the set of all subsets of I. Recall that a *filter F over I* is defined to be a set $F \subseteq 2^I$ such that

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$, and
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

A filter F is *proper* if and only if F is not 2^I and it is an *ultrafilter* if and only if

$$X \in F \text{ if and only if } (I \setminus X) \notin F$$

for all $X \in 2^I$. Notice that ultrafilters are proper filters.

A set $S \subseteq 2^I$ *has the finite intersection property* if $J_1 \cap J_2 \cap \cdots \cap J_n \neq \emptyset$ for all $J_1, J_2, \ldots, J_n \in S$. The following classical result is known as the 'Ultrafilter Theorem':

Theorem 2.1. [7] If $S \subseteq 2^I$ has the finite intersection property, then there exists an ultrafilter U over I such that $S \subseteq U$.

2.2. Categories

This work assumes some familiarity with category theory, and generally uses the same notations and terminology as Mac Lane [23], except that composition is denoted by ";" and written in the diagrammatic order. The application of functions (functors) to arguments may be written either normally using parentheses, or else in diagrammatic order without parentheses, or, more rarely, by using sub-scripts or super-scripts. The category of sets is denoted as $\mathbb{S}et$, and the category of categories[1] as $\mathbb{C}at$. The opposite of a category $\mathbb{C}$ is denoted by $\mathbb{C}^{\mathrm{op}}$. The class of objects of a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$; also the set of arrows in $\mathbb{C}$ having the object a as source and the object b as target is denoted as $\mathbb{C}(a,b)$. The isomorphism of objects in categories is denoted by $\simeq$. A *diagram* D in a category $\mathbb{C}$ is just a functor $J \stackrel{D}{\to} \mathbb{C}$ when J is a small category.

For any object a, the comma category $a/\mathbb{C}$ has

- arrows $f \in \mathbb{C}(a,b)$, as objects, and
- arrows $h \in \mathbb{C}(b,b')$ such that $f;h = f'$, as arrows between $f \in \mathbb{C}(a,b)$ and $f' \in \mathbb{C}(a,b')$.

A functor $L\colon\ J' \to J$ is called *final* if for each object $j \in |J|$ the comma category j/L is non-empty and connected. Consequently, a subcategory $J' \subseteq J$ is final when the corresponding inclusion functor is final.

2.2.1. Finiteness

A category J is *directed* if to any two objects i and j there exist arrows $i \to k \leftarrow j$. A limit (colimit) of a functor $D\colon\ J \to \mathbb{C}$ is *directed* if the category J is a directed poset.

We say that an object a in a category $\mathbb{C}$ is *finitely presented* [2] if and only if for any arrow $f\colon\ a \to d$ to the vertex of a colimiting co-cone $\mu\colon\ D \Rightarrow d$ of a directed diagram $D\colon\ J \to \mathbb{C}$ there exists $i \in |J|$ and an arrow $f_i\colon\ a \to D(i)$ such that $f = f_i;\mu_i$. This is equivalent to the fact that the hom-functor $\mathbb{C}(a,-)\colon\ \mathbb{C} \to \mathbb{S}et$ preserves directed colimits. For example, a set is finitely presented in $\mathbb{S}et$ if and only if it is finite, and an algebra is finitely presented in the category of algebras of a signature if and only if it can be presented (in the usual general algebra sense) by finitely many generators and finitely many equations.

2.3. Institutions

In this section besides briefly reviewing some of the basic concepts of institution theory, we also introduce some novel concepts necessary for this work.

From a logic perspective, institutions are much more abstract than Tarski's model theory, and also have another basic ingredient, namely signatures and the possibility of translating sentences and models across signature morphisms. A special case of this translation is familiar in first-order model theory: if $\Sigma \to \Sigma'$ is an inclusion of first-order signatures[2] and M is a Σ'-model, then we can form the *reduct* of M to Σ, denoted $M\!\upharpoonright_\Sigma$. Similarly, if e is a Σ-sentence, we can always view it as a Σ'-sentence (but there is no standard notation for this).

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[1]We steer clear of any foundational problem related to the "category of all categories"; several solutions can be found in the literature, see, for example [23].

[2]Called "languages" in [7].

Institutions formalize the concept of 'logic' from a categorical abstract model-theoretic perspective. The key axiom, called the *satisfaction condition*, says that the meaning of a sentence does not depend on the context in which it is interpreted, which is surely a very basic intuition for classical logic.

Definition 2.1. An *institution* $\Im = (\mathbb{S}ign, Sen, \text{MOD}, \models)$ consists of

1. a category $\mathbb{S}ign$, whose objects are called *signatures*,
2. a functor $Sen\colon\ \mathbb{S}ign \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\text{MOD}\colon\ \mathbb{S}ign^{\text{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-*model homomorphisms*, and
4. a relation $\models_\Sigma \subseteq |\text{MOD}(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign|$, called Σ-*satisfaction*,

such that for each morphism $\varphi\colon\ \Sigma \to \Sigma'$ in $\mathbb{S}ign$, the *satisfaction condition*

$$M' \models_{\Sigma'} Sen(\varphi)(e) \ \text{ iff } \ \text{MOD}(\varphi)(M') \models_\Sigma e$$

holds for each $M' \in |\text{MOD}(\Sigma')|$ and $e \in Sen(\Sigma)$. We may denote the reduct functor $\text{MOD}(\varphi)$ by $\_\!\upharpoonright_\varphi$ and the sentence translation $Sen(\varphi)$ by $\varphi(\_)$. Also, we will sometimes say that the signature morphism φ has a certain property 'P' if $\text{MOD}(\varphi)$ has the property 'P'. When $M = M'\!\upharpoonright_\varphi$, we will say that M' is an *expansion of M along* φ.

Definition 2.2. Let Σ be a signature in an institution $(\mathbb{S}ign, Sen, \text{MOD}, \models)$.

- For each set of Σ-sentences E, let $E^* = \{M \in \text{MOD}(\Sigma) \mid M \models_\Sigma e \text{ for each } e \in E\}$, and
- For each class $\mathcal{M}$ of Σ-models, let $\mathcal{M}^* = \{e \in Sen(\Sigma) \mid M \models_\Sigma e \text{ for each } M \in \mathcal{M}\}$.

If E is a set of sentences and e is a single sentence, then $e \in E^{**}$ is denoted by $E \models e$.

Two models M and M' of the same signature are *elementarily equivalent* (denoted as $M \equiv M'$) if they satisfy the same set of sentences, i.e. $\{M\}^* = \{M'\}^*$.

Two sentences e and e' of the same signature are *semantically equivalent* (denoted as $e \equiv e'$) if they are satisfied by the same class of models, i.e., $\{e\}^* = \{e'\}^*$.

Definition 2.3. In any institution, a class $\mathcal{K}$ of models for a signature is *elementary* if it is closed, i.e., $\mathcal{K}^{**} = \mathcal{K}$.

Remark 2.1. Each elementary class of models is closed under elementary equivalence.

Definition 2.4. Let $(\mathbb{S}ign, Sen, \text{MOD}, \models)$ be an institution. (Σ, E) is a *theory* when Σ is a signature and E is *closed* set of Σ-sentences, i.e., $E = E^{**}$.

A theory E is *presented by* E_0 if $E_0 \subseteq E$ and $E_0 \models e$ for each $e \in E$, and is *finitely presented* if there exists a finite E_0 which presents E.

A *theory morphism* $\varphi\colon\ (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi\colon\ \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Let $\mathbb{T}h$ denote the category of all theories in $\Im$.

Remark 2.2. For any institution, model functor MOD extends from the category of its signatures $\mathbb{S}ign$ to the category of its theories $\mathbb{T}h$, by mapping a theory (Σ, E) to the full subcategory $\mathrm{MOD}(\Sigma, E)$ of $\mathrm{MOD}(\Sigma)$ formed by the Σ-models which satisfy E.

Definition 2.5. [12] A theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is *conservative* if and only if each (Σ, E)-model has an expansion to a (Σ', E')-model, i.e., for each Σ-model M satisfying E, there exists a Σ'-model M' satisfying E' such that $M' \restriction_\varphi = M$.

Example 2.1. An important particular case for this work is that of *conservative* signature morphisms. In conventional model theory, a signature morphism $\varphi : \Sigma \to \Sigma'$ is conservative when φ is injective (on sort, function, and relation symbols) and does not add new operations of sorts that are ‘empty’ (i.e., without terms) in Σ. Consequently, if Σ has only ‘non-empty’ sorts, then each injective signature morphism $\varphi : \Sigma \to \Sigma'$ is conservative.

Definition 2.6. A theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is *liberal* if and only if the reduct functor $\_\restriction_\varphi : \mathrm{MOD}(\Sigma', E') \to \mathrm{MOD}(\Sigma, E)$ has a left-adjoint, denoted $(\_)^\varphi$.

$$\begin{array}{cccccc} M \models_\Sigma E & \quad & M & \longrightarrow & (M^\varphi)\restriction_\varphi & \quad M^\varphi \\ & & {\scriptstyle h}\downarrow & \swarrow {\scriptstyle h'\restriction_\varphi} & & \swarrow \text{ there exists a unique } h' \\ M' \models_{\Sigma'} E' & & M'\restriction_\varphi & & M' & \end{array}$$

Consequently, a signature morphism is *liberal* when it is liberal as theory morphism (between the corresponding empty theories).

While in the actual institutions liberality of theory morphisms is a non-trivial property, the liberality of signature morphisms for typical institution of interest here.

Definition 2.7. An institution $(\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ is *exact* if and only if the model functor $\mathrm{MOD} : \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ preserves finite limits. The institution is *semi-exact* if and only if MOD preserves pullbacks.

Fact 2.1. Consider a semi-exact institution, a pushout of signatures

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi_1} & \Sigma_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi_1'} \\ \Sigma_2 & \xrightarrow[\phi_2']{} & \Sigma' \end{array}$$

and two models, a Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1\restriction_{\phi_1} = M_2\restriction_{\phi_2}$. Then by the semi-exactness, there exists a unique Σ'-model M' such that $M'\restriction_{\phi_1'} = M_1$ and $M'\restriction_{\phi_2'} = M_2$. We call this model the *amalgamation* of M_1 and M_2 and denote it by $M_1 \otimes_{\phi_1,\phi_2} M_2$.

A similar amalgamation concept can also be defined for model homomorphisms.

Exactness properties for institutions formalise the possibility of amalgamating models of different signatures when they are consistent on some kind of ‘intersection’ of the signatures (formalised as a pushout square).

2.3.1. Representable signature morphisms

The topic of this subsection represents a novel categorical generalisation of the concept of first-order variables from conventional logic to the framework of institutions.

Let us have a brief look at the conventional concept of variable in general algebra or first-order logic (Appendix A). Given a signature Σ and a set of variables X for Σ, we may consider the extended signature $\Sigma \cup X$ by regarding the variables as constants. Then each $\Sigma \cup X$-model is just a Σ-model M plus an interpretation of the elements of X into M. But the interpretations of X into M are in canonical bijection with the Σ-model homomorphisms $T_\Sigma(X) \to M$, where $T_\Sigma(X)$ is the free Σ-algebra over X. Therefore, a $\Sigma \cup X$-model is the same with a Σ-model homomorphism $T_\Sigma(X) \to M$ with M a Σ-model. This can be regarded as a categorical property of the signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$, suggesting the following institution-independent definition:

Definition 2.8. Let $(\mathbb{S}ign, Sen, \text{MOD}, \models)$ be an institution. A signature morphism $\phi : \Sigma \to \Sigma'$ is *representable* if and only if there exists a Σ-model M_ϕ (called the *representation of* ϕ) and an isomorphism i_ϕ of categories such that the following diagram commutes:

$$
\begin{array}{ccc}
\text{MOD}(\Sigma') & \xrightarrow{\;i_\phi\;} & (M_\phi / \text{MOD}(\Sigma)) \\
& \searrow^{\text{MOD}(\phi)} & \downarrow \textit{forgetful} \\
& & \text{MOD}(\Sigma)
\end{array}
$$

If the representation M_ϕ is finitely presented in $\text{MOD}(\Sigma)$, we say that ϕ is *finitary representable.*

Informally, this definition says that each Σ'-model is just a Σ-model plus an interpretation of the representation model into the Σ-model.

Example 2.2. In the institution of first-order logic (Appendix A) each extension of signatures $\phi : (S, F, P) \hookrightarrow (S, F \cup X, P)$ only adding constants X to F is representable by the free (S, F, P)-model over the added constants X. If X is finite, then ϕ is finitary representable.

Similarly, signature morphisms only adding constants are also representable in rewriting logic (Appendix B), partial algebra (Appendix C), and hidden algebra (Appendix D).

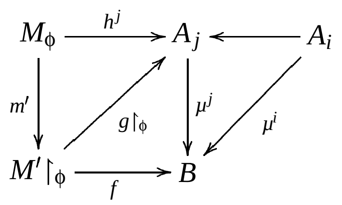
Representable signature morphisms capture exactly the idea of first order variables in an abstract institution-independent setting. Extensions of the signatures with higher order variables (such as sort, operation, relation symbols, etc.) are *not* representable signature morphisms.

Although the next results are not used anywhere in this paper, it shows some basic properties of representable signature morphisms.

Proposition 2.1. The model reduct functor $\text{MOD}(\phi)$ corresponding to a finitary representable signature morphism $\phi : \Sigma \to \Sigma'$ preserve finitely presented models, i.e. $M' \restriction_\phi$ is finitely presented for each finitely presented Σ'-model M'.

Proof:
Let $(J, \leq)$ be a directed poset, let $A : (J, \leq) \to \text{MOD}(\Sigma)$ be a $(J, \leq)$-diagram of Σ-models, and let $\mu : A \Rightarrow B$ be its colimit. Assume a Σ-model homomorphism $f : M' \restriction_\phi \to B$.

Define $m' = i_\phi(M') : M_\phi \to M'\upharpoonright_\phi$ and the Σ'-model $B' = i_\phi^{-1}(m'; f)$. Notice that B' is an expansion of B along ϕ and that $f' = i_\phi^{-1}(f) : M' \to B'$.



On the other hand, because M_ϕ is finitely presented, there exists $i \in J$ such that for each $i \leq j$ there exists $h^j : M_\phi \to A_j$ such that $h^j; \mu^j = m'; f$. This determines a final sub-poset $J_i \subseteq J$ consisting of all elements greater than i. For each $j \in J_i$ let $A'_j = i_\phi^{-1}(h^j)$, hence each A'_j is an expansion of A_j along ϕ. Therefore the sub-diagram of A determined by J_i lifts to a $(J_i, \leq)$-diagram $A' : (J_i, \leq) \to \mathrm{MOD}(\Sigma')$, and also the part of the co-cone μ determined by J_i lifts to a co-cone $\mu' : A' \Rightarrow B'$.

Because $(J_i, \leq)$ is a final sub-poset of $(J, \leq)$, the part of the co-cone μ determined by J_i is still a colimiting co-cone (see [23]), and because the forgetful functor $M_\phi/\mathrm{MOD}(\Sigma) \to \mathrm{MOD}(\Sigma)$ creates directed colimits, we deduce that $\mu' : A' \Rightarrow B'$ is a directed colimit.

Now, because M' is finitely presented we have that there exists $j \in J_i$ and $g : M' \to A'_j$ such that $g; \mu'^j = f'$. Therefore $g\upharpoonright_\phi : M'\upharpoonright_\phi \to A_j$ such that $g; \mu^j = f$. □

The properties described in Proposition 2.2 below are expected composability properties for representable signature morphisms. These properties are stated in two versions: a general one and a 'finitary' one. The informal meaning of the first three items of Proposition 2.2 below is that the 'union of variables' exists, is associative, and has the 'empty set' as identity, while the meaning of the fifth item is that this 'union' is commutative too.

Proposition 2.2. In any institution

1. the composition of [finitary] representable signature morphisms is [finitary] representable,
2. the identity signature morphism is [finitary] representable if and only if the corresponding signature has initial models,
3. if each signature of the institution has initial models then the [finitary] representable signature morphisms form a subcategory of the category of signatures,
4. if $\varphi; \phi$ is [finitary] representable and φ is representable, then ϕ is [finitary] representable, and
5. if the institution is semi-exact and its categories of models have finite coproducts, then the subcategory of [finitary] representable signature morphisms creates pushouts.

Proof:
1. Consider the following representable signature morphisms $\Sigma \xrightarrow{\phi} \Sigma' \xrightarrow{\phi'} \Sigma''$. We show that $\phi; \phi'$ is represented by $M_{\phi'}\upharpoonright_\phi$ where $M_{\phi'}$ is the representation of ϕ'.

For each Σ''-model M'', we define $i_{\phi;\phi'}(M'') = i_{\phi'}(M'')\upharpoonright_\phi$.

On the other hand, given any $m : (M_{\phi'})\upharpoonright_\phi \to M$, we define $i_{\phi;\phi'}^{-1}(m) = i_{\phi'}^{-1}(i_\phi^{-1}(m))$ (notice that $m : h \to h; m$ in $M_\phi/\mathrm{MOD}(\Sigma)$ where $h = i_\phi(M_{\phi'})$).

Moreover, if both ϕ and ϕ' are finitary representable, then by Proposition 2.1 we have that $M_{\phi'}\restriction_{\phi}$ is finitely presented, hence $\phi;\phi'$ is finitary representable.

2. is immediate and 3. follows immediately from 1. and 2.

4. Let $\varphi\colon\ \Sigma \to \Sigma'$ and $\phi\colon\ \Sigma' \to \Sigma''$. Let the Σ-model M_φ be the representation of φ and $M_{\varphi;\phi}$ be the representation of $\varphi;\phi$. Let $m_\phi = i_\varphi(i^{-1}_{\varphi;\phi}(1_{M_{\varphi;\phi}})\restriction_\phi)\colon\ M_\varphi \to M_{\varphi;\phi}$. The conclusion follows by noticing that $M_{\varphi;\phi}/\mathrm{MOD}(\Sigma)$ is isomorphic to $m_\phi/(M_\varphi/\mathrm{MOD}(\Sigma))$.

The finitary case follows by noticing that m_ϕ is finitely presented in $M_\varphi/\mathrm{MOD}(\Sigma)$ as an immediate consequence of the fact that $M_{\varphi;\phi}$ is finitely presented in $\mathrm{MOD}(\Sigma)$.

5. Consider the following pushout of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi_1} & \Sigma_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi'_1} \\ \Sigma_2 & \xrightarrow[\phi'_2]{} & \Sigma' \end{array}$$

We need to prove that ϕ'_1 and ϕ'_2 are representable when ϕ_1 and ϕ_2 are representable.

Let the Σ-model M_{ϕ_1} represent ϕ_1 and the Σ-model M_{ϕ_2} represent ϕ_2. Let $M_{\phi_1} \stackrel{m_1}{\to} M_{\phi_1} + M_{\phi_2} \stackrel{m_2}{\leftarrow} M_{\phi_2}$ be the coproduct of M_{ϕ_1} and M_{ϕ_2}. Then the conclusion follows because

- the following diagram of forgetful functors is a pullback (we leave this simple calculation as exercise to the reader):

$$\begin{array}{ccc} \mathrm{MOD}(\Sigma) & \longleftarrow & M_{\phi_1}/\mathrm{MOD}(\Sigma) \\ \uparrow & & \uparrow \\ M_{\phi_2}/\mathrm{MOD}(\Sigma) & \longleftarrow & m_2/(M_{\phi_2}/\mathrm{MOD}(\Sigma)) \simeq (M_{\phi_1}+M_{\phi_2})/\mathrm{MOD}(\Sigma) \simeq m_1/(M_{\phi_1}/\mathrm{MOD}(\Sigma)) \end{array}$$

  [For the finitary case we notice that $M_{\phi_1} + M_{\phi_2}$ is finitely presented as a coproduct of finitely presented models, which further implies that m_1 and m_2 are finitely presented.] and

- by 4. the mediating morphisms for representable co-cones are [finitary] representable.

□

3. Internal Logic

In this section we define a method of describing the sentences supporting our results on institution-independent ultraproducts. If we fix the institution, these are sentences of this institution rather than being new sentences constructed on top of the sentences of the original institution via some institution-independent building operations such as logical connectives and some form of quantification (such as [36] does for the sentences defining the quasi-varieties of models in arbitrary institutions). This description of the sentences supporting our results on institution-independent ultraproducts will allow us to notice easily that, in most of the institutions used in algebraic specification or computing science theory, these sentences are in fact *all* the sentences of the institution; this gives a wide range of applications to the results of this paper.

At the basic level we have the *basic* sentences which are the basic constituents for the sentences supporting our results on institution-independent ultraproducts. The complex level is obtained from the basic level by iterations of *logical connectives* and an abstract form of *quantification* (both universally and existentially). Although this description might have a strong first-order flavour, it goes well beyond conventional first order logic because of two reasons. On the one hand, this is done in an *arbitrary* institution, almost without any technical restrictions. On the other hand, the level of generality of our concept of quantification is much higher than the conventional quantification with first-order variables, particular conventional cases including second order quantification, for example.

3.1. Basic sentences

In the actual institutions, the basic sentences are the simplest sentences matching the structure of the models of the institution, i.e. which are preserved by the model homomorphisms, and they usually constitute the bricks from which the complex sentences are constructed by using logical connectives and quantification.

Notice that the satisfaction of basic sentences is a particular case of 'injectivity' satisfaction in the sense of [4].

Definition 3.1. Given a signature Σ, a Σ-sentence e is *basic* if there exists a Σ-model M_e such that for each Σ-model M, $M \models_\Sigma e$ if and only if there exists a model homomorphism $M_e \to M$.

We say that an basic sentence e is *finitary* if the model M_e is finitely presented in the category $\mathrm{MOD}(\Sigma)$ of Σ-models.

Remark 3.1. In any institution basic sentences are preserved by model homomorphisms, i.e., $N \models e$ whenever $M \models e$ and there exists a model homomorphism $h : M \to N$.

Example 3.1. *First-order logic.*
In the case of first-order predicate logic with equality (Appendix A), the ground atoms are finitary basic. Recall that a ground atom is either an equality between ground terms or a relation (predicate) with ground terms as arguments.

If we consider a ground equation $(\forall\emptyset)\ t = t'$ for an algebraic signature (S,F), then let $T_{(S,F)}/E$ be the (quotient) initial (S,F)-algebra satisfying $(\forall\emptyset)\ t = t'$. In this case E is the congruence generated by the pair (t,t'). Then, an algebra A satisfies $(\forall\emptyset)\ t = t'$ if and only if there exists a homomorphism $T_{(S,F)}/E \to A$.

If we consider a ground atomic relation $\pi(t_1 \ldots t_n)$ for a first-order logic signature (S,F,P), where $t_1,\ldots,t_n$ is a list of F-terms, then we consider the (S,F,P)-model T such that as an algebra, T is the initial term (S,F)-algebra $T_{(S,F)}$, and which interprets all relation symbols as the empty relation except $T_\pi = \{(t_1,\ldots,t_n)\}$. Then $M \models \pi(t_1 \ldots t_n)$ if and only if there exists a homomorphism $T \to M$, for each (S,F,P)-model M.

Finite conjunctions of ground atoms would also be finitary basic, but in the case of infinitary logic, infinite conjunctions of ground atoms would be only basic.

Example 3.2. *Rewriting logic.*
In the case of the rewriting logic (see Appendix B), the (atomic) ground equations and the ground transitions are finitary basic.

For example, given a ground transition $(\forall\emptyset)\ t\ \ =>\ \ t'$ for a signature Σ, let $(T_\Sigma, \leq)$ be the preorder model where T_Σ is the initial term Σ-algebra, and $\leq$ is the preorder compatible with the Σ-operations generated by the pair (t, t'). One can notice easily that for each preorder model M, $M \models_\Sigma (\forall\emptyset)\ t\ \ =>\ \ t'$ if and only if there exists a preorder model homomorphism $(T_\Sigma, \leq) \to M$.

Example 3.3. *Partial algebra.*
In the case of partial algebra (Appendix C), we show that the strong ground equations are basic. Let $(\forall\emptyset)\ t \overset{s}{=} t'$ be a strong ground (S, TF, PF)-equation for a partial algebraic signature (S, TF, PF). By Proposition C.1, consider the initial total $TF \cup PF \cup \perp$-algebra $T_{TF\cup PF\cup\perp, \Gamma\cup\{t=t'\}}$ for the theory $\Gamma \cup \{(\forall\emptyset)\ t\ \texttt{=}\ t'\}$. Its corresponding partial (S, TF, PF)-algebra by Proposition C.1 is (the total algebra) $T_{TF\cup PF, E}$, where E is the $TF \cup PF$-congruence generated by (t, t') on the initial $TF \cup PF$-algebra $T_{TF\cup PF}$.

Notice that, by Proposition C.1, for each partial algebra A, $A \models (\forall\emptyset)\ t \overset{s}{=} t'$ if and only if $\overline{A} \models (\forall\emptyset)\ t\ \texttt{=}\ t'$ if and only if there exists a total $(TF \cup PF \cup \perp)$-homomorphism $\overline{h}:\ T_{TF\cup PF\cup\perp, \Gamma\cup\{t=t'\}} \to \overline{A}$ if and only if there exists a partial (S, TF, PF)-homomorphism $h:\ T_{TF\cup PF, E} \to A$.

Finally, a strong ground equation is finitary basic if the signature is finite. The reason for this is that in this case Γ is finite and therefore $T_{TF\cup PF\cup\perp, \Gamma\cup\{t=t'\}}$ is finitely presented, which means that $T_{TF\cup PF, E}$ is finitely presented and thus finitely presented in the category of partial (S, TF, PF)-algebras.

In the weak version of partial algebra, the ground existence or strong equations are basic by considering the initial partial algebra too.

3.2. Logical connectives

The institution-independent approach to logical connectives is straightforward. We only give here the definitions for negation and conjunction because all other logical connectives can be generated from these.

Definition 3.2. Given a signature Σ in an institution $\mathfrak{I}$,

the Σ-sentence $\neg e$ is the *negation* of e when $M \models_\Sigma \neg e$ if and only if $M \not\models_\Sigma e$, for each Σ-model M, and

the Σ-sentence $e \wedge e'$ is the *conjunction* of the Σ-sentences e and e' when $M \models_\Sigma e \wedge e'$ if and only if ($M \models_\Sigma e$ and $M \models_\Sigma e'$) for each Σ-model M.

The institution

has negation if and only if for each $e \in Sen(\Sigma)$ there exists $e' \in Sen(\Sigma)$ such that $e' \equiv \neg e$, and

has conjunction if and only if for each $e', e'' \in Sen(\Sigma)$ there exists $e \in Sen(\Sigma)$ such that $e \equiv e' \wedge e''$

for each signature Σ.

The institution-independent semantics of other logical connectives, such as disjunction, implication, equivalence, etc. can be defined directly in a same way. This can also be extended to infinitary versions of the logical connectives, such as infinitary conjunctions and infinitary disjunctions.

3.3. Quantifiers

Definition 3.3. Given a signature morphism $\chi : \Sigma \to \Sigma'$, a Σ-sentence $(\forall\chi)e'$ is *universal χ-quantification* of the Σ'-sentence e' if and only if for each Σ-model M

$$M \models_\Sigma (\forall\chi)e' \text{ if and only if } (M' \models_{\Sigma'} e' \text{ for all } \Sigma'\text{-models } M' \text{ with } M'\restriction_\chi = M)$$

Existential quantification $(\exists\chi)e'$ can be defined similarly by replacing 'all' by 'some' in the definition of the universal quantification.

This very abstract and general concept of quantification, introduced first time by [36], for example, in the particular case of classical model theory includes the second order quantification. Notice that this internalisation of the quantification does not use the ordinary concepts of open formulæ and valuations (of unbounded variables), but rather considers the "variables" as part of the signature and treats the "valuations" as model expansions along the signature extension defined by the addition of the "variables" to the signature. This is exactly what happens in applications because each valuation of variables into a model can be regarded as an expansion of the model to the signature extended with the variables. Otherwise said, for quantification we need only to mark a part of the signature over which the quantification is done. Although this way of thinking about variables and quantification is well known in conventional mathematical logic [33, 21] it is quite rare in the usual presentations of classical logic.

Notice that although in classical one sorted first order logic universal and existential quantifiers are interdefinable by $(\exists\chi)\rho \equiv \neg(\forall\chi)\neg\rho$, in general this is not always true.

Remark 3.2. In each institution $((\exists\chi)\rho)^* \subseteq (\neg(\forall\chi)\neg\rho)^*$ for each Σ'-sentence ρ and each signature morphism $\chi : \Sigma \to \Sigma'$, but

$(\exists\chi)\rho$ and $\neg(\forall\chi)\neg\rho$ are semantically equivalent if and only if χ is conservative.

Example 3.4. Given a signature (S,F,P) in first order logic, the ordinary first order quantification by a set X of variables is the same with the χ-quantification, where $\chi : (S,F,P) \hookrightarrow (S,F\cup X,P)$. Notice that in this case χ is representable (Definition 2.8), and is finitary representable when X is finite.

The cases when $\chi : (S,F,P) \hookrightarrow (S',F',P')$ is an arbitrary signature inclusion correspond to the second order quantification by the operations $F' \setminus F$ and predicates (relations) $P' \setminus P$ when $S = S'$, and extends also to sort quantification when $S \subseteq S'$. 'Weak' second order quantification only over *finite* subsets of the models can be obtained by enriching the signatures with (1-ary) symbols denoting finite subsets of models and, and consequently the models have to interpret them accordingly.

Quantifications higher than second order can be modelled by Definition 3.3 provided that the classical concept of first order predicate logic signature is extended in order to accommodate symbols denoting higher order structures.

While quantification in rewriting logic (Appendix B) and hidden algebra (Appendix D) are modelled by Definition 3.3 in the same way as in first-order predicate logic, some special notice is needed for the case of partial algebra.

Given a partial algebra signature (S,TF,PF) (Appendix C), the ordinary first order quantification by a set X of variables is the same with the χ-quantification, where $\chi : (S,TF,PF) \hookrightarrow (S,TF\cup X,PF)$ is the signature inclusion. Notice that the variables X are treated as total rather than partial constant symbols because the valuations of the variables in partial algebra are total. This is possible due to having explicit declarations for total operations as part of the partial algebra signatures.

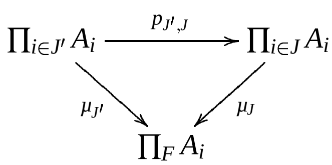
4. Model Ultraproducts in Institutions

4.1. Categorical reduced products

The reduced product construction from classical model theory (see [7]) has been probably defined categorically for the first time in [26] and has been intensively used in abstract model theoretic [3, 4] or categorical logic [24, 25] works. The equivalence between the category theoretic and the set theoretic definitions of the reduced products is shown in [19]. Let us recall here the category theoretic definition of the reduced products:

Definition 4.1. Let $\mathbb{C}$ be a category with small products and directed colimits. Consider a family of objects $\{A_i\}_{i\in I}$. Each filter F over the set of indices I determines a functor $A_F: F \to \mathbb{C}$ such that $A_F(J \subset J') = p_{J',J}: \prod_{i\in J'} A_i \to \prod_{i\in J} A_i$ for each $J, J' \in F$ with $J \subset J'$, and with $p_{J',J}$ being the canonical projection.

Then the *reduced product of* $\{A_i\}_{i\in I}$ *modulo* F is the colimit $\mu: A_F \Rightarrow \prod_F A_i$ of the functor A_F.



If F is ultrafilter then the reduced product modulo F is called an *ultraproduct.*

Remark 4.1. Notice that F is a directed poset, hence under the assumptions of Definition 4.1 the reduced products always exist.

Example 4.1. For each signature in first-order predicate logic (Appendix A), rewriting logic (Appendix B), partial algebra (Appendix C), or hidden algebra (Appendix D), its category of models has reduced products.

In all these cases, the forgetful functor from the category of models to the category of many-sorted sets mapping each model to its underlying carrier creates small products and directed colimits. While this observation is obvious in the case of the products, in the case of the directed colimits it is a direct consequence of the finiteness of the arities of the operation or relation symbols of the signature (see Proposition 2, Chapter IX of [23] for the case of [varieties of] many-sorted algebra).

Notice also that in the case of the partial algebras, this argument is obtained via Proposition C.1.

Definition 4.2. Let $G: \mathbb{C}' \to \mathbb{C}$ be a functor and F be a filter over a set I. Then

- *G preserves the reduced product* $\mu': B_F \Rightarrow \prod_F B_i$ (for $\{B_i\}_{i\in I}$ a family of objects in $\mathbb{C}'$), if $\mu' G: B_F; G \Rightarrow \prod_F G(B_i)$ is also a reduced product in $\mathbb{C}$ of $\{G(B_i)\}_{i\in I}$, and
- *G lifts the reduced product* $\mu: A_F \Rightarrow \prod_F A_i$ (for $\{A_i\}_{i\in I}$ a family of objects in $\mathbb{C}$), if for each object B in $\mathbb{C}'$ such that $G(B) = \prod_F A_i$, there exists $\{B_i\}_{i\in I}$ a family of objects in $\mathbb{C}'$ such that $G(B_i) = A_i$ for each $i \in I$ and there exists a reduced product $\mu': B_F \Rightarrow B$ such that $\mu' G = \mu$.

Given a class $\mathcal{F}$ of filters, we say that functor *preserves/lifts $\mathcal{F}$-reduced products* if it preserves/lifts all reduced products modulo F for each filter $F \in \mathcal{F}$.

In general, in the applications, the preservation of reduced products is an easy property that holds naturally without other conditions.

Fact 4.1. Any functor preserving small products and directed colimits preserves reduced products.

Example 4.2. Any signature morphism in first-order predicate logic (Appendix A), rewriting logic (Appendix B), partial algebra (Appendix C), and hidden algebra (Appendix D), preserve[3] the reduced products of models.

The model products are preserved by the signature morphisms because in all these institutions the signature morphisms are liberal and all limits are preserved by right-adjoint functors [23].

The directed colimits of models are created, and thus preserved, by the signature morphisms by the generalisation of the argument that the forgetful functors from the categories of models to the categories of many-sorted sets mapping each model to its underlying carrier creates directed colimits (see Example 4.1).

Although in the literature there are quite established concepts of lifting of colimits (see [1] for example), there seems to be no standard categorical notion for the lifting notion of Definition 4.2. Also, by contrast to the preservation of the reduced products, in general, only a restricted class of signature morphisms lift the reduced products in the applications. The following result gives a general class of signature morphisms that lift the reduced products in any institution.

Proposition 4.1. In any institution the finitary representable conservative signature morphisms lift all reduced products.

Proof:
Consider a finitary representable conservative signature morphism $\phi\colon\ \Sigma \to \Sigma'$. Let M_ϕ be the Σ-model representing ϕ. Recall that there exists an canonical isomorphism i_ϕ of categories such that the following diagram commutes:

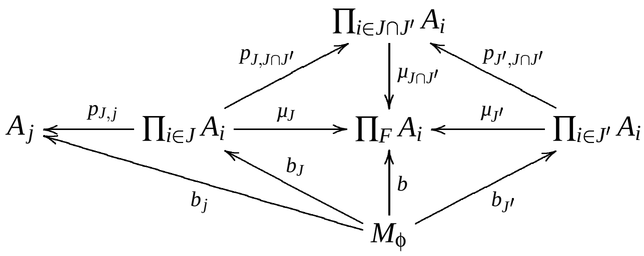
$$\begin{array}{ccc} \mathrm{MOD}(\Sigma') & \xrightarrow{\ i_\phi\ } & (M_\phi/\mathrm{MOD}(\Sigma)) \\ & \searrow^{\mathrm{MOD}(\phi)} & \downarrow \mathit{forgetful} \\ & & \mathrm{MOD}(\Sigma) \end{array}$$

Consider a family $\{A_i\}_{i\in I}$ of Σ-models and a filter F over I. Let $\mu\colon\ A_F \Rightarrow \prod_F A_i$ be the corresponding reduced product and let B be a Σ'-model such that $B\upharpoonright_\phi = \prod_F A_i$.

Let $i_\phi(B) = b\colon\ M_\phi \to \prod_F A_i$. Because M_ϕ is finitely presented, there exists $J \in F$ and $b_J\colon\ M_\phi \to \prod_{i\in J} A_i$ such that $b_J;\mu_J = b$. For each $j \in J$, let $b_j = b_J; p_{J,j}$, where $p_{J,j}\colon\ \prod_{i\in J} A_i \to A_j$ is the projection from the product to its j-th component. Then we define $B_j = i_\phi^{-1}(b_j)$ for each $j \in J$ and, because ϕ is conservative, let B_i be an arbitrary expansion of A_i to a Σ'-model if $i \notin J$. Let $b_i = i_\phi(B_i)$ for each $i \in I$, and for each $J' \in F$ let (by the universal property of the product) $b_{J'}\colon\ M_\phi \to \prod_{i\in J'} A_i$ be the unique arrow such that $b_{J'}; p_{J',i} = b_i$ for each $i \in J'$. If we show that μ is a colimiting co-cone defining the reduced

[3]Notice that for reasons of simplicity of terminology we inaccurately attribute the preservation of reduced products of models to signature morphisms rather than to their corresponding model reduct functors. Such abbreviation of terminology has been introduced more generally by Definition 2.1.

product $b = \prod_F b_i$ in $M_\phi/\mathrm{MOD}(\Sigma)$, then this proposition is proved because of the canonical isomorphism between $M_\phi/\mathrm{MOD}(\Sigma)$ and $\mathrm{MOD}(\Sigma')$.



We first show that $\mu:\ b_F \Rightarrow b$ is a co-cone, where $b_F:\ F \to M_\phi/\mathrm{MOD}(\Sigma)$ is the functor with $b_F(J') = b_{J'}$ for each $J' \in F$ and with $b_F(J' \subseteq J'') = p_{J'',J'}$. Consider an arbitrary $J' \in F$. Then $b_{J'};\mu_{J'} = b_{J'};p_{J',J\cap J'};\mu_{J\cap J'} = b_{J\cap J'};\mu_{J\cap J'} = b_J;p_{J,J\cap J'};\mu_{J\cap J'} = b_J;\mu_J = b$. (Notice that here we have used the crucial fact that $J\cap J' \in F$ because both $J, J' \in F$ and each filter is closed under finite intersections.)

Now consider another co-cone $\nu:\ b_F \Rightarrow b'$ with $b':\ M_\phi \to A'$. By the forgetful functor $M_\phi/\mathrm{MOD}(\Sigma) \to \mathrm{MOD}(\Sigma)$, $\nu:\ A_F \Rightarrow A'$ is a co-cone, therefore by the colimit property in $\mathrm{MOD}(\Sigma)$, there exists a unique $h:\ \prod_F A_i \to A'$ such that $\mu;h = \nu$. All we still have to prove is that $h:\ b \to b'$ in $M_\phi/\mathrm{MOD}(\Sigma)$. But we have that $b;h = b_J;\mu_J;h = b_J;\nu_J = b'$. □

Example 4.3. Any signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$ in first-order predicate logic (Appendix A), rewriting logic (Appendix B), partial algebra (Appendix C), and hidden algebra (Appendix D) lifts the reduced products of models where X is a finite set of arbitrary constants and when X does not introduce a constant on a sort which does not have terms in Σ.

Such signature inclusions are finitary representable by the free Σ-model over X, and the fact that X does not introduce a constant on a sort which does not have terms in Σ guarantees that $\Sigma \hookrightarrow \Sigma \cup X$ is conservative too.

4.2. The fundamental theorem

For this section we assume a fixed institution $(\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ such that all its categories of models have small products and directed colimits.

Definition 4.3. Let $\mathcal{F}$ be a class of filters. For each signature Σ, a Σ-sentence e is

- *preserved by $\mathcal{F}$-reduced factors* if $\prod_F A_i \models_\Sigma e$ implies $\{i \in I \mid A_i \models_\Sigma e\} \in F$,
- *preserved by $\mathcal{F}$-reduced products* if $\{i \in I \mid A_i \models_\Sigma e\} \in F$ implies $\prod_F A_i \models_\Sigma e$, and

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{A_i\}_{i\in I}$ of Σ-models.

A sentence is a *Łoś-sentence* when is preserved by all ultrafactors and all ultraproducts.

The following theorem is the fundamental result of this paper.

Theorem 4.1. For any class $\mathcal{F}$ of filters,

1. The basic sentences are preserved by all reduced products.

2. The finitary basic sentences are preserved by all reduced products and all reduced factors.
3. The set of sentences preserved by $\mathcal{F}$-reduced products is closed under existential χ-quantification, when χ is conservative and preserves $\mathcal{F}$-reduced products.
4. The set of sentences preserved by $\mathcal{F}$-reduced factors is closed under existential χ-quantification, when χ lifts $\mathcal{F}$-reduced products.
5. The set of sentences preserved by $\mathcal{F}$-reduced factors and the set of sentences preserved by $\mathcal{F}$-reduced products are both closed under conjunction.
6. The set of sentences preserved by $\mathcal{F}$-reduced products is closed under infinite conjunctions.
7. If a sentence is preserved by $\mathcal{F}$-reduced factors then its negation is preserved by $\mathcal{F}$-reduced products.

And finally, if we further assume that $\mathcal{F}$ contains only ultrafilters,

8. If a sentence is preserved by $\mathcal{F}$-reduced products then its negation is preserved by $\mathcal{F}$-reduced factors.
9. The set of sentences preserved by both $\mathcal{F}$-reduced products and factors is closed under negation.

Proof:
1. Let F be any filter over I and let $\{A_i\}_{i\in I}$ be a family of of Σ-models for a signature Σ.

Let e be a basic sentence and consider $J = \{i \in I \mid A_i \models_\Sigma e\}$. There exists a model homomorphism $M_e \to A_i$ for each $i \in J$, therefore by the universal property of the products, there exists a model homomorphism $M_e \to \prod_{i\in J} A_i$. When composing this with $\prod_{i\in J} A_i \xrightarrow{\mu_J} \prod_F A_i$, we get a model homomorphism $M_e \to \prod_F A_i$, which implies that $\prod_F A_i \models e$.

2. Consider a finitary basic Σ-sentence e. By 1. we have to prove only that e is preserved by reduced factors. If $\prod_F A_i \models e$, then there exists a model homomorphism $M_e \to \prod_F A_i$. Since M_e is finitely presented, there exists a model homomorphism $M_e \to \prod_{i\in J} A_i$ for some nonempty $J \in F$, which, by the product projections, means that $A_i \models e$ for all $i \in J$. Therefore $\{i \in I \mid A_i \models_\Sigma e\} \in F$ because $J \subseteq \{i \in I \mid A_i \models_\Sigma e\}$.

3. Let $\chi\colon\ \Sigma \to \Sigma'$ be a signature morphism which is conservative and preserves reduced products. Let e' be a Σ'-sentence preserved by reduced products, and let e be an existential χ-quantification of e'. Consider a filter $F \in \mathcal{F}$ over a set I, and let $\{A_i\}_{i\in I}$ be a family of Σ-models such that $J = \{i \in I \mid A_i \models_\Sigma e\} \in F$. We have to prove that $\prod_F A_i \models_\Sigma e$.

For each $i \in J$ let B_i be a Σ'-model such that $B_i\upharpoonright_\chi = A_i$ and $B_i \models_{\Sigma'} e'$. Because χ is conservative, for each $i \notin J$, let B_i be a Σ'-model such that $B_i\upharpoonright_\chi = A_i$. Because e' is preserved by reduced products and because $J \subseteq \{i \in I \mid B_i \models_{\Sigma'} e'\}$ and F is filter, we have that $\prod_F B_i \models_{\Sigma'} e'$. Because χ preserves reduced products, we have that $(\prod_F B_i)\upharpoonright_\chi = \prod_F A_i$, which implies that $\prod_F A_i \models_\Sigma e$.

4. Let $\chi\colon\ \Sigma \to \Sigma'$ be a signature morphism which lifts reduced products. Let e' be a Σ'-sentence preserved by reduced factors, and let e be an existential χ-quantification of e'. Consider a filter $F \in \mathcal{F}$ over a set I, and let $\{A_i\}_{i\in I}$ be a family of Σ-models such that $\prod_F A_i \models_\Sigma e$. We have to prove that $\{i \in I \mid A_i \models_\Sigma e\} \in F$.

Let B be a χ-expansion of $\prod_F A_i$ such that $B \models_{\Sigma'} e'$. Because χ lifts reduced products, for each $i \in I$ there exists a Σ'-model B_i such that $B_i\upharpoonright_\chi = A_i$ and such that $\prod_F B_i = B$. Because e' is preserved by reduced factors, $J = \{i \in I \mid B_i \models_{\Sigma'} e'\} \in F$. But $J \subseteq \{i \in I \mid A_i \models_\Sigma e\}$, therefore $\{i \in I \mid A_i \models_\Sigma e\} \in F$ because F is filter.

5. This follows from the following:

- $\{i \in I \mid A_i \models e\} = \{i \in I \mid A_i \models e'\} \cap \{i \in I \mid A_i \models e''\}$,
- $J' \cap J'' \in F$ if and only if $J', J'' \in F$, and
- $\prod_F A_i \models e$ if and only if $\prod_F A_i \models e'$ and $\prod_F A_i \models e''$.

where the sentence e is the conjunction of e' and e'' in a signature Σ, $F \in \mathcal{F}$ is a filter over a set I, and $\{A_i\}_{i\in I}$ is any family of Σ-models.

6. Given a signature Σ, for each family $\{e_l\}_{l\in L}$ of Σ-sentences preserved by $\mathcal{F}$-reduced products, assume that $\{i \in I \mid A_i \models e_l \text{ for each } l \in L\} \in F$, where $F \in \mathcal{F}$ is any filter over a set I and $\{A_i\}_{i\in I}$ is any family of Σ-models. Then for each $l \in L$, $\{i \in I \mid A_i \models e_l\} \supseteq \{i \in I \mid A_i \models e_l \text{ for each } l \in L\} \in F$, thus $\{i \in I \mid A_i \models e_l\} \in F$, therefore $\prod_F A_i \models e_l$ for each $l \in L$.

7. Let e be the negation of a Σ-sentence e' for a signature Σ such that e' is preserved by $\mathcal{F}$-reduced factors. Let F be any filter in $\mathcal{F}$ over a set I and let $\{A_i\}_{i\in I}$ be a family of models such that $J = \{j \in I \mid A_j \models e\} \in F$.

We have to prove that $\prod_F A_i \models e$. If we assume the contrary, it means that $\prod_F A_i \models e'$. Since e' is preserved by $\mathcal{F}$-reduced factors, $J' = \{j \in I \mid A_j \models e'\} \in F$. Because F is a proper filter $J \cap J' \in F$ is not empty, hence we can find j such that $A_j \models e$ and $A_j \models e'$, which is impossible.

8. Let e be the negation of e' such that e' is preserved by $\mathcal{F}$-reduced products. Let F be any ultrafilter in $\mathcal{F}$ over a set I and let $\{A_i\}_{i\in I}$ be a family of models such that $\prod_F A_i \models e$. If $\{j \in I \mid A_j \models e\} \notin F$ then its complement $\{j \in I \mid A_j \models e'\}$ belongs to F (because F is ultrafilter). Because e' is preserved by $\mathcal{F}$-reduced products, this would imply $\prod_F A_i \models e'$ which contradicts $\prod_F A_i \models e$, therefore $\{j \in I \mid A_j \models e\} \in F$.

9. From 7. and 8. □

The following Corollary can be regarded as an institution-independent generalisation of the so-called 'Fundamental Theorem on Ultraproducts' for first-order predicate logic [7], originally due to Łoś [22].

Corollary 4.1. The Łoś-sentences contain all finitary basic sentences and are closed under logical connectives and any χ-quantification for which χ is conservative and preserves and lifts reduced products.

Proof:
Although this Corollary follows directly from Theorem 4.1, a special notice is needed for the case when the institution does not have negations of its sentences. This is needed because the universal quantification can be expressed in terms of existential quantification and negation, and all logical connectives can also be expressed in terms of conjunctions and negation.

For example, consider $(\forall\chi)e$ an universal χ-quantification of a Łoś-sentence e. Then $(\forall\chi)e \equiv \neg(\exists\chi)\neg e$ in the closure $\mathfrak{I}^{\neg}$ under negation[4] of the original institution $\mathfrak{I}$. Therefore $(\forall\chi)e$ is a Łoś-sentence in $\mathfrak{I}^{\neg}$, which implies that it is a Łoś-sentence in the institution $\mathfrak{I}$ too. □

Corollary 4.1 can be specialised by using Proposition 4.1:

Corollary 4.2. The Łoś-sentences contain all finitary basic sentences and are closed under logical connectives and any χ-quantification for which χ is conservative finitary representable and preserves reduced products.

[4]Obtained by adding the negations of all sentences if necessary.

Definition 4.4. An institution is a *Łoś-institution* if and only if all its sentences are Łoś-sentences.

Example 4.4. First-order logic (Appendix A), rewriting logic (Appendix B), and partial algebra (Appendix C) are Łoś-institutions. This follows from Corollary 4.2 by noticing that:

- each sentence of first-order and rewriting logic and each strong equation in partial algebra is obtained from the finitary basic sentences (cf. Examples 3.1,3.2,3.3) by conjunction, implication, and universal quantification and, the existence equations in partial algebra are obtained via Proposition C.1 from finitary basic sentences by conjunction, negation, and universal quantification, where
- the quantification of all these sentences is finitary representable (cf. Example 3.4 and Example 4.3) and preserves the reduced products (cf. Example 4.2).

The same observation holds when the sentences of those institutions are extended to full first-order sentences built on top of the corresponding basic sentences.

On the other hand, in general, the behavioural sentences in the hidden algebra institution (Appendix D) are not Łoś-sentences. This is due to the infinitary nature of the behavioural satisfaction, because each ground behavioural equation $(\forall\emptyset)\ t \sim t'$ is semantically equivalent to the set of (universally quantified strict) equations $\{(\forall X)\ c[z/t] = c[z/t'] \mid c$ visible behavioural context$\}$ where X denotes the set of the variables of c.

However, we can prove that existentially quantified behavioural unconditional equations (also called *behavioural queries*) [16] are preserved by ultraproducts. This is a consequence of the fact that the universally quantified (strict) equations are Łoś-sentences and of 6. and 3. of Theorem 4.1 for $\mathcal{F}$ the class of ultrafilters.

5. Some Applications

In this section, the institutions are implicitly assumed to have small products and directed colimits for their categories of models.

5.1. Σ^1_1 sentences

Recall [7] that a Σ^1_1-sentence in first-order logic is a second-order sentence all of whose relation and operation quantifiers occur at the beginning and are existential. The following Definition generalises the concept of Σ^1_1-sentence to any institution:

Definition 5.1. In any institution, e is a Σ^1_1-sentence if it is an existential χ-quantification of a Łoś sentence, where χ is any conservative reduced products preserving signature morphism.

The following Corollary follows from Theorem 4.1 and can be regarded as an institution-independent generalisation of the result of [7] stating that the Σ^1_1-sentences in first-order logic (Appendix A) are preserved by ultraproducts. Notice that, unlike in the particular case of first-order logic, this result follows *directly* from the fundamental theorem 4.1 due to our general concept of quantification.

Corollary 5.1. In any institution each Σ^1_1-sentence is preserved by ultraproducts.

We encourage the interested readers to explore the significance of the Σ^1_1-sentences in other institutions of interest other than first-order logic.

5.2. Compactness

The following result is not only an institution-independent generalisation of the compactness via ultraproducts result of [14] (see also [7]), but it is also obtained for a more general class of sentences, which for example in the particular case of classical logic (Appendix A) include a class of second-order sentences (see Section 5.1) or in the case of hidden algebra (Appendix D) include behavioural queries (cf. Example 4.4).

Theorem 5.1. In any institution, let E be a set of sentences preserved by ultraproducts. Let I be the set of all finite subsets of E. Consider a model A_i for each finite subset $i \in I$. Then there exists an ultraproduct $\prod_U A_i$ such that $\prod_U A_i \models E$.

Proof:
Let $S = \{\{i \in I \mid \rho \in i\} \mid \rho \in E\}$. S has the finite intersection property because

$$\{\rho_1, \rho_2, \dots, \rho_n\} \in \{i \in I \mid \rho_1 \in i\} \cap \{i \in I \mid \rho_2 \in i\} \cap \dots \cap \{i \in I \mid \rho_n \in i\}$$

By the 'Ultrafilter Theorem' 2.1, let U be an ultrafilter such that $S \subseteq U$.

For each $\rho \in E$, we have that $\{i \in I \mid \rho \in i\} \subseteq \{i \in I \mid A_i \models \rho\}$. This means that $\{i \in I \mid A_i \models \rho\} \in U$. Because ρ is preserved by ultraproducts, it implies that $\prod_U A_i \models \rho$. Because $\rho \in E$ is arbitrary, it follows that $\prod_U A_i \models E$. □

Corollary 5.2. Let E be a set of sentences preserved by ultraproducts, and let e be a sentence preserved by ultrafactors such that $E \models e$. Then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.

Proof:
Let us assume the contrary, i.e., that for each finite $i \subseteq E$, $i \not\models e$. This means that there exist models A_i such that $A_i \models i$ but $A_i \not\models e$.

Let I be the set of all finite subsets of E. By Theorem 5.1, there exists an ultraproduct such that $\prod_U A_i \models E$. Therefore $\prod_U A_i \models e$. Because e is preserved by ultrafactors, $\{i \in I \mid A_i \models e\} \in U$. But $\{i \in I \mid A_i \models e\} = \emptyset$ which is a contradiction since as ultrafilter U is a proper filter. □

Definition 5.2. An institution is *compact* if for each set of sentences E and each sentence e, if $E \models e$ then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.

Corollary 5.3. Any Łoś-institution is compact.

Example 5.1. By Example 4.4, first-order logic, rewriting logic, and partial algebra are compact. These results are expected because, for example, first-order logic and rewriting logic are complete. The compactness results involving Σ_1^1-sentences are probably less usual (see Corollary 5.1): "if $E \models e$ where E is a set of Σ_1^1-sentences and e is any sentence, then there exists a finite subset $E' \subseteq E$ such that $E' \models e$."

5.3. Axiomatizability

The results (and proofs) of this section are institution-independent generalisations of the basic axiomatizability results in first-order logic of [14] (see also [7]).

Theorem 5.2. Consider an institution that has negation and conjunction and such that each sentence is preserved by ultraproducts. Then a class of models is elementary if and only if it is closed under ultraproducts and elementary equivalence.

Proof:
The implication that any elementary class of models is closed under elementary equivalence and ultraproducts follows immediately from Remark 2.1 and the fact that each sentence is preserved by ultraproducts.

For the opposite implication, consider a class of models $\mathcal{K}$ closed under ultraproducts and elementary equivalence. Let $E = \mathcal{K}^*$. We prove that $\mathcal{K} = \mathrm{MOD}(\Sigma, E)$.

Let $B \in \mathrm{MOD}(\Sigma, E)$. Consider I the set of the finite subsets of $\{B\}^*$. For each $i \in I$, there exists $A_i \in \mathcal{K}$ such that $A_i \models i$. (Otherwise for all $A \in \mathcal{K}$, $A \models \neg(e_1 \wedge \cdots \wedge e_n)$, where $i = \{e_1, \ldots, e_n\}$, which implies that $\neg(e_1 \wedge \cdots \wedge e_n) \in E$, which further implies that $B \models \neg(e_1 \wedge \cdots \wedge e_n)$ which contradicts the fact that $B \models e_1 \wedge \cdots \wedge e_n$.) By Theorem 5.1, there exists an ultrafilter U over I such that $\prod_U A_i \models \{B\}^*$. This implies that $\prod_U A_i \equiv B$ (otherwise if there exists a sentence e such that $\prod_U A_i \models e$ but $B \not\models e$, then $B \models \neg e$ and therefore $\prod_U A_i \models \neg e$ which is a contradiction). Because $\mathcal{K}$ is closed under ultraproducts and elementary equivalence, it follows that $B \in \mathcal{K}$. □

Corollary 5.4. Consider an institution which has negation and conjunction and each sentence is preserved by ultraproducts. Then a class of models for a signature is the class of models of a finitely presented theory if and only if both it and its complement are elementary.

Proof:
If $E = \{e_1, \ldots, e_n\}$ is a finite set of Σ-sentences, then the complement of $\mathrm{MOD}(\Sigma, E)$ is $\mathrm{MOD}(\Sigma, \neg(e_1 \wedge \cdots \wedge e_n))$.

For the opposite implication, consider $\mathrm{MOD}(\Sigma, E)$ an elementary class of models such that its complement is also elementary. We show by that there exists $E_0 \subseteq E$ finite such that $\mathrm{MOD}(\Sigma, E) = \mathrm{MOD}(\Sigma, E_0)$. If we assume the opposite, then for each $E_0 \subseteq E$ finite there exists a model A in the complement of $\mathrm{MOD}(\Sigma, E)$ such that $A \models E_0$. Let $I = \{E_0 \subseteq E \mid E_0 \text{ finite}\}$. Because each sentence is preserved by ultraproducts, by Theorem 5.1, there exists an ultraproduct $\prod_U A_i$ over I such that $\prod_U A_i \models E$ and $A_i \notin \mathrm{MOD}(\Sigma, E)$ and $A_i \models i$ for each $i \in I$. But because the complement of $\mathrm{MOD}(\Sigma, E)$ is closed under ultraproducts, we also get that $\prod_U A_i \notin \mathrm{MOD}(\Sigma, E)$, which is a contradiction. □

6. Conclusions and Future Research

We generalised the ultraproducts method from classical model theory to an institution-independent framework based on a very general institution-independent treatment of quantification, logical connectives, and basic sentences (simplest sentences preserved by model homomorphisms). We showed some immediate applications of the fundamental theorem on ultraproducts, such as institution-independent compactness, axiomatizability, and Σ^1_1-sentences. We illustrated the concepts and results of our work with examples from four different logics or institutions.

Our development of the institution-independent ultraproducts method also leads to several novel concepts in the theory of institutions, such as

- representable signature morphisms - used to abstract the concept of first-order variables to institutions, and
- a general institution-independent treatment of quantifiers, naturally including higher-order quantifiers - resulting in a simpler presentation of logics without open formulæ, valuations of free variables, etc.

This work opens up many future research directions, we mention several of them:

- Extend the area of the institution-independent applications of the ultraproducts method started in this paper by non-trivial generalisation of other results from classical or non-classical model theory.

  A recent important example is given by a novel institution-independent proof of Craig interpolation theorem [9] in dependence of a very general and abstract form of Birkhoff-style axiomatizability property of the actual institution. In that work the theory of institution-independent reduced products plays a crucial role.

  Another recently developed application consists of an institution-independent proof of preservation of modal sentences by frame ultraproducts, done for a very general possible worlds semantics [13].
- Study of the model theoretic properties of the various institutions in use in algebraic specification and computing science by applying the institution-independent ultraproducts method.
- Further explore the significance of our internal logic, especially our approach to quantification, and apply it for exporting other methods from classical model theory to an institution-independent framework.

  The ideas on internal logic already play a central important role in [8], they have also been used for the institution-independent approach to possible worlds semantics and modalities [13], which also extends our internal logic with various modal operators.

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In the Appendix we give very brief presentations of a number of institutions which are used in this paper as examples for illustrating some of the concepts introduced by this work and some of the applications of the main results. Although we assume some familiarity with these institutions, the reader is encouraged to consult the recommended references for more details. Also, some notations and terminology used in some sections of the Appendix rely on notations and terminology from previous sections.

A. First-order Logic

The role of this very brief presentation of (many-sorted) first-order logic with equality is mainly for fixing some notations and conventions. A detailed definition of the first-order logic institution can be found in [15].

Recall that a (many-sorted) signature in first-order logic is a tuple (S,F,P) where S is the set of sorts, F is the set of (S-sorted) operation symbols, and P is the set of (S-sorted) relation symbols. By $F_{w\to s}$ we denote the set of operations with arity w and sort s, and by P_w we denote the set of relations with arity w.

Given a signature (S,F,P), a *model* M of first-order logic interprets:

- each sort s as a set M_s,
- each operation symbol $\sigma \in F_{w\to s}$ as a function $M_\sigma : M_w \to M_s$, where M_w stands for $M_{s_1} \times \cdots \times M_{s_n}$ for $w = s_1 \ldots s_n$, and
- each relation symbol $\pi \in P_w$ as a relation $M_\pi \subseteq M_w$.

Any *ground* (i.e., without variables) F-term $t = \sigma(t_1 \dots t_n)$, where σ is an operation symbol and $t_1, \dots, t_n$ are subterms, gets interpreted as an element M_t in a (S,F,P)-model M by $M_t = M_\sigma(M_{t_1} \dots M_{t_n})$.

A (S,F,P)-*model homomorphism* $h : M \to M'$ is an indexed family of functions $\{h_s : M_s \to M'_s\}_{s \in S}$ such that

- h is a F-algebra homomorphism $M \to M'$, i.e., $h(M_\sigma(m)) = M'_\sigma(h(m))$ for each $\sigma \in F_{w \to s}$ and each $m \in M_w$,[5] and
- $h(m) \in M'_\pi$ if $m \in M_\pi$ for each relation $\pi \in P_w$ and each $m \in M_w$.

The sentences are the well-known first-order closed formulæ (including equations), and their satisfaction by the models is the well-known Tarskian satisfaction (see [15, 7] for details). In the many-sorted case we restrict the quantification of the sentences to *conservative* sets of variables for a signature, i.e., which do not have variables for the 'empty' sorts, which are sorts not having terms in the signature. This condition on the quantification in the many-sorted case was noticed for the first time in the context of the completeness of many-sorted equational logic [17].

A *signature morphism* $\phi = (\phi^{\mathrm{sort}}, \phi^{\mathrm{op}}, \phi^{\mathrm{rel}}) : (S,F,P) \to (S',F',P')$ consists of a function between the sets of sorts $\phi^{\mathrm{sort}} : S \to S'$, a function between the sets of operation symbols $\phi^{\mathrm{op}} : F \to F'$, and a function between the sets of relation symbols $\phi^{\mathrm{rel}} : P \to P'$ such that $\phi^{\mathrm{op}}(F_{w \to s}) \subseteq F'_{\phi^{\mathrm{sort}}(w) \to \phi^{\mathrm{sort}}(s)}$ and $\phi^{\mathrm{rel}}(P_w) \subseteq P'_{\phi^{\mathrm{sort}}(w)}$ for any string of sorts $w \in S^*$ and each sort $s \in S$.[6]

Given a signature morphism $\phi : (S,F,P) \to (S',F',P')$, the *reduct* $M'\restriction_\phi$ of a (S',F',P')-model M' is defined by $(M'\restriction_\phi)_s = M'_{\phi^{\mathrm{sort}}(s)}$ for each sort $s \in S$, $(M'\restriction_\phi)_\sigma = M'_{\phi^{\mathrm{op}}(\sigma)}$ for each operation symbol $\sigma \in F$, and $(M'\restriction_\phi)_\pi = M'_{\phi^{\mathrm{rel}}(\pi)}$ for each relation symbol $\pi \in P$.

The *sentence translation* along ϕ of any sentence is defined inductively on the structure of the sentences by replacing the symbols from (S,F,P) with symbols from (S',F',P') as defined by ϕ.

Notice that by discarding the relational part, we get the many-sorted algebra institution with full first-order equational sentences.

B. Rewriting Logic

Rewriting logic [28] is emerging as one of the most important new algebraic specification logics. Here we refer to a simplified variant of rewriting logic which is used for defining the CafeOBJ institution [11], however this example can be extended to the original definition of rewriting logic without any difficulty.

Recall (from [11]) that our rewriting logic signatures are just ordinary (many-sorted) algebraic signatures. The models are *preorder models* which are (algebraic) interpretations of the signatures into $\mathbb{P}re$ (the category of preorders) rather than in $\mathbb{S}et$ (the category of sets) as in the case of ordinary algebras. More precisely, given a signature (S,F) a model M interprets:

- each sort s as a preorder M_s, and
- each operation $\sigma \in F_{w \to s}$ as a preorder functor $M_\sigma : M_w \to M_s$, where M_w stands for $M_{s_1} \times \dots \times M_{s_n}$ for $w = s_1 \dots s_n$.

[5] By $h(m)$ we mean in fact $h_w(m)$, where $h_w : M_w \to M'_w$ is the canonical component-wise extension of h.
[6] For any string of sorts $w = s_1 \dots s_n$, by $\phi^{\mathrm{sort}}(w)$ we mean the string of sorts $\phi^{\mathrm{sort}}(s_1) \dots \phi^{\mathrm{sort}}(s_n)$.

The *sentences* are either ordinary equations or *transitions*, both in their unconditional or conditional form. For example, the unconditional Σ-transitions for a signature F, are sentences of the form

$$(\forall X)\, t \;=>\; t'$$

where X is a conservative many-sorted set of variables for (S,F) and t,t' are Σ-terms with variables X. Conditional sentences in rewriting logic are universally quantified implications where the hypotheses are finite conjunctions of transitions or equations and the conclusion is a transition or an equation.

The signature morphisms, the model reducts, and the sentence translations along signature morphisms are defined in the same way with ordinary (many-sorted) algebra (Appendix A).

A preorder model M *satisfies* a transition $M \models (\forall X)\, t \;=>\; t'$, if and only if $M'_t \leq M'_{t'}$ for each expansion M' of M along the signature inclusion $(S,F) \hookrightarrow (S,F \cup X)$. The satisfaction of conditional sentences extends the satisfaction of equations and transitions to the conditional case; we leave this as exercise to the reader.

More details of this institution of rewriting logic can be found in [11], while [28] has the details of the institution of full rewriting logic.

C. Partial Algebra

There are many approaches to partial algebra, two classical references being [6, 30]. Our formalisation of the partial algebra institution is tailored to the needs of this paper but without affecting the logic and model theory of partial algebra.

A *partial algebraic signature* is a pair (S,TF,PF), where TF is the set of the total operations and PF is the set of the partial operations.[7] A *partial* (S,TF,PF)*-algebra* A is just like a $TF \cup PF$-algebra but interpreting the operations of PF as partial functions rather than total functions. A *homomorphism* $h\colon\ A \to B$ between partial algebras, is a family of partial functions $\{h_s\colon\ A_s \rightharpoonup B_s\}_{s\in S}$ indexed by the set of sorts S of (S,TF,PF) such that either both $h(A_\sigma(a))$ and $B_\sigma(h(a))$ are undefined or they are defined and equal, for each operation $\sigma \in (\Sigma \cup PF)_{w\to s}$ and each argument $a \in A_w$.[8]

The *interpretation* A_t of a $TF \cup PF$-ground term t in a partial (S,TF,PF)-algebra is defined inductively by

- A_t is undefined if A_{t_k} is undefined for some $k \in \{1,\dots,n\}$ or $(A_{t_1},\dots,A_{t_n})$ does not belong to the definition domain of A_σ, otherwise
- $A_t = A_\sigma(A_{t_1},\dots,A_{t_n})$.

where $t = \sigma(t_1 \dots t_n)$ is a term with σ any (S,TF,PF)-operation and $t_1,\dots,t_n$ subterms.

Signature morphisms, model reducts, and sentence translations are defined similarly to the case of the total algebra (see Appendix A).

The sentences are either *strong* or *existence* equations, both in their conditional or unconditional form. For any unconditional strong $(TF \cup PF)$-equation $(\forall X)\, t \overset{s}{=} t'$, where X is a conservative many-sorted set of variables for (S,TF,PF), a partial (S,TF,PF)-algebra A satisfies it if and only if

- A'_t and $A'_{t'}$ are both undefined, or

[7] In this notation we ignore the set of sorts, which are of course common to the total and the partial operations.
[8] Notice that by convention $h(a)$ is defined if and only if is defined on all components of a.

- A'_t and $A'_{t'}$ are both defined and $A'_t = A'_{t'}$.

for each expansion A' of the partial algebra A along the signature inclusion $(S,TF,PF) \hookrightarrow (TF \cup X, PF)$.

For any unconditional existence $(TF \cup PF)$-equation $(\forall X)\ t \overset{e}{=} t'$, where X is a many-sorted set of variables for (S,TF,PF), a partial (S,TF,PF)-algebra A satisfies it if and only if

- A'_t and $A'_{t'}$ are both defined and $A'_t = A'_{t'}$.

for each expansion A' of the partial algebra A along the signature inclusion $(S,TF,PF) \hookrightarrow (TF \cup X, PF)$. These definitions extend without any problems to the conditional case. We leave it as exercise to the reader.

The following result show how this version of partial algebra is equivalent to an equationally defined class (i.e. variety) of total algebras, which is very useful for establishing some properties of partial algebras. We omit here its straightforward proof.

Proposition C.1. For any partial algebra signature (S,TF,PF) with S the set of sorts, let $\perp = \{\perp_s\}_{s \in S}$ be an indexed set of new constant symbols and let Γ be set of the equations

$$(\forall x_1 \dots \forall x_n)\sigma(x_1 \cdots \perp_s \dots x_n) = \perp_{s'}$$

for all operations $\sigma \in \Sigma \cup PF$.

Then the functor mapping each partial (S,TF,PF)-algebra A to the total $(TF \cup PF \cup \perp, \Gamma)$-algebra $\overline{A}$ such that

- $\overline{A}_s = A_s \cup \{\perp_s\}$ for each sort $s \in S$,
- for each operation $\sigma \in \Sigma \cup PF$, $\overline{A}_\sigma(a) = A_\sigma(a)$ if a belongs to the definition domain of A_σ, and
- $\overline{A}_\sigma(a) = \perp_s$ otherwise, where s is the sort of σ,

and mapping each partial algebra homomorphism $h : A \to B$ to the total algebra homomorphism $\overline{h} : \overline{A} \to \overline{B}$ such that for each sort s,

- $\overline{h}_s(a) = h_s(a)$ if a belongs to the definition domain of h_s, and
- $\overline{h}_s(a) = \perp_s$ otherwise.

is an isomorphism between the category of partial (S,TF,PF)-algebras and the category of total $(TF \cup PF \cup \perp, \Gamma)$-algebras.

Moreover,

$$A \models_{S,TF,PF} (\forall X)\ t \overset{s}{=} t' \text{ iff } \overline{A} \models_{TF \cup PF \cup \perp} (\forall X)\ t = t'$$

for each strong equation $(\forall X)\ t \overset{s}{=} t'$, and

$$A \models_{S,TF,PF} (\forall X)\ t \overset{e}{=} t' \text{ iff } \overline{A} \models_{TF \cup PF \cup \perp} ((\forall X)\ t = t' \text{ and } \neg(\exists X) t = \perp)$$

for each existence equation $(\forall X)\ t \overset{e}{=} t'$.

We may also consider a weak variant of partial algebra which adopts weak homomorphisms between partial algebras. A *weak homomorphism* between partial algebras $A \to B$ preserves but not necessarily reflects definedness, i.e. $h : A \to B$ is a weak homomorphism when it consists of an S-indexed family of total functions such that $A_\sigma(a)$ defined implies $B_\sigma(h(a))$ defined and equal to $h(A_\sigma(a))$ for each operation $\sigma \in (TF \cup PF)_{w \to s}$ and each $a \in A_w$.

D. Hidden Algebra

Hidden algebra is the institution underlying behavioural specification, which is one of the most important new algebraic specification formalisms. In the literature there are several versions of hidden algebra, with only slight technical differences between them [10, 20, 18]. Here we adopt a slightly modified version of *coherent hidden algebra* (abbreviated *CHA*) of [10].

A *CHA signature* is a tuple (H,V,F,F^{b}), where

- H and V are disjoint sets of *hidden* sorts and *visible* sorts, respectively,
- F is a $H \cup V$-sorted signature,
- $F^{\mathrm{b}} \subseteq F$ is a subset of *behavioural operations* such that $\sigma \in F^{\mathrm{b}}_{w \to s}$ has *exactly* one hidden sort in w.

A CHA model M for a signature (H,V,F,F^{b}) is just an ordinary F-algebra.

CHA sentences can be ordinary (strict) equations, *behavioural equations* (both in conditional or unconditional format), or *coherence declarations* (see [10, 11] for details). Recall ([10, 11]) that coherence declarations are semantically equivalent to conditional behavioural equations and that the strict equations are treated in the same way as in the case of the ordinary algebra. An unconditional *behavioural equation* is a sentence of the form

$$(\forall X)\, t \ \sim\ t'$$

where X is a conservative set of variables and t,t' are F-terms over X.

Recall that a *F-context* $c[z]$ is any F-term c with a marked variable z occurring only once in c. A context $c[z]$ is *behavioural* iff all operations above[9] z are behavioural.

Given a F-algebra A, two elements (of the same sort s) a and a' are called *behaviourally equivalent*, denoted $a \sim_s a'$ (or just $a \sim a'$), iff $A^a_c = A^{a'}_c$ for each *visible* behavioural context c, where A^a and $A^{a'}$ are any expansions of A along the signature inclusion $F \hookrightarrow F \cup Y$, where Y is the set of variables of c, and such that $A^a_y = A^{a'}_y$ for each $y \in Y \setminus \{z\}$, $A^a_z = a$, and $A^{a'}_z = a'$.

Then, a F-algebra A satisfies an (unconditional) behavioural equation $A \models (\forall X)\, t \ \sim\ t'$, iff $A'_t \sim A'_{t'}$ for each A' expansion of the algebra A along the signature inclusion $F \hookrightarrow F \cup X$.
This definition extends without any problems to the conditional case. We leave it as exercise to the reader.

Recall also that a *CHA signature morphism* $\phi :\ (H,V,F,F^{\mathrm{b}}) \to (H',V',F',F'^{\mathrm{b}})$ is an many-sorted signature morphism $(H \cup V,F) \to (H' \cup V',F')$ such that

(M1) $\phi(V) \subseteq V'$ and $\phi(H) \subseteq H'$,
(M2) $\phi(F^{\mathrm{b}}) = F'^{\mathrm{b}}$ and $\phi^{-1}(F'^{\mathrm{b}}) \subseteq F^{\mathrm{b}}$,

Finally, model reducts and sentence translations along CHA signature morphisms are the same with those from ordinary many-sorted algebra (Appendix A).

[9]Meaning that z is in the subterm determined by the operation.