On quasi-varieties of multiple valued logic models

Răzvan Diaconescu[*1]

[1] Institute of Mathematics "Simion Stoilow" of the Romanian Academy

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We extend the concept of quasi-variety of first-order models from classical logic to multiple valued logic (MVL) and study the relationship between quasi-varieties and existence of initial models in MVL. We define a concept of 'Horn sentence' in MVL and based upon our study of quasi-varieties of MVL models we derive the existence of initial models for MVL 'Horn theories'.

1 Introduction

Multiple valued logic (abbreviated MVL and also known as 'many-valued' or 'multi-valued' logic) has a long tradition [17, 15, 7] and needs no presentation. In this paper we lift the concept of quasi-variety of first-order models from classical (two valued) logic to MVL and study some of its important properties. Our results can be seen as MVL extensions or generalizations of corresponding classical results.

In the classical two valued framework quasi-varieties of models have been studied rather extensively by pioneers such as Mal'cev [16]; in their algebraic version they also play an important role in general or universal algebra [10]. Because of their close relationship with initiality and because of the great role played by the latter concept in programming language semantics [9, 13] and logic programming (there known as 'least Herbrand models') [14], they have also been studied in computing science motivated works such as [18] within the very general categorical abstract model theoretic framework of the so-called 'theory of institutions' of Goguen and Burstall [8]. A more recent upgraded study of quasi-varieties at the level of abstract institutions can be found in [4].

The motivation for our investigation of quasi-varieties of MVL models and of their relationship to the existence of initial models (of theories) may be seen from two different but complementary directions.

- From the side of the theory of quasi-varieties, there is a legitimate interest to investigate the scope and limits of its most important concepts and results, how they apply to various less conventional frameworks. This motivation may be regarded as of pure theoretical nature.
- From the side of MVL, recently there have been efforts to develop its own first order model theory. However as the current literature seems to indicate (for example [11, 2, 6]) MVL first order model theory may still be at its beginning development stages, at least when one compares it with the classical one [1, 12]. In particular, in spite of its high relevance for formal specification or logic programming, the algebraic flavored style of model theory, such as the theory of (quasi-)varieties, seems to be absent from these developments. Our work can also be seen from the perspective of trying to fill this gap.

Understanding these motivations together seem to be very much on the side of the kind of relationship between the conventional logic and fuzzy logic research interests, that Petr Hájek advocates in his monograph [11].

The work reported here can be developed and presented in two different ways. One way would be by making use of the powerful and sophisticated institution-independent model theory machinery of [4]; however this would have made the presentation of this work rather heavy. We have therefore chosen a second way, that of staying

* Corresponding author: e-mail: Razvan.Diaconescu@imar.ro

within the concrete MVL framework, already quite abstract (but in a different way than institution theory). We believe that our choice of development fits better the current MVL/fuzzy logic culture, and therefore may be more useful for readers in the area.

The contents of our paper is as follows:

1. A preliminary section recalls the concept of residuated lattice and fixes notations and terminology for the MVL model theory framework used by our work.

2. In the next section we extend the concept of quasi-variety of models from classical logic to MVL, prove that each quasi-variety of MVL models has a reachable initial model, and prove a reciprocal of the above mentioned result. The domain of this reciprocal is however restricted to the classes of models of MVL 'theories'.

3. The last technical section introduces a concept of MVL 'Horn sentence', that involves also the residual connector characteristic to MVL, and proves that models of 'Horn theories' form a quasi-variety. Consequently we obtain that Horn theories admit initial models. From the perspective of fuzzy logic programming [19, 6] our 'Horn theories' correspond to (fuzzy) logic programs and our existence of initial model result corresponds to the 'least Herbrand model' construction there. So, this may be regarded as an alternative way to obtain the (same) denotational semantics for fuzzy logic programming, a way which we regard as more structural and which bridges the gap towards the formal (algebraic) specification culture (see [3] for a general algebraic oriented approach to logic programming).

2 Preliminaries

In this section we recall some basic notions from MVL model theory and fix our framework.

2.1 Residuated lattices

The characteristic feature of MVL, which appears explicitly in the terminology 'multiple valued', is that the truth values are not only the two classical ones, `true` and `false`, but they may be many. While in classical logic the truth values are structured as a Boolean algebra, in MVL these are structured as a 'residuated lattice' [20, 11, 5]. Recall from [11] that a *residuated lattice* L is a bounded lattice (with $\leq$ denoting the underlying partial order that has binary infimum $\wedge$, binary supremum $\vee$, biggest $\top$ and least $\perp$ elements) and which comes equipped with an additional commutative and associative binary operation $\otimes$ which has $\top$ as identity and such that for all elements x, y and z

1. $(x \otimes -)$ is monotonic, i.e. $(x \otimes y) \leq (x \otimes z)$ if $y \leq z$, and

2. there exists an element $x \Rightarrow z$ such that $y \leq (x \Rightarrow z)$ if and only if $x \otimes y \leq z$.

Other works (such as [5]) may define residuated lattices slightly more generally, for example without the commutativity of $\otimes$. From a category theoretic perspective the condition 1. just reads that $x \otimes -$ is a functor on the partial order $(L, \leq)$, and the condition 2. that this has a left adjoint $x \Rightarrow -$. For this reason, condition 2. is sometimes referred to as the *adjunction condition*.

The ordinary two valued situation of classical logic can be recovered when L is the two valued Boolean algebra with $\otimes$ being the conjunction. Then $\Rightarrow$ is the ordinary Boolean implication. There is a myriad of interesting examples of residuated lattices used for multiple valued logics for which $\otimes$ gets an interpretation rather different from the ordinary conjunction. One famous such example is the so-called *Łukasiewicz arithmetic conjunction* on the closed real numbers interval $[0, 1]$ defined by $x \otimes y = 1 - \max\{0, x + y - 1)\}$. In this example $x \Rightarrow y = \min\{1, 1 - x + y\}$.

For this work we consider that our residuated lattice are *complete*, i.e. each set X of elements of L has an infimum, denoted $\bigwedge X$. This implies that each $X \subseteq L$ has a supremum too, denoted $\bigvee X$.

2.2 MVL **model theory**

Let us fix a complete residuated lattice L.

An MVL *signature* is a tuple (C, P) with C set of symbols of *constants* and $P = (P_n)_{n\in\omega}$ a family of sets of *relation* symbols, with P_n denoting the set of symbols of arity n. Note that here we have not considered operation symbols other than constants. The only reason for this is the simplicity of the presentation, all our results can be presented without any significant additional effort within more refined frameworks including operation symbols, and even (crisp of similarity) equality, and many sortedness.

Given a signature (C, P), the set $\mathsf{Sen}(C, P)$ of the (C, P)-*sentences* is the least set

- containing $\perp$ and $\top$,
- containing the set $\mathsf{At}(C, P)$ of the (C, P)-*atoms* $\pi(c)$ where $n \in \omega$, $\pi \in P_n$ and $c \in C^n$,
- which is closed under the binary operators $\wedge$, $\vee$, $\Rightarrow$, and $\otimes$ and
- under universal and existential quantification by finite sets of variables (new constants), i.e. if ρ is a $(C \cup X, P)$-sentence then $(\forall X)\rho$ and $(\exists X)\rho$ are $(C \cup X, P)$-sentences.

For any signature (C, P), a (C, P)-*model* M consists of

- an underlying set, denoted M_s,
- an interpretation of each constant symbol $c \in C$ as an element $M_c \in M_s$, and
- an interpretation of each relation symbol $\pi \in P_n$ as a function $M_\pi : \ M_s^n \to L$.

As a matter of notation, for each $(c_1, \ldots, c_n) \in C^n$ we let $M_{(c_1,\ldots,c_n)}$ denote the tuple $(M_{c_1}, \ldots, M_{c_n})$.

When $C \subseteq C'$, we say that a (C', P)-model M' is a (C', P)-*expansion* of a (C, P)-model M when $M'_x = M_x$ for each $x \in C$ or $x \in P_n$ (for each $n \in \omega$). Alternatively, we say that M is the (C, P)-*reduct* of M'.

A (C, P)-*model homomorphism* $h : \ M \to N$ is function $h : \ M_s \to N_s$ such that

- $h(M_c) = N_c$ for each $c \in C$, and
- $M_\pi(m) \leq N_\pi(h^n(m))$ for each $\pi \in P_n$ and $m \in M_s^n$ (where $h^n(m)$ denotes the n-tuple $(h(m_1), \ldots, h(m_n))$ for $m = (m_1, \ldots, m_n)$).

M is a *sub-model* of N when $M_s \subseteq N_s$ and this inclusion is a homomorphism $M \to N$.

The satisfaction relation between models and sentences is a relation with four arguments that besides the signatures, the models, and the sentences it also involves the elements of the residuated lattice L. This is defined by induction on the structure of the sentences in the style of Tarski by means of the following an auxiliary mapping that for a given (C, P)-model M evaluates the 'truth value' of each (C, P)-sentence:

- $M[\perp] = \perp$ and $M[\top] = \top$,
- $M[\pi(c)] = M_\pi(M_c)$ for each atom $\pi(c)$,
- $M[\rho_1 \star \rho_2] = M[\rho_1] \star M[\rho_2]$ for any (C, P)-sentences ρ_1 and ρ_2 and each $\star \in \{\wedge, \vee, \otimes, \Rightarrow\}$,
- $M[(\forall X)\rho'] = \bigwedge\{M'[\rho'] \mid M' \text{ is } (C \cup X, P)\text{-expansion of } M\}$ for any $(C \cup X, P)$-sentence ρ', and
- $M[(\exists X)\rho'] = \bigvee\{M'[\rho'] \mid M' \text{ is } (C \cup X, P)\text{-expansion of } M\}$ for any $(C \cup X, P)$-sentence ρ'.

Then for any (C, P)-model M, any (C, P)-sentence ρ, and any $y \in L$, we define

$$M \models^y_{(C,P)} \rho \text{ if and only if } y \leq M[\rho].$$

Remark 2.1 Our definitions of MVL models, sentences, and satisfaction between them follow with those of the literature (such as [11] which is one of the basic references in the area of MVL/fuzzy model theory), however besides being based upon the same concept of 'truth degrees' ($M[\rho]$ in our notation) our satisfaction relation $\models$ is in addition parameterized by the elements of the residuated lattice. This is more a notational difference rather than a conceptual one, in the framework of [11] and of other works this being however recuperated at the level of theories.

3 Quasi-varieties versus initiality for MVL models

This section is structured as follows.

1. We introduce the concept of quasi-variety for MVL models.
2. We show that each quasi-variety of MVL models admits initial models.
3. We show a kind of reciprocal of the former item, which roughly means that the classes of models of 'theories' that admit initial models are quasi-varieties.

All these extend corresponding concepts and results from classical model theory or universal algebra to MVL.

Definition 3.1 (Closed homomorphisms) In any MVL signature (C, P), a model homomorphism $h: M \to N$ is *closed* if and only if for each $n \in \omega$, $\pi \in P_n$, and $m \in M_s^n$, we have $M_\pi(m) = N_\pi(h^n(m))$.

Consequently, M is a *closed sub-model* of N when M is a sub-model of N such that the corresponding inclusion homomorphism $M \to N$ is closed.

Note that in [6] our closed homomorphisms are called 'full homomorphisms'.

The following recalls the concept of categorical isomorphism within the particular framework of MVL models.

Definition 3.2 (Model isomorphisms) In any MVL signature (C, P), a model homomorphism $f: M \to N$ is *isomorphism* when it has an inverse, i.e. $g: N \to M$ such that both $f \circ g$ and $g \circ f$ are identities. The existence of isomorphisms $M \to N$ is denoted by $\simeq$.

The rather straightforward proofs of the following useful facts are omitted.

Fact 3.1 *A model homomorphism $h: M \to N$ is an isomorphism if and only if it is closed and $h: M_s \to N_s$ is a bijective function.*

Fact 3.2 *The satisfaction relation is invariant under model isomorphisms, i.e. if $M \simeq N$, then for each $y \in L$ and each sentence ρ, we have that $M \models^y_{(C,P)} \rho$ if and only if $N \models^y_{(C,P)} \rho$.*

Definition 3.3 (Direct products of models) Let $(M^i)_{i\in I}$ be any family of models for an MVL signature (C, P). A (C, P)-model M together with a family of homomorphisms $(p^i: M \to M^i)_{i\in I}$ is a *direct product* of $(M^i)_{i\in I}$ when for any other (C, P)-model N and family of homomorphisms $(q^i: N \to M^i)_{i\in I}$ there exists an unique homomorphism $h: N \to M$ such that $p^i \circ h = q^i$ for each $i \in I$.

$$\begin{tikzcd} M \arrow[r, "p^i"] & M^i \\ N \arrow[u, "h"] \arrow[ur, "q^i"'] & \end{tikzcd}$$

For those readers familiar with basic category theory, the definition above just gives the concept of (categorical) product in the category of the (C, P)-models. From general category theory, or simply directly from Dfn. 3.3, it follows that direct products (of MVL models) are unique up to isomorphisms. The following result shows their existence.

Proposition 3.4 (Existence of direct products of models) *For any* MVL *signature (C, P), any family of (C, P)-models has direct products.*

Proof. Let $(M^i)_{i\in I}$ be any family of (C, P)-models. We define a direct product $(p^i: M \to M^i)_{i\in I}$ as follows.

– M_s is the cartesian product $\prod_{i\in I} M_s^i$; for each $i \in I$ let us denote by $p^i: M_s \to M_s^i$ the corresponding projection,
– for each $c \in C$ we let $M_c = (M_c^i)_{i\in I}$, and
– for each $\pi \in P_n$ and each $m \in M_s^n$ we let $M_\pi(m) = \bigwedge_{i\in I} M_\pi^i(m^i)$, where $m^i = (p^i)^n(m)$ for each $i \in I$.

Note that each p^i is a homomorphism since by the infimum property we have that $M_\pi(m) \leq M^i_\pi((p^i)^n(m))$.

Now let us consider any other family of homomorphisms $(q^i : N \to M^i)_{i\in I}$. The condition $p^i \circ h = q^i$ determines an unique $h : N \to M$ by $h(x) = (q^i(x))_{i\in I}$; it remains to show that this is indeed a homomorphism $N \to M$. Because each q^i is homomorphism we have that $N_\pi(m) \leq M^i_\pi((q^i)^n(m))$ for each $\pi \in P_n$ and $m \in N^n$. Hence $N_\pi(m) \leq \bigwedge_{i\in I} M^i_\pi((q^i)^n(m)) = \bigwedge_{i\in I} M^i_\pi((p^i)^n(h^n(m))) = M_\pi(h^n(m))$. □

The definition below extends the well know concept of quasi-variety of models from classical logic to multiple valued logic.

Definition 3.5 (Quasi-varieties of MVL models) Let (C, P) be an MVL signature. A class $\mathcal{Q}$ of (C, P)-models is a *quasi-variety* if and only if it is closed under direct products and closed sub-models.

Closure under direct products means that for any family $(M^i)_{i\in I}$ if each $M^i \in \mathcal{Q}$ then any direct product of the family belongs to $\mathcal{Q}$ too. Closure under closed sub-models means that if M is a closed sub-model of N and $N \in \mathcal{Q}$ then $M \in \mathcal{Q}$ too. In the above definition it is crucial that the sub-models considered are closed, just plain sub-models instead would not do for the developments of our results.

Definition 3.6 (Reachable models) A (C, P)-model M is *reachable* when $M_s = \{M_c \mid c \in C\}$.

The following recalls the categorical concept of initiality for MVL models.

Definition 3.7 (Initial models) A (C, P)-model M is *initial* for a class $\mathcal{Q}$ of (C, P)-models when for each $N \in \mathcal{Q}$ there exists an unique homomorphism $M \to N$.

Fact 3.3 *If they belong to $\mathcal{Q}$, then any two initial models for $\mathcal{Q}$ are isomorphic.*

The following extends a well known result from classical model theory and universal algebra.

Proposition 3.8 *Each quasi-variety $\mathcal{Q}$ of* MVL *models contains a reachable initial model for $\mathcal{Q}$.*

Proof. For any model M of an MVL signature (C, P) let us denote by $\mathcal{R}(M)$ its *reachable closed sub-model* defined by

- $\mathcal{R}(M)_s = \{M_c \mid c \in C\}$,
- $\mathcal{R}(M)_c = M_c$ for each $c \in C$, and
- $\mathcal{R}(M)_\pi(m) = M_\pi(m)$ for each $\pi \in P_n$ and each $m \in \mathcal{R}(M)^n_s$.

Let us denote by I the isomorphism classes of $\{\mathcal{R}(M) \mid M \in \mathcal{Q}\}$. Then I is a *set* since it is less than the number of partitions of C times $\prod_{n\in\omega}\prod_{\pi\in P_n}[C^n \to L]$ where $[C^n \to L]$ is the set of all functions $C^n \to L$.

For each $i \in I$ we pick a model $A_i \in i$ and consider a direct product $(p^i : A \to A_i)_{i\in I}$ of the family $(A_i)_{i\in I}$. We define the candidate initial model for $\mathcal{Q}$ as $\mathcal{R}(A)$.

Since each $\mathcal{R}(M)$ is a closed sub-model of M, we have that $\{\mathcal{R}(M) \mid M \in \mathcal{Q}\} \subseteq \mathcal{Q}$. Note that quasi-varieties are closed under isomorphisms because any model that is isomorphic to a model M can be considered as a direct product of the family consisting only of the model M. It follows that each A_i belongs to $\mathcal{Q}$ and further that $A \in \mathcal{Q}$. Since $\mathcal{R}(A)$ is a closed sub-model of A it follows that $\mathcal{R}(A) \in \mathcal{Q}$.

Now let us consider any $M \in \mathcal{Q}$. Let i be the isomorphism class of $\mathcal{R}(M)$. A homomorphism $\mathcal{R}(A) \to M$ can be obtained as the composition of the following homomorphisms

$$\mathcal{R}(A) \longrightarrow A \xrightarrow{p^i} A_i \simeq \mathcal{R}(M) \longrightarrow M.$$

The uniqueness of the homomorphism $h : \mathcal{R}(A) \to M$ follows from the reachability of $\mathcal{R}(A)$ which implies that as homomorphism h is constrained to $h(\mathcal{R}(A)_c) = M_c$ for each $c \in C$. □

The remaining part of this section is devoted to a reciprocal of Prop. 3.8. This reciprocal is significantly harder than Prop. 3.8 and is restricted to classes of models of MVL 'theories'. For this we introduce the following notation.

Notation 3.9 For each signature (C, P), each set $\Gamma \subseteq L \times \mathsf{Sen}(C, P)$, and each set $E \subseteq C \times C$ let $\mathsf{Mod}_{(C,P)}(\Gamma, E)$ denote the class of (C, P)-models M satisfying $M \models^y_{(C,P} \rho$ for each $(y, \rho) \in \Gamma$ and $M_{c1} = M_{c2}$ for each $(c1, c2) \in E$. When E is empty we may denote $\mathsf{Mod}_{(C,P)}(\Gamma, \emptyset)$ simply by $\mathsf{Mod}_{(C,P)}(\Gamma)$.

Remark 3.10 Note that the subsets $\Gamma \subseteq L \times \mathsf{Sen}(C,P)$ correspond to 'fuzzy theories' $T: \mathsf{Sen}(C,P) \to L$ of the literature MVL ([6] for example) . This goes as follows: for each Γ as above we define $T(\Gamma): \mathsf{Sen}(C,P) \to L$ by $T(\Gamma)(\rho) = \bigvee\{x \mid (x,\rho) \in \Gamma\}$. On the other hand, each $T: \mathsf{Sen}(C,P) \to L$ determines $\Gamma(T) \subseteq L \times \mathsf{Sen}(C,P)$ by $\Gamma(T) = \{(x,\rho) \mid T(\rho) = x > 0\}$.

Fact 3.4 *The classes* $\mathsf{Mod}_{(C,P)}(\Gamma, E)$ *are closed under model isomorphisms.*

Theorem 3.11 *Let* (C,P) *be an* MVL *signature and* $\Gamma \subseteq L \times \mathsf{Sen}(C,P)$*. If for any* $C' \supseteq C$*, for any* $A \subseteq L \times \mathsf{At}(C',P)$ *and for any* $E \subseteq C' \times C'$ *the class of models* $\mathsf{Mod}_{(C',P)}(\Gamma \cup A, E)$ *has a reachable initial model, then* $\mathsf{Mod}_{(C,P)}(\Gamma)$ *is a quasi-variety.*

Proof. We have to show that under the hypothesis of the theorem $\mathsf{Mod}_{(C,P)}(E)$ is closed under (1) sub-models and (2) direct products. We first introduce a couple of notations in support of our proof. For any (C,P)-model M:

- Let M_M be the $(C \cup M_s, P)$-expansion defined by $(M_M)_m = m$ for each $m \in M_s$.
- Let $A(M) = \{(M_\pi(x), \pi(x)) \mid n \in \omega, \pi \in P_n, x \in M_s^n\} \subseteq L \times \mathsf{At}(C \cup M_s, P)$.
- Let $E(M) = \{(c, M_c) \mid c \in C\} \subseteq (C \cup M_s) \times (C \cup M_s)$.

Note that $M_M \in \mathsf{Mod}_{(C\cup M_s,P)}(A(M), E(M))$ and that M_M is initial for $\mathsf{Mod}_{(C\cup M_s,P)}(A(M), E(M))$. Indeed, for any $N' \in \mathsf{Mod}_{(C\cup M_s,P)}(A(M), E(M))$ the unique $(C \cup M_s, P)$-homomorphism $h: M_M \to N'$ is defined by $h(m) = N'_m$ for each $m \in (M_M)_s = M_s$. That h preserves the interpretations of the constants of M_s follows directly from the its definition, that it preserves the interpretations of the constants of C follows by $E(M)$, and the other homomorphism property follows by $A(M)$. Now we proceed with the proofs of the quasi-varieties properties for $\mathsf{Mod}_{(C,P)}(\Gamma)$.

(1) Let N be a closed sub-model of a model $M \in \mathsf{Mod}_{(C,P)}(\Gamma)$. We have to show that $N \in \mathsf{Mod}_{(C,P)}(\Gamma)$ too. Let A be a reachable initial model in $\mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$. Since $A \in \mathsf{Mod}_{(C\cup N_s,P)}(A(N), E(N))$, by the initiality property of N_N, let $h: N_N \to A$ be the unique $(C \cup N_s, P)$-homomorphism. Let M_N be the $(C \cup N_s, P)$-expansion of M defined by $M_x = N_x$ for each $x \in N_s$. Evidently, $M_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma)$. Moreover, $M_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$. Thus we let $f: A \to M_N$ be the unique $(C \cup N_s, P)$-homomorphism given by the initiality property of A.

$$N_N \xrightarrow{h} A \xrightarrow{f} M_N$$

Since f and h are $(C \cup N_s, P)$-homomorphisms for each $x \in N_s$ we have that $(M_N)_x = f(A_x)$ and $A_x = h((N_N)_x)$, which implies $(M_N)_x = f(h((N_N)_x))$. Since $(M_N)_x = (N_N)_x = x$ it follows that $f \circ h$ is identity. Because A is a reachable $(C \cup N_s, P)$-model it follows that h is also surjective, hence it is a bijective homomorphism. For each $n \in \omega$, $\pi \in P_n$ and $m \in N_s^n$ by the homomorphism property for f we have that $A_\pi(h^n(m)) \leq (M_N)_\pi(f^n(h^n(m)))$. Since $f \circ h$ is identity it follows that $A_\pi(h^n(m)) \leq (M_N)_\pi(m) = M_\pi(m)$. Since N is closed sub-model of M we have that $M_\pi(m) = N_\pi(m) = (N_N)_\pi(m)$, hence $A_\pi(h^n(m)) \leq (N_N)_\pi(m)$. This shows h is closed, and because we have already proved it is bijective too, from Fact 3.1 it follows that it is isomorphism. Now, because A and N_N are isomorphic, by Fact 3.4 we obtain that $N_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$ which implies $N_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma)$. Since ρ is a (C,P)-sentence for each $(y,\rho) \in \Gamma$, $N_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma)$ implies that $N \in \mathsf{Mod}_{(C,P)}(\Gamma)$.

(2) Let $(M^i)_{i\in I}$ be a family of models such that $M^i \in \mathsf{Mod}_{(C,P)}(\Gamma)$ for each $i \in I$ and let $(p^i: N \to M^i)_{i\in I}$ be a product of this family. We have to prove that $N \in \mathsf{Mod}_{(C,P)}(\Gamma)$ too.

For each $i \in I$ we define the $(C \cup N_s, P)$-expansion M_N^i of M^i by $(M_N^i)_x = p^i(x)$ for each $x \in N_s$. This makes each p^i into a $(C \cup N_s, P)$-homomorphism $N_N \to M_N^i$. Moreover the family $(p^i: N_N \to M_N^i)_{i\in I}$ is a product of $(C \cup N_s, P)$-models. Indeed for any other family $(q^i: B' \to M_N^i)_{i\in I}$ of $(C \cup N_s, P)$-homomorphisms, by the universal property of the product $(p^i: N \to M^i)_{i\in I}$, we consider the unique (C,P)-homomorphism $h: B \to N$ such that $p^i \circ h = q^i$ for each $i \in I$, where B is the (C,P)-reduct of B'. Then h is a $(C \cup N_s, P)$-homomorphism because for each $x \in N_s$ and each $i \in I$ we have that $p^i(h(B'_x)) = q^i(B'_x) = (M_N^i)_x = p^i((N_N)_x)$, which implies $h(B'_x) = (N_N)_x$.

Now let us show that $M_N^i \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$ for each $i \in I$. Since ρ is a (C,P)-sentence for each $(y,\rho) \in \Gamma$ and $M^i \in \mathsf{Mod}_{(C,P)}(\Gamma)$, we have that $M_N^i \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma)$. For $A(N)$ and $E(N)$ we have the following arguments.

- For each $(N_\pi(x), \pi(x)) \in A(N)$ by the definition of $(M^i_N)_x$ and by the homomorphism property for p^i we have that $(M^i_N)_\pi((M^i_N)_x) = M^i_\pi(p^i(x)) \geq N_\pi(x)$ which means $M^i_N \models^{N_\pi(x)} \pi(x)$.
- For each $(c, N_c) \in E(N)$, on the one hand since M^i_N is expansion of M^i and by the homomorphism property of p^i we have that $(M^i_N)_c = M^i_c = p^i(N_c)$, and on the other hand by definition we have that $(M^i_N)_{N_c} = p_i(N_c)$. Hence $(M^i_N)_c = (M^i_N)_{N_c}$.

Thus each $M^i_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$. Let A be the initial model of $\mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$ and for each $i \in I$ let $f^i : A \to M^i_N$ be the unique homomorphism given by the initiality of A. Let $h : A \to N_N$ be the unique homomorphism such that $p^i \circ h = f^i$ for each $i \in I$, given by the universal property of the product $(p^i : N_N \to M^i_N)_{i\in I}$.

$$\begin{tikzcd} A \arrow[r, shift right, "h"'] \arrow[dr, bend right, "f^i"'] & N_N \arrow[l, shift right, "g"'] \arrow[d, "p^i"] \\ & M^i_N \end{tikzcd}$$

Because N_N is initial in $\mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$ and $A \in \mathsf{Mod}_{(C\cup N_s,P)}(A(N), E(N)) \subseteq \mathsf{Mod}_{(C\cup N_s,P)}(A(N), E(N))$, there exists an unique homomorphism $g : N_N \to A$. By the initiality of N_N we obtain that $h \circ g$ is identity and by the initiality of A that $g \circ h$ is identity. Hence A and N_N are isomorphic which implies that $N_N \in \mathsf{Mod}_{(C\cup N_s,P)}(\Gamma \cup A(N), E(N))$. From this we obtain that $N \in \mathsf{Mod}_{(C,P)}(\Gamma)$. □

Remark 3.12 $A(M)$ above bears similarity to the concept of *diagram* of [2], the difference being that $A(M)$ considers all atoms rather than all sentences. In the particular two valued situation, [1] call $A(M)$ the *positive diagram* of M. $E(M)$ above just fills the gap given by the absence of (crisp) equalities in the language.

The following result helps putting together the conclusions of Thm. 3.11 and Prop. 3.8 as an 'if and only if' result.

Proposition 3.13 $\mathsf{Mod}_{(C,P)}(A, E)$ *is a quasi-variety for any* MVL *signature* (C, P)*, any* $A \subseteq \mathsf{At}(C, P)$ *and any* $E \subseteq C \times C$.

P r o o f. We have to prove that $\mathsf{Mod}_{(C,P)}(A, E)$ is closed under (1) closed sub-models and under (2) direct products.

(1) Let $M \in \mathsf{Mod}_{(C,P)}(A, E)$ and let N be a closed sub-model of M. We prove that $N \in \mathsf{Mod}_{(C,P)}(A, E)$ too.

- Let $(y, \pi(c)) \in A$. We have $M \models^y \pi(c)$ which means $y \leq M_\pi(M_c)$. Because N is a sub-model of M we have $N_c = M_c$, because it is closed we have $N_\pi(x) = M_\pi(x)$ for each $x \in N^n_s$. It follows that $N_\pi(N_c) = M_\pi(M_c)$, hence $N \models^y \pi(c)$.
- Let $(c1, c2) \in E$. We have that $M_{c1} = M_{c2}$. Because N is a sub-model of M we have $N_{c1} = M_{c1}$ and $N_{c2} = M_{c2}$ hence $N_{c1} = N_{c2}$.

(2) Let $(M^i)_{i\in I}$ be a family of (C, P)-models such that each $M^i \in \mathsf{Mod}_{(C,P)}(A, E)$. Let $(p^i : N \to M^i)_{i\in I}$ be a product of $(M^i)_{i\in I}$. We prove that $N \in \mathsf{Mod}_{(C,P)}(A, E)$ too.

- Let $(y, \pi(c)) \in A$. We have that $N_\pi(N_c) = \bigwedge_{i\in I} M^i_\pi((p^i)^n(N_c) = \bigwedge_{i\in I} M^i_\pi(M^i_c)$. Since $M^i \models^y \pi(c)$ for each $i \in I$ means $y \leq M^i_\pi(M^i_c)$ for each $i \in I$, it follows that $y \leq N_\pi(N_c)$ which means $N \models^y \pi(c)$.
- Let $(c1, c2) \in E$. We have that $N_{c1} = N_{c2}$ because for each $i \in I$ we have $p^i(N_{c1}) = M^i_{c1} = M^i_{c2} = p^i(N_{c2})$.

□

Corollary 3.14 *Let* (C, P) *be an* MVL *signature and* $\Gamma \subseteq L \times \mathsf{Sen}(C, P)$*. Then* $\mathsf{Mod}_{(C,P)}(\Gamma)$ *is a quasi-variety if and only if for any* $C' \supseteq C$*, for any* $A \subseteq L \times \mathsf{At}(C', P)$ *and for any* $E \subseteq C' \times C'$ *the class of models* $\mathsf{Mod}_{(C',P)}(\Gamma \cup A, E)$ *has a reachable initial model.*

P r o o f. One implication is given by Thm. 3.11. For the other implication, let us assume that $\mathsf{Mod}_{(C,P)}(\Gamma)$ is a quasi-variety. It is easy to note that $\mathsf{Mod}_{(C',P)}(\Gamma)$ is quasi-variety too. Since the intersection of quasi-varieties is quasi-variety (we omit the proof of this straightforward fact here), by using Prop. 3.13 it follows that $\mathsf{Mod}_{(C',P)}(\Gamma \cup A, E) = \mathsf{Mod}_{(C',P)}(\Gamma) \cap \mathsf{Mod}_{(C',P)}(A, E)$ is quasi-variety. Then $\mathsf{Mod}_{(C',P)}(\Gamma \cup A, E)$ has a reachable initial model by virtue of Prop. 3.8. □

4 Horn sentences

In this section we introduce the concept of Horn sentence in MVL and we show that the models satisfying Horn sentences form quasi-varieties. We draw the consequence that Horn 'theories' admit initial models.

Definition 4.1 (Horn sentence) Any (C, P)-sentence of the form $(\forall X)H \Rightarrow \rho$ is called a *Horn sentence* when ρ is an $(C \cup X, P)$-atom and H is a quantifier-free $(C \cup X, P)$-sentence formed from atoms and the connectives $\wedge$, $\vee$, and $\otimes$. Let $\mathsf{Horn}(C, P)$ denote the set of the Horn (C, P)-sentence.

Within the framework of fuzzy logic programming [19, 6] the Horn sentences defined above are the clauses of fuzzy logic programs. Note that fuzzy logic programming may involve several residuated operators, each of them paired with a corresponding fuzzy implication, each such pair satisfying the adjointness condition. Although meaningful in the applications, this multiple residuated operator aspect would be an inessential generalization of our Horn sentences.

The result below can also be found in [6].

Proposition 4.2 (Preservation by closed-submodels) *If M is a closed sub-model of N and ρ is a quantifier-free sentence then for each $y \in L$ we have that*

$$N \models^y_{(C,P)} (\forall X)\rho \text{ implies } M \models^y_{(C,P)} (\forall X)\rho.$$

P r o o f. Let us assume $N \models^y_{(C,P)} (\forall X)\rho$. We have to show that $y \leq M[(\forall X)\rho]$ which is equivalent to showing that $y \leq M'[\rho]$ for each $(C \cup X, P)$-expansion M' of M.

Let us consider any $(C \cup X, P)$-expansion M' of M. This determines a $(C \cup X, P)$-expansion N' of N defined by $N'_x = M'_x$ for each $x \in X$. By induction on the structure of ρ, we prove that $M'[\rho] = N'[\rho]$.

- If ρ is an atom $\pi(c)$, then we have that $M'[\rho] = M'[\pi(c)] = M'_\pi(M'_c) = M_\pi(M'_c)$. Similarly, $N'[\rho] = N_\pi(N'_c)$. The conclusion for this case follows by noting that $M'_c = N'_c$ and that $M_\pi(x) = N_\pi(x)$ for each $x \in M^n_s$ (where $\pi \in P_n$).
- If ρ is $\rho_1 \star \rho_2$ where $\star \in \{\wedge, \vee, \otimes\}$, then we have that $M'[\rho_1 \star \rho_2] = M'[\rho_1] \star M'[\rho_2]$ and $N'[\rho_1 \star \rho_2] = N'[\rho_1] \star N'[\rho_2]$. The conclusion follows by using the induction hypothesis $M'[\rho_k] = N'[\rho_k]$.

Thus $M'[\rho] = N'[\rho] \geq y$. □

Lemma 4.3 *Let $(M^i)_{i\in I}$ be any family of (C, P)-models and let $(p^i : N \to M^i)_{i\in I}$ be a product of this family. For any quantifier-free sentence H formed from atoms and the connectives $\wedge$, $\vee$, and $\otimes$, we have that $N[H] \leq \bigwedge_{i\in I} M^i[H]$.*

P r o o f. We prove the lemma by induction on the structure of H.

- If H is an atom $\pi(c)$ where $\pi \in P_n$, then we have

$$N[\pi(c)] = N_\pi(N_c) = \bigwedge_{i\in I} M^i_\pi((p^i)^n(N_c)) = \bigwedge_{i\in I} M^i_\pi(M^i_c) = \bigwedge_{i\in I} M^i[\pi(c)].$$

- If H is $\rho_1 \star \rho_2$ where $\star \in \{\wedge, \vee, \otimes\}$, then we have that $N[H] = N[\rho_1 \star \rho_2] = N[\rho_1] \star N[\rho_2]$. By the induction hypothesis, because each $\star \in \{\wedge, \vee, \otimes\}$ is monotonic as operation on L, it follows that

$$N[H] \leq (\bigwedge_{i\in I} M^i[\rho_1]) \star (\bigwedge_{i\in I} M^i[\rho_2]). \quad (1)$$

By using again the monotonicity of $\star$ we obtain that

$$(\bigwedge_{i\in I} M^i[\rho_1]) \star (\bigwedge_{i\in I} M^i[\rho_2]) \leq M^i[\rho_1] \star M^i[\rho_2] = M^i[H]. \tag{2}$$

From (1) and (2) it follows that $N[H] \leq M^i[H]$ for each $i \in I$, hence $N[H] \leq \bigwedge_{i\in I} M^i[H]$.

□

Proposition 4.4 (Preservation by products) *Let $(M^i)_{i\in I}$ be any family of (C,P)-models and let $(p^i : N \to M^i)_{i\in I}$ be a product of this family. Let $(\forall X)H \Rightarrow \rho$ be any Horn (C,P)-sentence. Let us assume that $M^i \models^{y^i} (\forall X)H \Rightarrow \rho$ for each $i \in I$. Then $N \models^{\bigwedge_{i\in I} y^i} (\forall X)H \Rightarrow \rho$.*

Proof. We let y denote $\bigwedge_{i\in I} y^i$. We have to show that $y \leq N'[H \Rightarrow \rho]$ for each $(C \cup X, P)$-expansion N' of N.

We consider an arbitrary $(C \cup X, P)$-expansion N' of N. This determines, for each $i \in I$, a $(C \cup X, P)$-expansion M'^i of M^i defined for each $x \in X$ by $M'^i_x = p^i(N'_x)$. By replicating an argument from the beginning of the second part of the proof of Thm. 3.11, we have that $(p^i : N' \to M'^i)_{i\in I}$ is a product of $(C\cup X, P)$-models.

Since $M^i \models^{y^i} (\forall X)H \Rightarrow \rho$ we have that $y^i \leq M'^i[H \Rightarrow \rho]$ which implies $y \leq M'^i[H \Rightarrow \rho]$. This means

$$y \leq M'^i[H] \Rightarrow M'^i[\rho]. \tag{3}$$

By the adjunction property of the residuated lattice, (3) means

$$y \otimes M'^i[H] \leq M'^i[\rho]. \tag{4}$$

Since the property (4) holds for each $i \in I$, it follows that

$$\bigwedge_{i\in I}(y \otimes M'^i[H]) \leq \bigwedge_{i\in I} M'^i[\rho]. \tag{5}$$

By Lemma 4.3 and by the monotonicity of $\otimes$ as operator on L (applied twice) we have

$$y \otimes N'[H] \leq y \otimes \bigwedge_{i\in I} M'^i[H] \leq y \otimes M'^i[H]. \tag{6}$$

Since (6) holds for each $i \in I$ it follows that

$$y \otimes N'[H] \leq \bigwedge_{i\in I}(y \otimes M'^i[H]). \tag{7}$$

From (7) and (5) we obtain that $y \otimes N'[H] \leq \bigwedge_{i\in I} M'^i[\rho]$ which because ρ is atom this means

$$y \otimes N'[H] \leq N'[\rho] \tag{8}$$

The desired conclusion, namely that $y \leq (N'[H] \Rightarrow N'[\rho])$, now follows from (8) by the adjunction property of the residuated lattice. □

From Prop. 3.8, Prop. 4.2 and Prop. 4.4 we derive immediately the following result.

Corollary 4.5 *For any $\Gamma \subseteq L \times \mathsf{Horn}(C,P)$, the class of models $\mathsf{Mod}_{(C,P)}(\Gamma)$ is a quasi-variety. Consequently $\mathsf{Mod}_{(C,P)}(\Gamma)$ has a reachable initial model.*

5 Conclusions and Future Work

We have lifted the concept of quasi-variety of models from classical logic to MVL and have developed a mutual relationship between MVL quasi-varieties and existence of initial models of theories. We have defined a concept of Horn sentence in MVL, that involves the residual connector ($\otimes$) and the MVL implication and which corresponds to clauses of fuzzy logic programs, and proved that the models of Horn theories form a quasi-variety and consequently admit initial models. This result provides a common semantic foundations for formal specification and logic programming with multiple truth values, in the tradition of initial semantics. From the fuzzy logic programming culture perspective, this result may be seen as an alternative way to obtain the 'least Herbrand model'.

There are several avenues for future research. An important one is to develop axiomatizability results for quasi-varieties of MVL models. Preliminary investigations showed that the nature of this problem may be different from that of the classical two valued case, and significantly more difficult, a situation that hints to the conclusion that the generalization of the concept of quasi-variety from classical logic to MVL is far from being a canonical process.

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