Outline

Contents

1 Theories and Models

Galois connection between syntax and semantics

$$\begin{tikzcd}
(\mathscr{P}(Sen(\Sigma)),\subseteq) \arrow[r, bend left, "\star"] & (\mathscr{P}(|\textsc{Mod}(\Sigma)|),\subseteq) \arrow[l, bend left, "\star"]
\end{tikzcd}$$

- $\mathscr{M}^\star = \{\rho \in Sen(\Sigma) \mid \mathscr{M} \models_\Sigma \rho\}$,
- $E^\star = \{M \in |\textsc{Mod}(\Sigma)| \mid M \models_\Sigma E\}$.

General properties

1. $E \subseteq E'$ implies $E'^\star \subseteq E^*$,
2. $\mathscr{M} \subseteq \mathscr{M}'$ implies $\mathscr{M}'^\star \subseteq \mathscr{M}^*$,
3. $X \subseteq X^{\star\star}$,
4. $X = X^{\star\star\star}$.

Definition 1 (Theory)**.** (Σ, E) such that $E \subseteq Sen(\Sigma)$ closed under semantic consequence, i.e. $E = E^{\star\star}$.

The category $\mathbb{P}res(\mathscr{I})$ of $\mathscr{I}$-presentations for an institution $\mathscr{I}$
Presentation (Σ, E):

- signature Σ,
- $E \subseteq Sen(\Sigma)$.

Presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$:

- $\varphi : \Sigma \to \Sigma'$, such that
- $E' \models_{\Sigma'} Sen(\varphi)(E)$.

Proposition 2. *$\mathscr{I}$-presentation morphisms form a category under the composition given by the composition of the underlying signature morphisms.*

Institutions of presentations

A general and simple yet very useful technical construction, especially for doing 'logic by translation', but not only.

Institution of $\mathscr{I}$-presentations $\mathscr{I}^{\mathrm{p}} = (\mathbb{S}ig^{\mathrm{p}}, Sen^{\mathrm{p}}, \mathrm{MOD}^{\mathrm{p}}, \models^{\mathrm{p}})$ over a given institution $\mathscr{I} = (\mathbb{S}ig, Sen, \mathrm{MOD}, \models)$:

- $\mathbb{S}ig^{\mathrm{p}} = \mathbb{P}res(\mathscr{I})$,
- $Sen^{\mathrm{p}}(\Sigma, E) = Sen(\Sigma)$,
- $|\mathrm{MOD}^{\mathrm{p}}(\Sigma, E)| = \{M \in |\mathrm{MOD}(\Sigma)| \mid M \models E\}$
  - $\mathrm{MOD}^{\mathrm{p}}(\varphi)(M') = \mathrm{MOD}(\varphi)(M')$ $(\mathrm{MOD}(\varphi)(M') \models E$ because $E' \models Sen(\varphi)(E))$,
- $M \models_{(\Sigma,E)} \rho$ if and only if $M \models_{\Sigma} \rho$.

Lifting signature co-limits to presentations

Many properties of a base institution $\mathscr{I}$ can be lifted to the institution $\mathscr{I}^{\mathrm{p}}$ (of $\mathscr{I}$-presentations) in a fully general way.

The following is such an example (very useful for specification theory, but also for pure model theoretic purposes).

Proposition 3. *If the category of $\mathscr{I}$-signatures has J-co-limits then the category of $\mathscr{I}^{\mathrm{p}}$-signatures (i.e. $\mathscr{I}$-presentations) has J-co-limits too.*

$$\begin{tikzcd}
(\Sigma_i, E_i) \arrow[rr, "\varphi_u"] \arrow[dr, "\mu_i"'] & & (\Sigma_j, E_j) \arrow[dl, "\mu_j"] \\
& (\Sigma, \bigcup_{i \in |J|} E_i) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma_i \arrow[rr, "\varphi_u"] \arrow[dr, "\mu_i"'] & & \Sigma_j \arrow[dl, "\mu_j"] \\
& \Sigma &
\end{tikzcd}$$

2 Model amalgamation

2.1 Definition

Model amalgamation

- A property pervading the development of most model theory results. So fundamental that it is one of the causes of the Satisfaction Condition in institutions with quantifiers (e.g. **FOL**).
- Holds implicitly in the conventional concrete institutions, therefore its (crucial) role quite hidden. It becomes explicit at the level of doing model theory in abstract institutions.
- Widely spread among logical systems, rather easy to establish.
- Not to be confused with much harder amalgamation from conventional **FOL** model theory that is something completely different (local to signatures, about elementary embeddings).

Model amalgamation: definition

$\mathscr{I}$ has model amalgamation when for each pushout of signature morphisms

$$\begin{tikzcd}
M & & & M_1 \\
& \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma_1 \arrow[d, "\theta'"] & \\
& \Sigma_2 \arrow[r, "\varphi'"'] & \Sigma' & \\
M_2 & & & M'
\end{tikzcd}$$

for any Σ_i models M_i such that $\textsc{Mod}(\varphi)(M_1) = \textsc{Mod}(\theta)(M_2)$

there exists an unique Σ'-model M' such that

$\textsc{Mod}(\theta')(M') = M_1$ and $\textsc{Mod}(\varphi')(M') = M_2$.

How to establish model amalgamation in concrete institutions

Can be done directly, however requires some straightforward but rather tedious work (pushouts of signature morphisms).

Here is a 'clever' general solution (can be applied to many concrete institutions too):

1. Define a 'super' signature Ω such that
   - each Σ-model appears as a signature morphism $\Sigma \to \Omega$, and
   - each φ-reduct $\textsc{Mod}(\varphi)(M')$ appears as a composition of signature morphisms $\Sigma \xrightarrow{\varphi} \Sigma' \xrightarrow{M'} \Omega$.
2. Use the pushout property:

$$\begin{tikzcd}
& \Sigma_1 \arrow[dr, "\theta'"] \arrow[drr, bend left, "M_1"] & & \\
\Sigma \arrow[ur, "\varphi"] \arrow[dr, "\theta"'] & & \Sigma' \arrow[r, "M'"] & \Omega \\
& \Sigma_2 \arrow[ur, "\varphi'"'] \arrow[urr, bend right, "M_2"'] & &
\end{tikzcd}$$

How to establish model amalgamation the definition of Ω in MSA

Now let us turn our attention to $\Omega = (S^\Omega, F^\Omega)$ which is defined as follows:

- $S^\Omega = |\mathbb{S}et|$, i.e., the class of all sets, and
- for any sets $s_1, \ldots, s_n, s$,
$$F^\Omega_{s_1 \ldots s_n \to s} = \mathbb{S}et(s_1 \times \cdots \times s_n, s)$$
i.e., the set of all functions $s_1 \times \cdots \times s_n \to s$.

Other useful forms of model amalgamation

Each of the following has its own applications.

- *Weak amalgmation*: requires only the existence of amalgamation M', not uniqueness. Quite often this is sufficient (such as for establising the Satisfaction Condition for quantifiers).
- *Semi-exactness*: amalgamation of model homomorphisms too.
- *J-amalgamation*: amalgamation from J-co-limits rather than just pushuts.

Model amalgamation from a more categorical perspective

For example, *semi-exactness* just means that

$$\textsc{Mod} : \mathbb{S}ig \to \mathbb{C}at^{\mathrm{op}} \ \text{ preserves pushouts.}$$

i.e. for any pushout in $\mathbb{S}ig$

$$\begin{tikzcd}
\Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma_1 \arrow[d, "\theta'"] \\
\Sigma_2 \arrow[r, "\varphi'"'] & \Sigma'
\end{tikzcd}$$

the following is a pullback in $\mathbb{C}at$:

$$\begin{tikzcd}
\textsc{Mod}(\Sigma) & \textsc{Mod}(\Sigma_1) \arrow[l, "\textsc{Mod}(\varphi)"'] \\
\textsc{Mod}(\Sigma_2) \arrow[u, "\textsc{Mod}(\theta)"] & \textsc{Mod}(\Sigma') \arrow[l, "\textsc{Mod}(\varphi')"] \arrow[u, "\textsc{Mod}(\theta')"']
\end{tikzcd}$$

2.2 Examples

Examples of concrete model amalgamation properties

J-exact institutions:

- the many-sorted forms of classical first order logic, its fragments (Horn clause logic, equational logic), intuitionistic logic, many-valued logics, modal logic with possible worlds semantics, etc.

Semi-exact institutions:

- all of the above in the single sorted form.

Weak model amalgamation:

- higher order logic with Henkin semantics,
- 'weak propositional logic' of Beziau (only half negation),
- some other interesting examples from computer science.

Lifting model amalgamation to presentations

Proposition 4. *If $\mathscr{I}$ has model amalgamation then $\mathscr{I}^{\mathrm{p}}$ has model amalgamation too.*

$$\begin{tikzcd} (\Sigma,E) \arrow[r, "\varphi"] \arrow[d, "\theta"'] & (\Sigma_1,E_1) \arrow[d, "\theta'"] \\ (\Sigma_2,E_2) \arrow[r, "\varphi'"'] & (\Sigma',E_1\cup E_2) \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \Sigma_1 \arrow[d, "\theta'"] \\ \Sigma_2 \arrow[r, "\varphi'"'] & \Sigma' \end{tikzcd}$$

The amalgamation M' of M_1 (Σ_1-model) and M_2 (Σ_2-model) satisfies $E_1 \cup E_2$.

3 The method of diagrams

3.1 Definition

The method of diagrams

- Much used in (conventional concrete) model theory, pervading many developments in
  - free constructions
  - axiomatizability theory
  - saturated model theory
  - interpolation and definability
  - etc.
- At the abstract institutions level appears a categorical property that displays some fundamental coherence between the syntax and the semantics of a given institution.

Conventional concrete diagrams (FOL)

Let M be an (S,F,P)-model.

1. We add the elements of M as new constants to the signature, thus (S,F_M,P),
2. The (positive) *diagram of M*:

$$E_M = \{(S,F_M,P)\text{-atoms } \rho \mid M_M \models \rho\}.$$

where M_M is the expansion of M interpreting the new constants by themselves, i.e. $(M_M)_m = m$ for each $m \in M$.

Institution-independent diagrams

An institution *has (elementary) diagrams* when for each Σ-model M there exists

1. $\iota_{\Sigma,M} : \Sigma \to \Sigma_M$, and
2. $E_M \subseteq Sen(\Sigma_M)$

such that

$$\begin{tikzcd} \mathrm{Mod}(\Sigma_M, E_M) \arrow[r, "iso"] \arrow[dr, "\mathrm{Mod}(\iota_\Sigma(M))"'] & (M/\mathrm{Mod}(\Sigma)) \arrow[d, "\text{forgetful}"] \\ & \mathrm{Mod}(\Sigma) \end{tikzcd}$$

Other axioms about coherence wrt signature/model translations omitted.

3.2 Examples

Some examples in FOL

Other concepts of diagrams may be obtained by changing the concept of model homomorphism:

homomorphisms	*diagram E_M*
ordinary	all atoms in M^*_M
injective	all atoms and neg. of atomic equations in M^*_M
closed	all atoms and neg. of atomic relations in M^*_M
closed and injective	all atoms and neg. of atoms in M^*_M
elem. embeddings	M^*_M

Some examples in other institutions

*Intuitionistic logic (***IPL***)*:

- for P-model $M : P \to A$ (A any Heyting algebra),
  - the elementary extension is $P \to P \uplus A$, and
  - $E_M = \{\rho, \rho_1 \Rightarrow \rho_2 \in M^*_M \mid \rho, \rho_1, \rho_2 \in P \uplus A\}$.

*Modal logic (***MFOL***, first order, Kripke semantics)*:

- no diagrams!

*Higher order logic (***HOL***)*:

- for (S,F)-model M
  - the elementary extension is (S, F_M),
  - $E_M = \{t = t' \mid M_M \models t = t'\}$.

Lifting diagrams to presentations

Proposition 5. *If $\mathscr{I}$ has diagrams then $\mathscr{I}^{\mathrm{p}}$ has diagrams too.*

For any (Σ, E)-model M:

- the elementary extension :

$$(\Sigma, E) \xrightarrow{\iota_\Sigma(M)} (\Sigma, E)_M = (\Sigma_M, Sen(\iota_\Sigma(M))(E))$$

- the diagram:

$$E_M \cup Sen(\iota_\Sigma(M))(E).$$

3.3 Using diagrams

An example of use of inst.-indep. diagrams establishing co-limits of models

In concrete situations, usually a difficult problem, e.g. co-limits of rings, etc. In essence, the result below reduces the problem to co-limits of signatures, which in concrete situations is much easier.

Theorem 6. *In any institution* $(\mathbb{S}ig, Sen, \mathrm{MOD}, \models)$ *such that*

1. *it has diagrams,*
2. *each presentation has initial models,*
3. $\mathbb{S}ig$ *has J-co-limits,*
4. *it has J-model amalgamation,*

then each category of Σ*-models has J-co-limits.*

An application co-limits of models of Horn theories

Corollary 7. *For any Horn theory (in a given* **FOL** *signature), the category of its models has (small) co-limits.*

The following is an instance of corollary above:

Corollary 8. *The category of rings has (small) co-limits.*

The proof

We set the institution to be $\mathbf{HCL}^{\mathrm{p}}$, where

- **HCL** is the sub-institution of **FOL** that restricts the sentences to the Horn sentences,
- $\mathbf{HCL}^{\mathrm{p}}$ is the institution of the **HCL**-presentations.

We can easily check the hypotheses of the general theorem above.

1. - **HCL** has the **FOL** diagrams, consisting of atomic sentences.
   - We lift the diagrams from **HCL** to $\mathbf{HCL}^{\mathrm{p}}$.
2. It is well known that Horn theories admit initial models.
3. - $\mathbb{S}ig^{\mathbf{HCL}} = \mathbb{S}ig^{\mathbf{FOL}}$ is (small) co-complete.
   - We lift this from **HCL**-signatures to $\mathbf{HCL}^{\mathrm{p}}$-signatures (i.e. **HCL**-presentations).
4. - **FOL**/**HCL** is exact (we have already proved).
   - We lift model amalgamation from **HCL** to $\mathbf{HCL}^{\mathrm{p}}$.