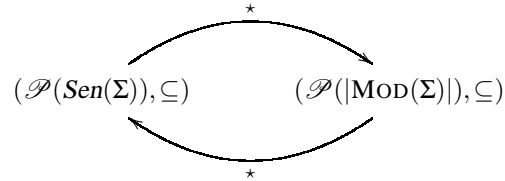


## Outline

## Contents

### 1 Theories and Models

#### Galois connection between syntax and semantics



- $\mathcal{M}^* = \{\rho \in \text{Sen}(\Sigma) \mid \mathcal{M} \models_{\Sigma} \rho\}$ ,
- $E^* = \{M \in |\text{MOD}(\Sigma)| \mid M \models_{\Sigma} E\}$ .

#### General properties

1.  $E \subseteq E'$  implies  $E'^* \subseteq E^*$ ,
2.  $\mathcal{M} \subseteq \mathcal{M}'$  implies  $\mathcal{M}'^* \subseteq \mathcal{M}^*$ ,
3.  $X \subseteq X^{**}$ ,
4.  $X = X^{***}$ .

**Definition 1** (Theory).  $(\Sigma, E)$  such that  $E \subseteq \text{Sen}(\Sigma)$  closed under semantic consequence, i.e.  $E = E^{**}$ .

#### The category $\mathbb{P}\text{res}(\mathcal{I})$ of $\mathcal{I}$ -presentations for an institution $\mathcal{I}$

*Presentation*  $(\Sigma, E)$ :

- signature  $\Sigma$ ,
- $E \subseteq \text{Sen}(\Sigma)$ .

*Presentation morphism*  $\varphi : (\Sigma, E) \rightarrow (\Sigma', E')$ :

- $\varphi : \Sigma \rightarrow \Sigma'$ , such that
- $E' \models_{\Sigma'} \text{Sen}(\varphi)(E)$ .

**Proposition 2.**  $\mathcal{I}$ -presentation morphisms form a category under the composition given by the composition of the underlying signature morphisms.

## Institutions of presentations

A general and simple yet very useful technical construction, especially for doing ‘logic by translation’, but not only.

*Institution of  $\mathcal{I}$ -presentations*  $\mathcal{I}^P = (\text{Sig}^P, \text{Sen}^P, \text{MOD}^P, \models^P)$  over a given institution  $\mathcal{I} = (\text{Sig}, \text{Sen}, \text{MOD}, \models)$ :

- $\text{Sig}^P = \text{Pres}(\mathcal{I})$ ,
- $\text{Sen}^P(\Sigma, E) = \text{Sen}(\Sigma)$ ,
- $|\text{MOD}^P(\Sigma, E)| = \{M \in |\text{MOD}(\Sigma)| \mid M \models E\}$ 
  - $\text{MOD}^P(\varphi)(M') = \text{MOD}(\varphi)(M')$  ( $\text{MOD}(\varphi)(M') \models E$  because  $E' \models \text{Sen}(\varphi)(E)$ ),
- $M \models_{(\Sigma, E)} \rho$  if and only if  $M \models_{\Sigma} \rho$ .

## Lifting signature co-limits to presentations

Many properties of a base institution  $\mathcal{I}$  can be lifted to the institution  $\mathcal{I}^P$  (of  $\mathcal{I}$ -presentations) in a fully general way.

The following is such an example (very useful for specification theory, but also for pure model theoretic purposes).

**Proposition 3.** *If the category of  $\mathcal{I}$ -signatures has  $J$ -co-limits then the category of  $\mathcal{I}^P$ -signatures (i.e.  $\mathcal{I}$ -presentations) has  $J$ -co-limits too.*

$$\begin{array}{ccc}
 (\Sigma_i, E_i) & \xrightarrow{\varphi_u} & (\Sigma_j, E_j) \\
 \searrow \mu_i & & \swarrow \mu_j \\
 & (\Sigma, \bigcup_{i \in |J|} E_i) & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Sigma_i & \xrightarrow{\varphi_u} & \Sigma_j \\
 \searrow \mu_i & & \swarrow \mu_j \\
 & \Sigma & 
 \end{array}$$

## 2 Model amalgamation

### 2.1 Definition

#### Model amalgamation

- A property pervading the development of most model theory results. So fundamental that it is one of the causes of the Satisfaction Condition in institutions with quantifiers (e.g. **FOL**).
- Holds implicitly in the conventional concrete institutions, therefore its (crucial) role quite hidden. It becomes explicit at the level of doing model theory in abstract institutions.
- Widely spread among logical systems, rather easy to establish.
- Not to be confused with much harder amalgamation from conventional **FOL** model theory that is something completely different (local to signatures, about elementary embeddings).

#### Model amalgamation: definition

$\mathcal{I}$  has model amalgamation when for each pushout of signature morphisms

$$\begin{array}{ccc}
 M & & M_1 \\
 & \begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma_1 \\ \theta \downarrow & & \downarrow \theta' \\ \Sigma_2 & \xrightarrow{\varphi'} & \Sigma' \end{array} & \\
 M_2 & & M'
 \end{array}$$

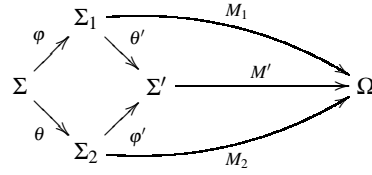
for any  $\Sigma_i$  models  $M_i$  such that  $\text{MOD}(\varphi)(M_1) = \text{MOD}(\theta)(M_2)$   
there exists an unique  $\Sigma'$ -model  $M'$  such that  
 $\text{MOD}(\theta')(M') = M_1$  and  $\text{MOD}(\varphi')(M') = M_2$ .

### How to establish model amalgamation in concrete institutions

Can be done directly, however requires some straightforward but rather tedious work (pushouts of signature morphisms).

Here is a ‘clever’ general solution (can be applied to many concrete institutions too):

1. Define a ‘super’ signature  $\Omega$  such that
  - each  $\Sigma$ -model appears as a signature morphism  $\Sigma \rightarrow \Omega$ , and
  - each  $\varphi$ -reduct  $\text{MOD}(\varphi)(M')$  appears as a composition of signature morphisms  $\Sigma \xrightarrow{\varphi} \Sigma' \xrightarrow{M'} \Omega$ .
2. Use the pushout property:



### How to establish model amalgamation the definition of $\Omega$ in MSA

Now let us turn our attention to  $\Omega = (S^\Omega, F^\Omega)$  which is defined as follows:

- $S^\Omega = |\text{Set}|$ , i.e., the class of all sets, and
- for any sets  $s_1, \dots, s_n, s$ ,

$$F_{s_1 \dots s_n \rightarrow s}^\Omega = \text{Set}(s_1 \times \dots \times s_n, s)$$

i.e., the set of all functions  $s_1 \times \dots \times s_n \rightarrow s$ .

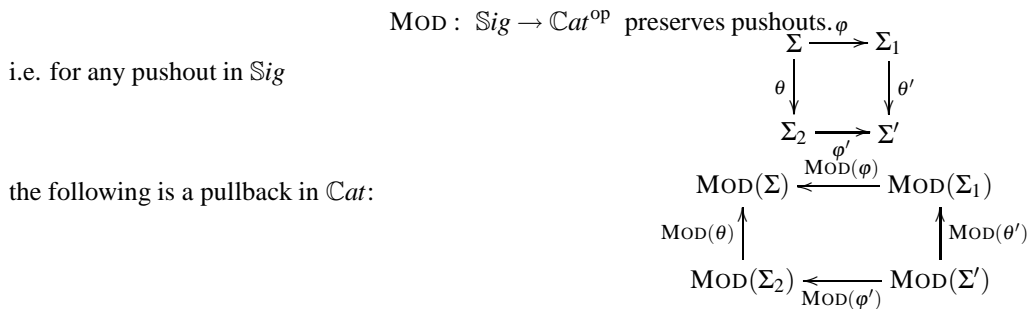
### Other useful forms of model amalgamation

Each of the following has its own applications.

- *Weak amalgamation*: requires only the existence of amalgamation  $M'$ , not uniqueness. Quite often this is sufficient (such as for establishing the Satisfaction Condition for quantifiers).
- *Semi-exactness*: amalgamation of model homomorphisms too.
- *J-amalgamation*: amalgamation from  $J$ -co-limits rather than just pushouts.

### Model amalgamation from a more categorical perspective

For example, *semi-exactness* just means that



## 2.2 Examples

### Examples of concrete model amalgamation properties

*J-exact institutions:*

- the many-sorted forms of classical first order logic, its fragments (Horn clause logic, equational logic), intuitionistic logic, many-valued logics, modal logic with possible worlds semantics, etc.

*Semi-exact institutions:*

- all of the above in the single sorted form.

*Weak model amalgamation:*

- higher order logic with Henkin semantics,
- ‘weak propositional logic’ of Beziau (only half negation),
- some other interesting examples from computer science.

### Lifting model amalgamation to presentations

**Proposition 4.** *If  $\mathcal{I}$  has model amalgamation then  $\mathcal{I}^P$  has model amalgamation too.*

$$\begin{array}{ccc}
 (\Sigma, E) & \xrightarrow{\varphi} & (\Sigma_1, E_1) \\
 \theta \downarrow & & \downarrow \theta' \\
 (\Sigma_2, E_2) & \xrightarrow{\varphi'} & (\Sigma', E_1 \cup E_2)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Sigma & \xrightarrow{\varphi} & \Sigma_1 \\
 \theta \downarrow & & \downarrow \theta' \\
 \Sigma_2 & \xrightarrow{\varphi'} & \Sigma'
 \end{array}$$

The amalgamation  $M'$  of  $M_1$  ( $\Sigma_1$ -model) and  $M_2$  ( $\Sigma_2$ -model) satisfies  $E_1 \cup E_2$ .

## 3 The method of diagrams

### 3.1 Definition

#### The method of diagrams

- Much used in (conventional concrete) model theory, pervading many developments in
  - free constructions
  - axiomatizability theory
  - saturated model theory
  - interpolation and definability
  - etc.
- At the abstract institutions level appears a categorical property that displays some fundamental coherence between the syntax and the semantics of a given institution.

#### Conventional concrete diagrams (FOL)

Let  $M$  be an  $(S, F, P)$ -model.

1. We add the elements of  $M$  as new constants to the signature, thus  $(S, F_M, P)$ ,
2. The (positive) *diagram* of  $M$ :

$$E_M = \{(S, F_M, P)\text{-atoms } \rho \mid M_M \models \rho\}.$$

where  $M_M$  is the expansion of  $M$  interpreting the new constants by themselves, i.e.  $(M_M)_m = m$  for each  $m \in M$ .

### Institution-independent diagrams

An institution *has (elementary) diagrams* when for each  $\Sigma$ -model  $M$  there exists

1.  $t_{\Sigma, M} : \Sigma \rightarrow \Sigma_M$ , and
2.  $E_M \subseteq \text{Sen}(\Sigma_M)$

such that

$$\begin{array}{ccc} \text{MOD}(\Sigma_M, E_M) & \xrightarrow{\text{iso}} & (M/\text{MOD}(\Sigma)) \\ & \searrow & \downarrow \text{forgetful} \\ \text{MOD}(t_{\Sigma}(M)) & & \text{MOD}(\Sigma) \end{array}$$

Other axioms about coherence wrt signature/model translations omitted.

## 3.2 Examples

### Some examples in FOL

Other concepts of diagrams may be obtained by changing the concept of model homomorphism:

homomorphisms	diagram $E_M$
ordinary	all atoms in $M_M^*$
injective	all atoms and neg. of atomic equations in $M_M^*$
closed	all atoms and neg. of atomic relations in $M_M^*$
closed and injective	all atoms and neg. of atoms in $M_M^*$
elem. embeddings	$M_M^*$

### Some examples in other institutions

*Intuitionistic logic (IPL):*

- for  $P$ -model  $M : P \rightarrow A$  ( $A$  any Heyting algebra),
  - the elementary extension is  $P \rightarrow P \uplus A$ , and
  - $E_M = \{\rho, \rho_1 \Rightarrow \rho_2 \in M_M^* \mid \rho, \rho_1, \rho_2 \in P \uplus A\}$ .

*Modal logic (MFOL, first order, Kripke semantics):*

- no diagrams!

*Higher order logic (HOL):*

- for  $(S, F)$ -model  $M$ 
  - the elementary extension is  $(S, F_M)$ ,
  - $E_M = \{t = t' \mid M_M \models t = t'\}$ .

### Lifting diagrams to presentations

**Proposition 5.** *If  $\mathcal{I}$  has diagrams then  $\mathcal{I}^{\text{P}}$  has diagrams too.*

For any  $(\Sigma, E)$ -model  $M$ :

- the elementary extension :

$$(\Sigma, E) \xrightarrow{t_{\Sigma}(M)} (\Sigma, E)_M = (\Sigma_M, \text{Sen}(t_{\Sigma}(M)))(E)$$

- the diagram:

$$E_M \cup \text{Sen}(t_{\Sigma}(M))(E).$$

### 3.3 Using diagrams

#### An example of use of inst.-indep. diagrams establishing co-limits of models

In concrete situations, usually a difficult problem, e.g. co-limits of rings, etc. In essence, the result below reduces the problem to co-limits of signatures, which in concrete situations is much easier.

**Theorem 6.** *In any institution  $(\text{Sig}, \text{Sen}, \text{MOD}, \models)$  such that*

1. *it has diagrams,*
2. *each presentation has initial models,*
3.  *$\text{Sig}$  has  $J$ -co-limits,*
4. *it has  $J$ -model amalgamation,*

*then each category of  $\Sigma$ -models has  $J$ -co-limits.*

#### An application co-limits of models of Horn theories

**Corollary 7.** *For any Horn theory (in a given **FOL** signature), the category of its models has (small) co-limits.*

The following is an instance of corollary above:

**Corollary 8.** *The category of rings has (small) co-limits.*

#### The proof

We set the institution to be  $\mathbf{HCL}^P$ , where

- $\mathbf{HCL}$  is the sub-institution of  $\mathbf{FOL}$  that restricts the sentences to the Horn sentences,
- $\mathbf{HCL}^P$  is the institution of the  $\mathbf{HCL}$ -presentations.

We can easily check the hypotheses of the general theorem above.

1.
  - $\mathbf{HCL}$  has the  $\mathbf{FOL}$  diagrams, consisting of atomic sentences.
  - We lift the diagrams from  $\mathbf{HCL}$  to  $\mathbf{HCL}^P$ .
2. It is well known that Horn theories admit initial models.
3.
  - $\text{Sig}^{\mathbf{HCL}} = \text{Sig}^{\mathbf{FOL}}$  is (small) co-complete.
  - We lift this from  $\mathbf{HCL}$ -signatures to  $\mathbf{HCL}^P$ -signatures (i.e.  $\mathbf{HCL}$ -presentations).
4.
  - $\mathbf{FOL}/\mathbf{HCL}$  is exact (we have already proved).
  - We lift model amalgamation from  $\mathbf{HCL}$  to  $\mathbf{HCL}^P$ .