# Institution Theory

basic methods

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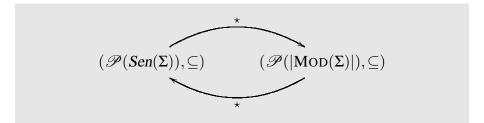
## Outline

#### 1 Theories and Models

2 Model amalgamation

- Definition
- Examples
- 3 The method of diagrams
  - Definition
  - Examples
  - Using diagrams

### Galois connection between syntax and semantics



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$$\mathcal{M}^{\star} = \{ \rho \in Sen(\Sigma) \mid \mathcal{M} \models_{\Sigma} \rho \},$$
$$\mathcal{E}^{\star} = \{ M \in |MOD(\Sigma)| \mid M \models_{\Sigma} E \}.$$

# General properties

1 
$$E \subseteq E'$$
 implies  $E'^* \subseteq E^*$ ,  
2  $\mathcal{M} \subseteq \mathcal{M}'$  implies  $\mathcal{M}'^* \subseteq \mathcal{M}^*$ ,  
3  $X \subseteq X^{**}$ ,

$$4 \quad X = X^{\star \star \star}.$$

#### Definition (Theory)

 $(\Sigma, E)$  such that  $E \subseteq Sen(\Sigma)$  closed under semantic consequence, i.e.  $E = E^{\star\star}$ .

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# The category $\mathbb{P}res(\mathscr{I})$ of $\mathscr{I}$ -presentations for an institution $\mathscr{I}$

- **Presentation**  $(\Sigma, E)$ :
  - signature  $\Sigma$ ,
  - $\blacksquare E \subseteq Sen(\Sigma).$

*Presentation morphism*  $\varphi$  :  $(\Sigma, E) \rightarrow (\Sigma', E')$ :

- $\varphi: \Sigma \to \Sigma'$ , such that
- $\blacksquare E' \models_{\Sigma'} Sen(\varphi)(E).$

#### Proposition

I -presentation morphisms form a category under the composition given by the composition of the underlying signature morphisms.

A general and simple yet very useful technical construction, especially for doing 'logic by translation', but not only.

Institution of  $\mathscr{I}$ -presentations  $\mathscr{I}^{p} = (\mathbb{S}ig^{p}, Sen^{p}, MOD^{p}, \models^{p})$ over a given institution  $\mathscr{I} = (\mathbb{S}ig, Sen, MOD, \models)$ :

 $\blacksquare \ \mathbb{S}ig^{p} = \mathbb{P}res(\mathscr{I}),$ 

• 
$$Sen^p(\Sigma, E) = Sen(\Sigma),$$

 $|\operatorname{MOD}^{p}(\Sigma, E)| = \{ M \in |\operatorname{MOD}(\Sigma)| \mid M \models E \}$ 

■ 
$$\operatorname{Mod}^{\operatorname{p}}(\varphi)(M') = \operatorname{Mod}(\varphi)(M')$$
  
 $(\operatorname{Mod}(\varphi)(M') \models E \text{ because } E' \models Sen(\varphi)(E)),$ 

• 
$$M \models_{(\Sigma,E)} \rho$$
 if and only if  $M \models_{\Sigma} \rho$ .

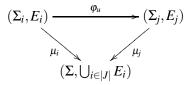
## Lifting signature co-limits to presentations

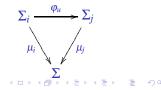
Many properties of a base institution  $\mathscr{I}$  can be lifted to the institution  $\mathscr{I}^p$  (of  $\mathscr{I}$ -presentations) in a fully general way.

The following is such an example (very useful for specification theory, but also for pure model theoretic purposes).

Proposition

If the category of  $\mathscr{I}$ -signatures has J-co-limits then the category of  $\mathscr{I}^{\mathrm{p}}$ -signatures (i.e.  $\mathscr{I}$ -presentations) has J-co-limits too.



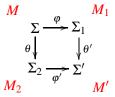


# Model amalgamation

- A property pervading the development of most model theory results. So fundamental that it is one of the causes of the Satisfaction Condition in institutions with quantifiers (e.g. FOL).
- Holds implicitly in the conventional concrete institutions, therefore its (crucial) role quite hidden. It becomes explicit at the level of doing model theory in abstract institutions.
- Widely spread among logical systems, rather easy to establish.
- Not to be confused with much harder amalgamation from conventional FOL model theory that is something completely different (local to signatures, about elementary embeddings).

# Model amalgamation: definition

*I has model amalgamation* when for each pushout of signature morphisms



for any  $\Sigma_i$  models  $M_i$  such that  $MOD(\varphi)(M_1) = MOD(\theta)(M_2)$ there exists an unique  $\Sigma'$ -model M' such that  $MOD(\theta')(M') = M_1$  and  $MOD(\varphi')(M') = M_2$ .

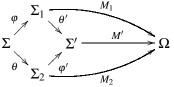
# How to establish model amalgamation in concrete institutions

Can be done directly, however requires some straightforward but rather tedious work (pushouts of signature morphisms).

Here is a 'clever' general solution (can be applied to many concrete institutions too):

**1** Define a 'super' signature  $\Omega$  such that

- each  $\Sigma$ -model appears as a signature morphism  $\Sigma \to \Omega$ , and
- each  $\varphi$ -reduct MOD $(\varphi)(M')$  appears as a composition of signature morphisms  $\Sigma \xrightarrow{\varphi} \Sigma' \xrightarrow{M'} \Omega$ .
- **2** Use the pushout property:



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# How to establish model amalgamation the definition of $\Omega$ in MSA

Now let us turn our attention to  $\Omega = (S^{\Omega}, F^{\Omega})$  which is defined as follows:

- $S^{\Omega} = |\mathbb{S}et|$ , i.e., the class of all sets, and
- for any sets  $s_1, \ldots, s_n, s$ ,

$$F_{s_1\ldots s_n\to s}^{\mathbf{\Omega}} = \mathbb{S}et(s_1\times\cdots\times s_n, s)$$

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i.e., the set of all functions  $s_1 \times \cdots \times s_n \rightarrow s$ .

# Other useful forms of model amalgamation

Each of the following has its own applications.

- *Weak amalgmation*: requires only the existence of amalgamation *M'*, not uniqueness. Quite often this is sufficient (such as for establising the Satisfaction Condition for quantifiers).
- Semi-exactness: amalgamation of model homomorphisms too.
- *J-amalgamation*: amalgamation from *J*-co-limits rather than just pushuts.

# Model amalgamation from a more categorical perspective

For example, semi-exactness just means that

MOD :  $\mathbb{S}ig \to \mathbb{C}at^{op}$  preserves pushouts.

 $\begin{array}{c} \Sigma \xrightarrow{\varphi} \Sigma_1 \\ \theta \downarrow & \downarrow \theta' \\ \Sigma_2 \xrightarrow{\varphi'} \Sigma' \end{array}$ i.e. for any pushout in Sig  $MOD(\Sigma) \stackrel{MOD(\varphi)}{\longleftarrow} MOD(\Sigma_1)$ the following is a pullback in  $\mathbb{C}at$ :  $MOD(\theta)$   $MOD(\theta')$  $\operatorname{MOD}(\Sigma_2)_{\operatorname{MOD}(\sigma')} \operatorname{MOD}(\Sigma')$ ◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 シのへの

# Examples of concrete model amalgamation properties

#### J-exact institutions:

the many-sorted forms of classical first order logic, its fragments (Horn clause logic, equational logic), intuitionistic logic, many-valued logics, modal logic with possible worlds semantics, etc.

#### Semi-exact institutions:

■ all of the above in the single sorted form.

#### Weak model amalgamation:

- higher order logic with Henkin semantics,
- 'weak propositional logic' of Beziau (only half negation),
- some other interesting examples from computer science.

# Lifting model amalgamation to presentations

#### Proposition

If  $\mathscr{I}$  has model amalgamation then  $\mathscr{I}^p$  has model amalgamation too.

$$\begin{array}{ccc} (\Sigma, E) & \xrightarrow{\varphi} (\Sigma_1, E_1) & \Sigma \xrightarrow{\varphi} \Sigma_1 \\ \theta \downarrow & \downarrow \theta' & \theta \downarrow & \downarrow \theta' \\ (\Sigma_2, E_2) & \xrightarrow{\varphi'} (\Sigma', E_1 \cup E_2) & \Sigma_2 & \xrightarrow{\varphi'} \Sigma' \end{array}$$

The amalgamation M' of  $M_1$  ( $\Sigma_1$ -model) and  $M_2$  ( $\Sigma_2$ -model) satisfies  $E_1 \cup E_2$ .

# The method of diagrams

- Much used in (conventional concrete) model theory, pervading many developments in
  - free constructions
  - axiomatizability theory
  - saturated model theory
  - interpolation and definability
  - etc.
- At the abstract institutions level appears a categorical property that displays some fundamental coherence between the syntax and the semantics of a given institution.

# Conventional concrete diagrams (FOL)

Let M be an (S, F, P)-model.

- We add the elements of M as new constants to the signature, thus  $(S, F_M, P)$ ,
- **2** The (positive) *diagram of M*:

$$E_M = \{(S, F_M, P) \text{-atoms } \rho \mid M_M \models \rho\}.$$

where  $M_M$  is the expansion of M interpreting the new constants by themselves, i.e.  $(M_M)_m = m$  for each  $m \in M$ .

## Institution-independent diagrams

An institution *has (elementary) diagrams* when for each  $\Sigma$ -model *M* there exists

- **1**  $\iota_{\Sigma,M}$ :  $\Sigma \to \Sigma_M$ , and
- $E_M \subseteq Sen(\Sigma_M)$

such that

Other axioms about coherence wrt signature/model translations omitted.

# Some examples in FOL

# Other concepts of diagrams may be obtained by changing the concept of model homomorphism:

homomorphisms	diagram $E_M$
ordinary	all atoms in $M_M^*$
injective	
closed	all atoms and neg. of atomic relations in $M_M^*$
closed and injective	all atoms and neg. of atoms in $M_M^*$
elem. embeddings	$M_M^*$

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## Some examples in other institutions

#### Intuitionistic logic (IPL):

- for *P*-model  $M: P \rightarrow A$  (*A* any Heyting algebra),
  - the elementary extension is  $P \rightarrow P \uplus A$ , and
  - $\blacksquare E_M = \{\rho, \rho_1 \Rightarrow \rho_2 \in M_M^* \mid \rho, \rho_1, \rho_2 \in P \uplus A\}.$

#### Modal logic (MFOL, first order, Kripke semantics): no diagrams!

#### *Higher order logic* (HOL):

- for (S, F)-model M
  - the elementary extension is  $(S, F_M)$ ,

 $\blacksquare E_M = \{t = t' \mid M_M \models t = t'\}.$ 

## Lifting diagrams to presentations

#### Proposition

If  $\mathscr{I}$  has diagrams then  $\mathscr{I}^p$  has diagrams too.

For any  $(\Sigma, E)$ -model *M*:

• the elementary extension :

$$(\Sigma, E) \xrightarrow{\iota_{\Sigma}(M)} (\Sigma, E)_M = (\Sigma_M, Sen(\iota_{\Sigma}(M))(E))$$

• the diagram:

 $E_M \cup Sen(\iota_{\Sigma}(M))(E).$ 

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# An example of use of inst.-indep. diagrams establishing co-limits of models

In concrete situations, usually a difficult problem, e.g. co-limits of rings, etc.

In essence, the result below reduces the problem to co-limits of signatures, which in concrete situations is much easier.

#### Theorem

In any institution (Sig, Sen, MOD,  $\models$ ) such that

- 1 it has diagrams,
- 2 each presentation has initial models,
- **3** Sig has J-co-limits,
- 4 it has J-model amalgamation,

then each category of  $\Sigma$ -models has J-co-limits.

#### An application co-limits of models of Horn theories

#### Corollary

For any Horn theory (in a given **FOL** signature), the category of its models has (small) co-limits.

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The following is an instance of corollary above:

#### Corollary

The category of rings has (small) co-limits.



We set the institution to be **HCL**<sup>p</sup>, where

• HCL is the sub-institution of FOL that restricts the sentences to the Horn sentences,

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**HCL**<sup>p</sup> is the institution of the **HCL**-presentations.

# The proof II

We can easily check the hypotheses of the general theorem above.

- HCL has the FOL diagrams, consisting of atomic sentences.
  - We lift the diagrams from **HCL** to **HCL**<sup>p</sup>.
- 2 It is well known that Horn theories admit initial models.
- 3  $\mathbb{S}ig^{\mathbf{HCL}} = \mathbb{S}ig^{\mathbf{FOL}}$  is (small) co-complete.
  - We lift this from HCL-signatures to HCL<sup>p</sup>-signatures (i.e. HCL-presentations).
- FOL/HCL is exact (we have already proved).
   We lift model amalgamation from HCL to HCL<sup>p</sup>.