

## Outline

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## 1 Internal Logic

### 1.1 Boolean and other connectives

#### Conjunction

A  $\Sigma$ -sentence  $\rho$  is a *conjunction*  $(\rho_1 \wedge \rho_2)$  of  $\Sigma$ -sentences  $\rho_1$  and  $\rho_2$  when

$$\rho^* = \rho_1^* \cap \rho_2^*$$

The *institution has conjunctions* when any  $\Sigma$ -sentences  $\rho_1$  and  $\rho_2$  have a conjunction.

#### Disjunction

A  $\Sigma$ -sentence  $\rho$  is a *disjunction*  $(\rho_1 \vee \rho_2)$  of  $\Sigma$ -sentences  $\rho_1$  and  $\rho_2$  when

$$\rho^* = \rho_1^* \cup \rho_2^*$$

The *institution has disjunctions* when any  $\Sigma$ -sentences  $\rho_1$  and  $\rho_2$  have a disjunction.

#### Implication

A  $\Sigma$ -sentence  $\rho$  is an *implication*  $(\rho_1 \Rightarrow \rho_2)$  of  $\Sigma$ -sentences  $\rho_1$  and  $\rho_2$  when

$$\rho^* = \overline{\rho_1^*} \cup \rho_2^*$$

The *institution has implications* when any  $\Sigma$ -sentences  $\rho_1$  and  $\rho_2$  have an implication.

#### Negation

A  $\Sigma$ -sentence  $\rho$  is a *negation*  $(\neg \rho')$  of a  $\Sigma$ -sentence  $\rho'$  when

$$\rho^* = \overline{\rho'^*}$$

The *institution has negations* when any  $\Sigma$ -sentence  $\rho'$  has a negation.

### Abstract connectives

A (semantic logical) connective  $c$  of arity  $n$  consists of a family  $(c_\Sigma)_{\Sigma \in \text{Sig}}$  of functions

$$c_\Sigma : \mathcal{P}(|\text{MOD}(\Sigma)|)^n \rightarrow \mathcal{P}(|\text{MOD}(\Sigma)|).$$

- A connective is *Boolean* when it is a (derived) operation of the Boolean algebra  $(\mathcal{P}(|\text{MOD}(\Sigma)|), \cap, \cup, \neg, \emptyset)$ .
- $\rho$  is a  $c$ -connection of  $\rho_i$ ,  $1 \leq i \leq n$ , ( $\rho = c(\rho_1, \dots, \rho_n)$ ) when  $\rho^* = c_\Sigma(\rho_1^*, \dots, \rho_n^*)$ .

### Examples

institution	$\wedge$	$\neg$	$\vee$	$\Rightarrow$	$\Leftrightarrow$
<b>FOL, PL, HOL, HNK</b>	✓	✓	✓	✓	✓
<b>WPL(Béziau)</b>	✓		✓	✓	✓
<b>FOL<sup>+</sup></b>	✓		✓		
<b>EQL, HCL, MVL</b>					
<b>EQLN</b>		✓			
<b>MFOL, MPL</b>	✓				
<b>IPL</b>	✓				

## 1.2 Quantifiers

### Quantifiers

Given signature morphism  $\chi : \Sigma \rightarrow \Sigma'$ ,  $\rho \in \text{Sen}(\Sigma)$  and  $\rho' \in \text{Sen}(\Sigma')$ ,

- $\rho$  is a *universal  $\chi$ -quantification* of  $\rho'$  when

$$\rho^* = \overline{\text{MOD}(\chi)(\rho'^*)}$$

- $\rho$  is a *existential  $\chi$ -quantification* of  $\rho'$  when

$$\rho^* = \text{MOD}(\chi)(\rho'^*)$$

The *institution has universal/existential  $\mathcal{D}$ -quantifiers* when for each  $(\chi : \Sigma \rightarrow \Sigma') \in \mathcal{D}$ , any  $\Sigma'$ -sentence  $\rho'$  has a universal/existential  $\chi$ -quantification.

### Examples

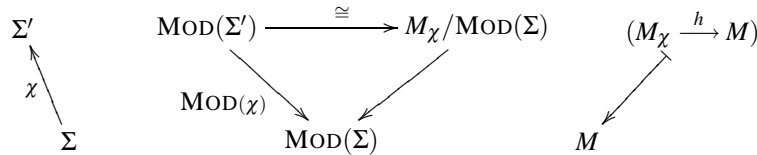
institution	$\mathcal{D}$	$\forall$	$\exists$
<b>FOL, MVL</b>	fin. inj. sign. ext. with constants	✓	✓
<b>SOL</b>	fin. inj. sign. ext.	✓	✓
<b>PA</b>	fin. inj. sign. ext. with total constants	✓	✓
<b>EQL, HCL</b>	fin. inj. sign. ext. with constants	✓	
<b>MFOL</b>	fin. inj. sign. ext. with rigid constants	✓	
<b>HOL, HNK</b>	fin. inj. sign. ext.	✓	✓

fin. inj. sign. ext. = finitary injective signature extension

### Representable signature morphisms

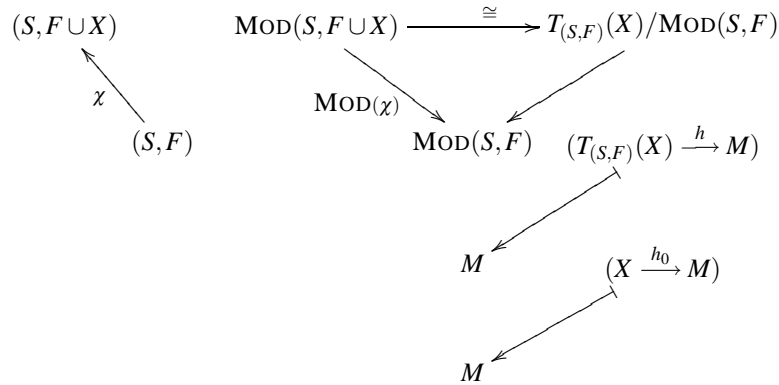
Many results depend on the quantification being *first order*.

At the level of abstract institutions this is captured by the condition that the signature morphism  $\chi$  is *representable*:



$\chi$  is *finitary representable* when  $M_\chi$  is finitely presented.

### A concrete example: MSA first order quantifiers



### Quasi-representable signature morphisms

A weaker very useful version of representability:

$\chi: \Sigma \rightarrow \Sigma'$  is *quasi-representable* if and only if

$$M'/\text{MOD}(\Sigma') \cong (M' \upharpoonright_{\chi})/\text{MOD}(\Sigma)$$

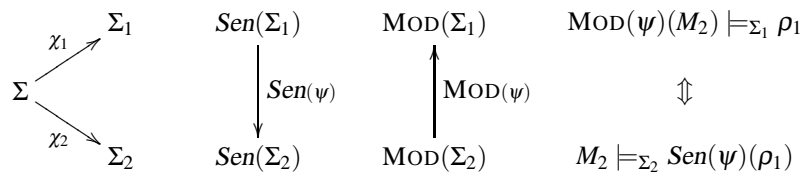
**Proposition 1.** A signature morphism  $\chi: \Sigma \rightarrow \Sigma'$  is representable if and only if it is quasi-representable and  $\text{MOD}(\Sigma')$  has initial models.

### Examples

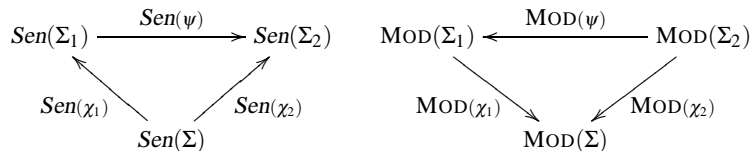
institution	$\chi$	
<b>FOL</b>	(fin.) inj. sign. ext. with const.	(fin.) rep.
<b>MVL</b>	(fin.) inj. sign. ext. with const.	(fin.) rep.
<b>PA</b>	(fin.) inj. sign. ext. with total const.	(fin.) rep.
<b>E(FOL)</b>	(fin.) inj. sign. ext. with const.	(fin.) quasi-rep.
<b>MFOL</b>	(fin.) inj. sign. ext. with rigid const.	(fin.) quasi-rep.
<b>HOL</b>	(fin.) inj. sign. ext.	(fin.) quasi-rep.

### 1.3 General substitutions

**Substitution**  $\psi: \chi_1 \rightarrow \chi_2$



such that



## Examples

- First order substitutions  $\psi : X \rightarrow T_{(S,F)}(Y)$ :
  - $\psi : ((S,F) \rightarrow (S,F \cup X)) \rightarrow ((S,F) \rightarrow (S,F \cup Y))$
  - $\text{Sen}(\psi) : \text{Sen}(S,F \cup X) \rightarrow \text{Sen}(S,F \cup Y)$ ,
  - $\text{MOD}(\psi) : \text{MOD}(S,F \cup Y) \rightarrow \text{MOD}(S,F \cup X)$

$$\text{MOD}(\psi)(M)_x = M_{\psi(x)}$$

- Second order substitutions mapping operations to terms (such that arity preserved).
- In **HOL,HNK**, higher order substitutions.

## Representable substitutions

Capture abstractly the concept ‘first order’ substitutions.

$\psi : \chi_1 \rightarrow \chi_2$  with  $\chi_1$  and  $\chi_2$  representable.

**Proposition 2.** Any substitution  $\psi : \chi_1 \rightarrow \chi_2$  between representable signature morphisms  $\chi_1 : \Sigma \rightarrow \Sigma_1$  and  $\chi_2 : \Sigma \rightarrow \Sigma_2$  determines canonically a  $\Sigma$ -model homomorphism  $M_\psi : M_{\chi_1} \rightarrow M_{\chi_2}$ . Moreover, the mapping  $\psi \mapsto M_\psi$  is functorial and faithful [modulo substitution equivalence].

**Example:**

$$\begin{array}{ccc} ((S,F) \rightarrow (S,F \cup X)) & \xrightarrow{\psi} & ((S,F) \rightarrow (S,F \cup Y)) \\ \downarrow & & \downarrow \\ T_{(S,F)}(X) & \xrightarrow{M_\psi} & T_{(S,F)}(Y) \\ \updownarrow & & \updownarrow \\ (X \rightarrow T_{(S,F)}(Y)) & & \end{array}$$

## 1.4 Basic sentences

### Abstract capture of atomic sentences

In **MSA**, categorical characterization of satisfaction of atoms in the style of ‘satisfaction by injectivity’ (Nemeti, Andreka, ...):

**Proposition 3.**  $M \models_{(S,F)} t = t'$  if and only if there exists a homomorphism  $T_{(S,F)}/_{\{t=t'\}} \rightarrow M$ .

### Basic sentences

In any institution, a  $\Sigma$ -sentence  $\rho$  is (*finitary*) *basic* when there exists a (finitely presented)  $\Sigma$ -model  $M_\rho$  such that for any  $\Sigma$ -model  $M$

$$M \models \rho \text{ if and only if there exists homomorphism } M_\rho \rightarrow M$$

In actual institutions atoms are finitary basic, but also:

**Proposition 4.** *Basic sentences are closed under existential quasi-representable quantification.*

### A tighter approximation of atoms

The following rules out some ‘non-atomic’ basic sentences (e.g.  $(\exists \chi)\rho$ , for atomic  $\rho$ ):

In any institution with initial models of signatures (denoted  $0_\Sigma$ ), a basic sentence  $\rho$  is *epic basic* when the unique homomorphism  $0_\Sigma \rightarrow M_\rho$  is epi.

Epic basic are a tighter capture of ‘atomic’ sentences, yet not a perfect one.

## 2 Ultraproducts

### The method of ultraproducts

One of the most powerful model theory methods, much model theory may be developed through this method (see Bell and Slomson classic book).

Applications include:

- (semantic) compactness,
- preservation and axiomatizability,
- Keisler-Shelah Isomorphism Theorem,
- interpolation and definability,
- applications to algebra (fields, algebraic geometry, etc.).

### 2.1 Categorical ultraproducts

#### Filters and Ultrafilters

$F \subseteq \mathcal{P}(I)$  is *filter over I* when

- $I \in F$ ,
- $X \cap Y \in F$  if  $X \in F$  and  $Y \in F$ ,
- $Y \in F$  if  $X \subseteq Y$  and  $X \in F$ .

*Ultrafilter* when in addition, for all  $X \subseteq I$ , we have that

$$X \in F \text{ if and only if } I \setminus X \notin F$$

If  $F$  filter over  $I$ , and  $I' \subseteq I$ , then the *reduction of  $F$  to  $I'$* :

$$F|_{I'} = \{I' \cap X \mid X \in F\}$$

#### Concrete filtered products

In **MSA**: given  $F$  filter over  $I$ , and  $(M_i)_{i \in I}$  family of  $(S, F)$ -algebras, the  *$F$ -filtered product of  $(M_i)_{i \in I}$*  is defined as  $\prod_F M_i = (\prod_{i \in I} M_i) / \sim_F$  where

1.  $\prod_{i \in I} M_i$  is direct product of  $(M_i)_{i \in I}$ , and
2.  $\sim_F$  is the congruence defined by

$$m \sim_F m' \text{ if and only if } \{i \in I \mid m_i \sim_F m'_i\} \in F.$$

#### Categorical filtered products

Co-limit of the diagram of projections ( $J, J' \subseteq I$ ):

$$\begin{array}{ccc} \prod_{i \in J'} M_i & \xrightarrow{p_{J', J}} & \prod_{i \in J} M_i \\ & \searrow \mu_{J'} & \swarrow \mu_J \\ & \prod_F M_i & \end{array}$$

Idea much exploited by approaches to categorical model theory (Nemeti, Andreka, Makkai, etc.)

### Some examples

<i>institution</i>	direct prod.	directed co-lim.	filt.prod.	ultraprod.
<b>FOL</b>	✓	✓	✓	✓
<b>PA</b>	✓	✓	✓	✓
<b>IPL</b>	✓	✓	✓	✓
<b>MFOL</b>	✓	✓	✓	✓
<b>MVL</b>	✓	?	✓	✓
<b>HNK</b>	✓			✓
<b>MA</b>				

## 2.2 Fundamental Ultraproducts Theorem

### Preservation of sentences by filtered factors/products

For a signature  $\Sigma$  in an institution, for each filter  $F \in \mathcal{F}$  over a set  $I$  and for each family  $\{A_i\}_{i \in I}$  of  $\Sigma$ -models, a  $\Sigma$ -sentence  $e$  is

- preserved by  $\mathcal{F}$ -filtered factors:  $f_{\mathcal{F}}(e)$

$$\text{if } \prod_F A_i \models_{\Sigma} e \text{ implies } \{i \in I \mid A_i \models_{\Sigma} e\} \in F,$$

- preserved by  $\mathcal{F}$ -filtered products:  $p_{\mathcal{F}}(e)$

$$\text{if } \{i \in I \mid A_i \models_{\Sigma} e\} \in F \text{ implies } \prod_F A_i \models_{\Sigma} e.$$

Preservation by ultrafactors/ultraproducts when  $\mathcal{F}$  is the class of ultrafilters.

### Fundamental Ultraproducts Theorem (Los)

In any institution:

<i>preservation property</i>	<i>condition</i>
$p_{\mathcal{F}}$ (basic)	
$f_{\mathcal{F}}$ (finitary basic)	
$p_{\mathcal{F}}(\rho) \Rightarrow p_{\mathcal{F}}((\exists \chi)\rho)$	MOD( $\chi$ ) pres. $\mathcal{F}$ -filtered prod.
$f_{\mathcal{F}}(\rho) \Rightarrow f_{\mathcal{F}}((\exists \chi)\rho)$	MOD( $\chi$ ) lifts $\mathcal{F}$ -filtered prod.
$p_{\mathcal{F}}(\rho_1), p_{\mathcal{F}}(\rho_2) \Rightarrow p_{\mathcal{F}}(\rho_1 \wedge \rho_2)$	
$f_{\mathcal{F}}(\rho_1), f_{\mathcal{F}}(\rho_2) \Rightarrow f_{\mathcal{F}}(\rho_1 \wedge \rho_2)$	
$(p_{\mathcal{F}}(\rho_i))_{i \in I} \Rightarrow p_{\mathcal{F}}(\bigwedge_{i \in I} \rho_i)$	
$f_{\mathcal{F}}(\rho) \Rightarrow p_{\mathcal{F}}(\neg \rho)$	
$p_{\mathcal{F}}(\rho) \Rightarrow f_{\mathcal{F}}(\neg \rho)$	$\mathcal{F} \subseteq \text{Ultrafilters}$

### Corollary

**Corollary 5.** In any institution, any sentence which is accessible from the finitary basic sentences by

- Boolean connectives,
- finitary representable quantification, and
- projectively representable quantification (assuming that the institution has epi model projections)

is preserved by ultraproducts and ultrafactors.

Examples includes **FOL**, **PA**, **IPL**, etc. In **MFOL**, **FOL** $_{\infty}$  sentences preserved only by ultraproducts.

## 2.3 Compactness by ultraproducts

### (Semantic) compactness

An institution is

1. *m-compact* when each set of sentences has a model if and only if any of its *finite* subsets has a model.
2. *compact* when  $E \models \rho$  implies that there exists *finite*  $E_0 \subseteq E$  such that  $E_0 \models \rho$ .

**Proposition 6.** – *Each compact institution having false is m-compact.*

– *Each m-compact institution having negations is compact.*

### Compactness by ultraproducts

**Corollary 7.** *Any institution in which each sentence is preserved by ultraproducts is m-compact.*

Examples include **FOL**, **PA**, **IPL**, etc. but also **MFOL**.

**Corollary 8.** *Let  $E$  be a set of sentences preserved by ultraproducts, and let  $e$  be a sentence preserved by ultrafactors such that  $E \models e$ . Then there exists a finite subset  $E' \subseteq E$  such that  $E' \models e$ .*