Outline

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1 Internal Logic

1.1 Boolean and other connectives

Conjunction

A Σ-sentence ρ is a *conjunction* $(\rho_1 \wedge \rho_2)$ of Σ-sentences ρ_1 and ρ_2 when

$$\rho^\star = \rho_1^\star \cap \rho_2^\star$$

The *institution has conjunctions* when any Σ-sentences ρ_1 and ρ_2 have a conjunction.

Disjunction

A Σ-sentence ρ is a *disjunction* $(\rho_1 \vee \rho_2)$ of Σ-sentences ρ_1 and ρ_2 when

$$\rho^\star = \rho_1^\star \cup \rho_2^\star$$

The *institution has disjunctions* when any Σ-sentences ρ_1 and ρ_2 have a disjunction.

Implication

A Σ-sentence ρ is an *implication* $(\rho_1 \Rightarrow \rho_2)$ of Σ-sentences ρ_1 and ρ_2 when

$$\rho^\star = \overline{\rho_1^\star} \cup \rho_2^\star$$

The *institution has implications* when any Σ-sentences ρ_1 and ρ_2 have a implication.

Negation

A Σ-sentence ρ is a *negation* $(\neg\rho')$ of a Σ-sentence ρ' when

$$\rho^\star = \overline{\rho'^\star}$$

The *institution has negations* when any Σ-sentence ρ' has a negation.

Abstract connectives

A *(semantic logical) connective c of arity n* consists of a family $(c_\Sigma)_{\Sigma\in\mathbb{S}ig}$ of functions

$$c_\Sigma : \ \mathscr{P}(|\mathrm{Mod}(\Sigma)|)^n \to \mathscr{P}(|\mathrm{Mod}(\Sigma)|).$$

- A connective is *Boolean* when it is a (derived) operation of the Boolean algebra $(\mathscr{P}(|\mathrm{Mod}(\Sigma)|),\cap,\cup,\neg,\emptyset)$.
- *ρ is a c-connection of ρ_i*, $1 \le i \le n$, $(\rho = c(\rho_1,\dots,\rho_n))$ when $\rho^\star = c_\Sigma(\rho_1^\star,\dots,\rho_n^\star)$.

Examples

institution	$\wedge$	$\neg$	$\vee$	$\Rightarrow$	$\Leftrightarrow$
FOL, **PL**, **HOL**, **HNK**	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
WPL(Béziau)	$\surd$		$\surd$	$\surd$	$\surd$
FOL$^+$	$\surd$		$\surd$		
EQL, **HCL**, **MVL**					
EQLN		$\surd$			
MFOL, **MPL**	$\surd$				
IPL	$\surd$				

1.2 Quantifiers

Quantifiers

Given signature morphism $\chi : \ \Sigma \to \Sigma'$, $\rho \in Sen(\Sigma)$ and $\rho' \in Sen(\Sigma')$,

- ρ is a *universal χ-quantification* of ρ' when

$$\rho^\star = \overline{\mathrm{Mod}(\chi)(\overline{\rho'^\star})}$$

- ρ is a *existential χ-quantification* of ρ' when

$$\rho^\star = \mathrm{Mod}(\chi)(\rho'^\star)$$

The *institution has universal/existential $\mathscr{D}$-quantifiers* when for each $(\chi : \ \Sigma \to \Sigma') \in \mathscr{D}$, any Σ'-sentence ρ' has a universal/existential χ-quantification.

Examples

institution	$\mathscr{D}$	$\forall$	$\exists$
FOL, **MVL**	fin. inj. sign. ext. with constants	$\surd$	$\surd$
SOL	fin. inj. sign. ext.	$\surd$	$\surd$
PA	fin. inj. sign. ext. with total constants	$\surd$	$\surd$
EQL, **HCL**	fin. inj. sign. ext. with constants	$\surd$	
MFOL	fin. inj. sign. ext. with rigid constants	$\surd$	
HOL, **HNK**	fin. inj. sign. ext.	$\surd$	$\surd$

fin. inj. sign. ext. = finitary injective signature extension

Representable signature morphisms

Many results depend on the quantification being *first order*.

At the level of abstract institutions this is captured by the condition that the signature morphism χ is *representable*:

$$\begin{tikzcd} \Sigma' \\ \Sigma \arrow[u, "\chi"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mod}(\Sigma') \arrow[rr, "\cong"] \arrow[dr, "\mathrm{Mod}(\chi)"'] & & M_\chi/\mathrm{Mod}(\Sigma) \arrow[dl] \\ & \mathrm{Mod}(\Sigma) & \end{tikzcd} \qquad \begin{tikzcd} (M_\chi \xrightarrow{h} M) \arrow[d, mapsto] \\ M \end{tikzcd}$$

χ is *finitary representable* when M_χ is finitely presented.

A concrete example: MSA first order quantifiers

$$
\begin{tikzcd}
(S, F \cup X) & \\
& (S,F) \arrow[ul, "\chi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{MOD}(S, F \cup X) \arrow[rr, "\cong"] \arrow[dr, "\mathrm{MOD}(\chi)"'] & & T_{(S,F)}(X)/\mathrm{MOD}(S,F) \arrow[dl] \\
& \mathrm{MOD}(S,F) &
\end{tikzcd}
$$

$$
(T_{(S,F)}(X) \xrightarrow{h} M) \mapsto M
\qquad
(X \xrightarrow{h_0} M) \mapsto M
$$

Quasi-representable signature morphisms

A weaker very useful version of representability:

$\chi : \ \Sigma \to \Sigma'$ is *quasi-representable* if and only if

$$M'/\mathrm{MOD}(\Sigma') \cong (M'\!\restriction_{\chi})/\mathrm{MOD}(\Sigma)$$

Proposition 1. *A signature morphism $\chi : \ \Sigma \to \Sigma'$ is representable if and only if it is quasi-representable and* $\mathrm{MOD}(\Sigma')$ *has initial models.*

Examples

institution	χ	
FOL	(fin.) inj. sign. ext. with const.	*(fin.) rep.*
MVL	(fin.) inj. sign. ext. with const.	*(fin.) rep.*
PA	(fin.) inj. sign. ext. with total const.	*(fin.) rep.*
E(**FOL**)	(fin.) inj. sign. ext. with const.	(fin.) quasi-rep.
MFOL	(fin.) inj. sign. ext. with rigid const.	(fin.) quasi-rep.
HOL	(fin.) inj. sign. ext.	(fin.) quasi-rep.

1.3 General substitutions

Substitution $\psi : \ \chi_1 \to \chi_2$

$$
\begin{tikzcd}
& \Sigma_1 \\
\Sigma \arrow[ur, "\chi_1"] \arrow[dr, "\chi_2"'] & \\
& \Sigma_2
\end{tikzcd}
\qquad
\begin{tikzcd}
Sen(\Sigma_1) \arrow[d, "Sen(\psi)"] \\
Sen(\Sigma_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{MOD}(\Sigma_1) \\
\mathrm{MOD}(\Sigma_2) \arrow[u, "\mathrm{MOD}(\psi)"']
\end{tikzcd}
\qquad
\begin{array}{c}
\mathrm{MOD}(\psi)(M_2) \models_{\Sigma_1} \rho_1 \\
\Updownarrow \\
M_2 \models_{\Sigma_2} Sen(\psi)(\rho_1)
\end{array}
$$

such that

$$
\begin{tikzcd}
Sen(\Sigma_1) \arrow[rr, "Sen(\psi)"] & & Sen(\Sigma_2) \\
& Sen(\Sigma) \arrow[ul, "Sen(\chi_1)"] \arrow[ur, "Sen(\chi_2)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{MOD}(\Sigma_1) \arrow[dr, "\mathrm{MOD}(\chi_1)"'] & & \mathrm{MOD}(\Sigma_2) \arrow[ll, "\mathrm{MOD}(\psi)"'] \arrow[dl, "\mathrm{MOD}(\chi_2)"] \\
& \mathrm{MOD}(\Sigma) &
\end{tikzcd}
$$

Examples

- First order substitutions $\psi: X \to T_{(S,F)}(Y)$:
  - $\psi: ((S,F) \to (S,F\cup X)) \to ((S,F) \to (S,F\cup Y))$
  - $Sen(\psi): Sen(S,F\cup X) \to Sen(S,F\cup Y)$,
  - $\mathrm{MOD}(\psi): \mathrm{MOD}(S,F\cup Y) \to \mathrm{MOD}(S,F\cup X)$

$$\mathrm{MOD}(\psi)(M)_x = M_{\psi(x)}$$

- Second order substitutions mapping operations to terms (such that arity preserved).
- In **HOL**,**HNK**, higher order substitutions.

Representable substitutions

Capture abstractly the concept 'first order' substitutions.

$$\psi: \chi_1 \to \chi_2 \text{ with } \chi_1 \text{ and } \chi_2 \text{ representable.}$$

Proposition 2. *Any substitution $\psi: \chi_1 \to \chi_2$ between representable signature morphisms $\chi_1: \Sigma \to \Sigma_1$ and $\chi_2: \Sigma \to \Sigma_2$ determines canonically a Σ-model homomorphism $M_\psi: M_{\chi_1} \to M_{\chi_2}$. Moreover, the mapping $\psi \mapsto M_\psi$ is functorial and faithful [modulo substitution equivalence].*

Example:

$$\begin{array}{c} ((S,F) \to (S,F\cup X)) \xrightarrow{\psi} ((S,F) \to (S,F\cup Y)) \\ \big\downarrow \\ T_{(S,F)}(X) \xrightarrow{M_\psi} T_{(S,F)}(Y) \\ \big\updownarrow \\ (X \to T_{(S,F)}(Y)) \end{array}$$

1.4 Basic sentences

Abstract capture of atomic sentences

In **MSA**, categorical characterization of satisfaction of atoms in the style of 'satisfaction by injectivity' (Nemeti, Andreka, ...):

Proposition 3. *$M \models_{(S,F)} t = t'$ if and only if there exists a homomorphism $T_{(S,F)}/_{=_{\{t=t'\}}} \to M$.*

Basic sentences

In any institution, a Σ-sentence ρ is *(finitary) basic* when there exists a (finitely presented) Σ-model M_ρ such that for any Σ-model M

$$M \models \rho \text{ if and only if there exists homomorphism } M_\rho \to M$$

In actual institutions atoms are finitary basic, but also:

Proposition 4. *Basic sentences are closed under existential quasi-representable quantification.*

A tighter approximation of atoms

The following rules out some 'non-atomic' basic sentences (e.g. $(\exists\chi)\rho$, for atomic ρ):

In any institution with initial models of signatures (denoted 0_Σ), a basic sentence ρ is *epic basic* when the unique homomorphism $0_\Sigma \to M_\rho$ is epi.

Epic basic are a tighter capture of 'atomic' sentences, yet not a perfect one.

2 Ultraproducts

The method of ultraproducts

One of the most powerful model theory methods, much model theory may be developed through this method (see Bell and Slomson classic book).

Applications include:

- (semantic) compactness,
- preservation and axiomatizability,
- Keisler-Shelah Isomorphism Theorem,
- interpolation and definability,
- applications to algebra (fields, algebraic geometry, etc.).

2.1 Categorical ultraproducts

Filters and Ultrafilters

$F \subseteq \mathscr{P}(I)$ is *filter over I* when

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$,
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

Ultrafilter when in addition, for all $X \subseteq I$, we have that

$$X \in F \text{ if and only if } I \setminus X \not\in F$$

If F filter over I, and $I' \subseteq I$, then the *reduction of F to I'*:

$$F|_{I'} = \{I' \cap X \mid X \in F\}$$

Concrete filtered products

In **MSA**: given F filter over I, and $(M_i)_{i\in I}$ family of (S,F)-algebras, the *F-filtered product of $(M_i)_{i\in I}$* is defined as $\prod_F M_i = (\prod_{i\in I} M_i)/_{\sim_F}$ where

1. $\prod_{i\in I} M_i$ is direct product of $(M_i)_{i\in I}$, and
2. $\sim_F$ is the congruence defined by

$$m \sim_F m' \text{ if and only if } \{i \in I \mid m_i \sim_F m'_i\} \in F.$$

Categorical filtered products

Co-limit of the diagram of projections ($J, J' \subseteq I$):

$$\begin{tikzcd} \prod_{i\in J'} M_i \arrow[rr, "p_{J',J}"] \arrow[dr, "\mu_{J'}"'] & & \prod_{i\in J} M_i \arrow[dl, "\mu_J"] \\ & \prod_F M_i & \end{tikzcd}$$

Idea much exploited by approaches to categorical model theory (Nemeti, Andreka, Makkai, etc.)

Some examples

institution	direct prod.	directed co-lim.	filt.prod.	ultraprod.
FOL	$\surd$	$\surd$	$\surd$	$\surd$
PA	$\surd$	$\surd$	$\surd$	$\surd$
IPL	$\surd$	$\surd$	$\surd$	$\surd$
MFOL	$\surd$	$\surd$	$\surd$	$\surd$
MVL	$\surd$	?	$\surd$	$\surd$
HNK	$\surd$			$\surd$
MA				

2.2 Fundamental Ultraproducts Theorem

Preservation of sentences by filtered factors/products

For a signature Σ in an institution, for each filter $F \in \mathscr{F}$ over a set I and for each family $\{A_i\}_{i\in I}$ of Σ-models, a Σ-sentence e is

- *preserved by $\mathscr{F}$-filtered factors:* $f_{\mathscr{F}}(e)$

$$\text{if } \prod_F A_i \models_\Sigma e \text{ implies } \{i \in I \mid A_i \models_\Sigma e\} \in F,$$

- *preserved by $\mathscr{F}$-filtered products:* $p_{\mathscr{F}}(e)$

$$\text{if } \{i \in I \mid A_i \models_\Sigma e\} \in F \text{ implies } \prod_F A_i \models_\Sigma e.$$

Preservation by ultrafactors/ultraproducts when $\mathscr{F}$ is the class of ultrafilters.

Fundamental Ultraproducts Theorem (Łos)

In any institution:

preservation property	*condition*
$p_{\mathscr{F}}(\text{basic})$	
$f_{\mathscr{F}}(\text{finitary basic})$	
$p_{\mathscr{F}}(\rho) \Rightarrow p_{\mathscr{F}}((\exists\chi)\rho)$	$\mathrm{Mod}(\chi)$ pres. $\mathscr{F}$-filtered prod.
$f_{\mathscr{F}}(\rho) \Rightarrow f_{\mathscr{F}}((\exists\chi)\rho)$	$\mathrm{Mod}(\chi)$ lifts $\mathscr{F}$-filtered prod.
$p_{\mathscr{F}}(\rho_1), p_{\mathscr{F}}(\rho_2) \Rightarrow p_{\mathscr{F}}(\rho_1 \wedge \rho_2)$	
$f_{\mathscr{F}}(\rho_1), f_{\mathscr{F}}(\rho_2) \Rightarrow f_{\mathscr{F}}(\rho_1 \wedge \rho_2)$	
$(p_{\mathscr{F}}(\rho_i))_{i\in I} \Rightarrow p_{\mathscr{F}}(\wedge_{i\in I}\rho_i)$	
$f_{\mathscr{F}}(\rho) \Rightarrow p_{\mathscr{F}}(\neg\rho)$	
$p_{\mathscr{F}}(\rho) \Rightarrow f_{\mathscr{F}}(\neg\rho)$	$\mathscr{F} \subseteq Ultrafilters$

Corollary

Corollary 5. *In any institution, any sentence which is accessible from the finitary basic sentences by*

– *Boolean connectives,*

– *finitary representable quantification, and*

– *projectively representable quantification (assuming that the institution has epi model projections)*

is preserved by ultraproducts and ultrafactors.

Examples includes **FOL**, **PA**, **IPL**, etc. In **MFOL**, $\mathbf{FOL}_\infty$ sentences preserved only by ultraproducts.

2.3 Compactness by ultraproducts

(Semantic) compactness

An institution is

1. *m-compact* when each set of sentences has a model if and only if any of its *finite* subsets has a model.
2. *compact* when $E \models \rho$ implies that there exists *finite* $E_0 \subseteq E$ such that $E_0 \models \rho$.

Proposition 6. – *Each compact institution having false is m-compact.*

– *Each m-compact institution having negations is compact.*

Compactness by ultraproducts

Corollary 7. *Any institution in which each sentence is preserved by ultraproducts is m-compact.*

Examples include **FOL**, **PA**, **IPL**, etc. but also **MFOL**.

Corollary 8. *Let E be a set of sentences preserved by ultraproducts, and let e be a sentence preserved by ultrafactors such that $E \models e$. Then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.*