Institution Theory

internal logic

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Outline

Conjunction

A Σ-sentence ρ is a *conjunction* $(\rho_1 \wedge \rho_2)$ of Σ-sentences ρ_1 and ρ_2 when

$$\rho^\star = \rho_1^\star \cap \rho_2^\star$$

The *institution has conjunctions* when any Σ-sentences ρ_1 and ρ_2 have a conjunction.

Disjunction

A Σ-sentence ρ is a *disjunction* $(\rho_1 \vee \rho_2)$ of Σ-sentences ρ_1 and ρ_2 when

$$\rho^\star = \rho_1^\star \cup \rho_2^\star$$

The *institution has disjunctions* when any Σ-sentences ρ_1 and ρ_2 have a disjunction.

Implication

A Σ-sentence ρ is an *implication* $(\rho_1 \Rightarrow \rho_2)$ of Σ-sentences ρ_1 and ρ_2 when

$$\rho^\star = \overline{\rho_1^\star} \cup \rho_2^\star$$

The *institution has implications* when any Σ-sentences ρ_1 and ρ_2 have a implication.

Negation

A Σ-sentence ρ is a *negation* $(\neg\rho')$ of a Σ-sentence ρ' when

$$\rho^\star = \overline{\rho'^\star}$$

The *institution has negations* when any Σ-sentence ρ' has a negation.

Abstract connectives

A *(semantic logical) connective c of arity n* consists of a family $(c_\Sigma)_{\Sigma \in \mathbb{S}ig}$ of functions

$$c_\Sigma : \ \mathscr{P}(|\mathrm{Mod}(\Sigma)|)^n \to \mathscr{P}(|\mathrm{Mod}(\Sigma)|).$$

- A connective is *Boolean* when it is a (derived) operation of the Boolean algebra $(\mathscr{P}(|\mathrm{Mod}(\Sigma)|), \cap, \cup, \neg, \emptyset)$.
- ρ *is a c-connection of* ρ_i, $1 \le i \le n$, ($\rho = c(\rho_1, \ldots, \rho_n)$) when $\rho^\star = c_\Sigma(\rho_1^\star, \ldots, \rho_n^\star)$.

Examples

institution	$\wedge$	$\neg$	$\vee$	$\Rightarrow$	$\Leftrightarrow$
FOL, PL, HOL, HNK	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
WPL(Béziau)	$\surd$		$\surd$	$\surd$	$\surd$
$\mathbf{FOL}^{+}$	$\surd$		$\surd$		
EQL, HCL, MVL					
EQLN		$\surd$			
MFOL, MPL	$\surd$				
IPL	$\surd$				

Quantifiers

Given signature morphism $\chi : \Sigma \to \Sigma'$, $\rho \in Sen(\Sigma)$ and $\rho' \in Sen(\Sigma')$,

- ρ is a *universal χ-quantification* of ρ' when

$$\rho^\star = \overline{\mathrm{Mod}(\chi)(\overline{\rho'^\star})}$$

- ρ is a *existential χ-quantification* of ρ' when

$$\rho^\star = \mathrm{Mod}(\chi)(\rho'^\star)$$

The *institution has universal/existential $\mathscr{D}$-quantifiers* when for each $(\chi : \Sigma \to \Sigma') \in \mathscr{D}$, any Σ'-sentence ρ' has a universal/existential χ-quantification.

Examples

institution	$\mathscr{D}$	$\forall$	$\exists$
FOL, MVL	fin. inj. sign. ext. with constants	$\surd$	$\surd$
SOL	fin. inj. sign. ext.	$\surd$	$\surd$
PA	fin. inj. sign. ext. with total constants	$\surd$	$\surd$
EQL, HCL	fin. inj. sign. ext. with constants	$\surd$	
MFOL	fin. inj. sign. ext. with rigid constants	$\surd$	
HOL, HNK	fin. inj. sign. ext.	$\surd$	$\surd$

fin. inj. sign. ext. = finitary injective signature extension

Representable signature morphisms

Many results depend on the quantification being *first order*.

At the level of abstract institutions this is captured by the condition that the signature morphism χ is *representable*:

$$
\begin{tikzcd}
\Sigma' \\
\Sigma \arrow[u, "\chi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Mod}(\Sigma') \arrow[rr, "\cong"] \arrow[dr, "\mathrm{Mod}(\chi)"'] & & M_\chi/\mathrm{Mod}(\Sigma) \arrow[dl] \\
& \mathrm{Mod}(\Sigma) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(M_\chi \xrightarrow{h} M) \arrow[d, mapsto] \\
M
\end{tikzcd}
$$

χ is *finitary representable* when M_χ is finitely presented.

A concrete example: **MSA** first order quantifiers

$$
\begin{tikzcd}
(S, F \cup X) & & \textsc{Mod}(S, F \cup X) \arrow[rr, "\cong"] \arrow[dr, "\textsc{Mod}(\chi)"'] & & T_{(S,F)}(X)/\textsc{Mod}(S,F) \arrow[dl] \\
& (S,F) \arrow[ul, "\chi"] & & \textsc{Mod}(S,F) & (T_{(S,F)}(X) \xrightarrow{h} M) \arrow[dl, mapsto] \\
& & & M & (X \xrightarrow{h_0} M) \arrow[dl, mapsto] \\
& & & M &
\end{tikzcd}
$$

Quasi-representable signature morphisms

A weaker very useful version of representability:

$\chi : \Sigma \to \Sigma'$ is *quasi-representable* if and only if

$$M'/\mathrm{MOD}(\Sigma') \cong (M' \restriction_\chi)/\mathrm{MOD}(\Sigma)$$

Proposition

A signature morphism $\chi : \Sigma \to \Sigma'$ is representable if and only if it is quasi-representable and $\mathrm{MOD}(\Sigma')$ has initial models.

Examples

institution	χ	
FOL	(fin.) inj. sign. ext. with const.	*(fin.) rep.*
MVL	(fin.) inj. sign. ext. with const.	*(fin.) rep.*
PA	(fin.) inj. sign. ext. with total const.	*(fin.) rep.*
E(**FOL**)	(fin.) inj. sign. ext. with const.	*(fin.) quasi-rep.*
MFOL	(fin.) inj. sign. ext. with rigid const.	*(fin.) quasi-rep.*
HOL	(fin.) inj. sign. ext.	*(fin.) quasi-rep.*

Substitution $\psi : \chi_1 \to \chi_2$

$$\begin{tikzcd}[row sep=small] & \Sigma_1 \\ \Sigma \arrow[ur, "\chi_1"] \arrow[dr, "\chi_2"'] & \\ & \Sigma_2 \end{tikzcd} \qquad \begin{tikzcd} Sen(\Sigma_1) \arrow[d, "Sen(\psi)"] \\ Sen(\Sigma_2) \end{tikzcd} \qquad \begin{tikzcd} \textsc{Mod}(\Sigma_1) \\ \textsc{Mod}(\Sigma_2) \arrow[u, "\textsc{Mod}(\psi)"'] \end{tikzcd} \qquad \begin{array}{c} \textsc{Mod}(\psi)(M_2) \models_{\Sigma_1} \rho_1 \\ \Updownarrow \\ M_2 \models_{\Sigma_2} Sen(\psi)(\rho_1) \end{array}$$

such that

$$\begin{tikzcd} Sen(\Sigma_1) \arrow[rr, "Sen(\psi)"] & & Sen(\Sigma_2) \\ & Sen(\Sigma) \arrow[ul, "Sen(\chi_1)"] \arrow[ur, "Sen(\chi_2)"'] & \end{tikzcd} \qquad \begin{tikzcd} \textsc{Mod}(\Sigma_1) \arrow[dr, "\textsc{Mod}(\chi_1)"'] & & \textsc{Mod}(\Sigma_2) \arrow[ll, "\textsc{Mod}(\psi)"'] \arrow[dl, "\textsc{Mod}(\chi_2)"] \\ & \textsc{Mod}(\Sigma) & \end{tikzcd}$$

Examples

- First order substitutions $\psi: \ X \rightarrow T_{(S,F)}(Y)$:
  - $\psi: \ ((S,F) \rightarrow (S,F \cup X)) \rightarrow ((S,F) \rightarrow (S,F \cup Y))$
  - $Sen(\psi): \ Sen(S,F \cup X) \rightarrow Sen(S,F \cup Y)$,
  - $\textsc{Mod}(\psi): \ \textsc{Mod}(S,F \cup Y) \rightarrow \textsc{Mod}(S,F \cup X)$

$$\textsc{Mod}(\psi)(M)_x = M_{\psi(x)}$$

- Second order substitutions mapping operations to terms (such that arity preserved).
- In **HOL**,**HNK**, higher order substitutions.

Representable substitutions I

Capture abstractly the concept ‘first order’ substitutions.

$\psi : \chi_1 \rightarrow \chi_2$ with χ_1 and χ_2 representable.

Proposition

Any substitution $\psi : \chi_1 \rightarrow \chi_2$ between representable signature morphisms $\chi_1 : \Sigma \rightarrow \Sigma_1$ and $\chi_2 : \Sigma \rightarrow \Sigma_2$ determines canonically a Σ-model homomorphism $M_\psi : M_{\chi_1} \rightarrow M_{\chi_2}$. Moreover, the mapping $\psi \mapsto M_\psi$ is functorial and faithful [modulo substitution equivalence].

Representable substitutions II

Example:

$$\begin{array}{c}
((S,F) \longrightarrow (S,F\cup X)) \xrightarrow{\psi} ((S,F) \longrightarrow (S,F\cup Y)) \\
\Big\downarrow \\
T_{(S,F)}(X) \xrightarrow{M_\psi} T_{(S,F)}(Y) \\
\Big\updownarrow \\
(X \longrightarrow T_{(S,F)}(Y))
\end{array}$$

Abstract capture of atomic sentences

In **MSA**, categorical characterization of satisfaction of atoms in the style of 'satisfaction by injectivity' (Nemeti, Andreka, ...):

Proposition

$M \models_{(S,F)} t = t'$ *if and only if there exists a homomorphism* $T_{(S,F)}/_{=_{\{t=t'\}}} \to M$.

Basic sentences

In any institution, a Σ-sentence ρ is *(finitary) basic* when there exists a (finitely presented) Σ-model M_ρ such that for any Σ-model M

$$M \models \rho \text{ if and only if there exists homomorphism } M_\rho \to M$$

In actual institutions atoms are finitary basic, but also:

Proposition

Basic sentences are closed under existential quasi-representable quantification.

A tighter approximation of atoms

The following rules out some ‘non-atomic’ basic sentences (e.g. $(\exists\chi)\rho$, for atomic ρ):

In any institution with initial models of signatures (denoted 0_Σ), a basic sentence ρ is *epic basic* when the unique homomorphism $0_\Sigma \to M_\rho$ is epi.

Epic basic are a tighter capture of ‘atomic’ sentences, yet not a perfect one.

The method of ultraproducts

One of the most powerful model theory methods, much model theory may be developed through this method (see Bell and Slomson classic book).

Applications include:

- (semantic) compactness,
- preservation and axiomatizability,
- Keisler-Shelah Isomorphism Theorem,
- interpolation and definability,
- applications to algebra (fields, algebraic geometry, etc.).

Filters and Ultrafilters

$F \subseteq \mathscr{P}(I)$ is *filter over I* when

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$,
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

Ultrafilter when in addition, for all $X \subseteq I$, we have that

$$X \in F \text{ if and only if } I \setminus X \notin F$$

If F filter over I, and $I' \subseteq I$, then the *reduction of F to I'*:

$$F|_{I'} = \{I' \cap X \mid X \in F\}$$

Concrete filtered products

In **MSA**: given F filter over I, and $(M_i)_{i\in I}$ family of (S,F)-algebras, the *F-filtered product of $(M_i)_{i\in I}$* is defined as $\prod_F M_i = (\prod_{i\in I} M_i)/_{\sim_F}$ where

1. $\prod_{i\in I} M_i$ is direct product of $(M_i)_{i\in I}$, and
2. $\sim_F$ is the congruence defined by

$$m \sim_F m' \text{ if and only if } \{i \in I \mid m_i \sim_F m'_i\} \in F.$$

Categorical filtered products

Co-limit of the diagram of projections ($J, J' \subseteq I$):

$$\begin{tikzcd} \prod_{i \in J'} M_i \arrow[rr, "p_{J',J}"] \arrow[dr, "\mu_{J'}"'] & & \prod_{i \in J} M_i \arrow[dl, "\mu_J"] \\ & \prod_F M_i & \end{tikzcd}$$

Idea much exploited by approaches to categorical model theory (Nemeti, Andreka, Makkai, etc.)

Some examples

institution	direct prod.	directed co-lim.	filt.prod.	ultraprod.
FOL	$\surd$	$\surd$	$\surd$	$\surd$
PA	$\surd$	$\surd$	$\surd$	$\surd$
IPL	$\surd$	$\surd$	$\surd$	$\surd$
MFOL	$\surd$	$\surd$	$\surd$	$\surd$
MVL	$\surd$	?	$\surd$	$\surd$
HNK	$\surd$			$\surd$
MA				

Preservation of sentences by filtered factors/products

For a signature Σ in an institution, for each filter $F \in \mathscr{F}$ over a set I and for each family $\{A_i\}_{i \in I}$ of Σ-models, a Σ-sentence e is

- *preserved by $\mathscr{F}$-filtered factors:* $f_{\mathscr{F}}(e)$

$$\text{if } \prod_F A_i \models_\Sigma e \text{ implies } \{i \in I \mid A_i \models_\Sigma e\} \in F,$$

- *preserved by $\mathscr{F}$-filtered products:* $p_{\mathscr{F}}(e)$

$$\text{if } \{i \in I \mid A_i \models_\Sigma e\} \in F \text{ implies } \prod_F A_i \models_\Sigma e.$$

Preservation by ultrafactors/ultraproducts when $\mathscr{F}$ is the class of ultrafilters.

Fundamental Ultraproducts Theorem (Łos)

In any institution:

preservation property	*condition*
$p_{\mathscr{F}}(\text{basic})$	
$f_{\mathscr{F}}(\text{finitary basic})$	
$p_{\mathscr{F}}(\rho) \Rightarrow p_{\mathscr{F}}((\exists\chi)\rho)$	$\text{Mod}(\chi)$ pres. $\mathscr{F}$-filtered prod.
$f_{\mathscr{F}}(\rho) \Rightarrow f_{\mathscr{F}}((\exists\chi)\rho)$	$\text{Mod}(\chi)$ lifts $\mathscr{F}$-filtered prod.
$p_{\mathscr{F}}(\rho_1), p_{\mathscr{F}}(\rho_2) \Rightarrow p_{\mathscr{F}}(\rho_1 \wedge \rho_2)$	
$f_{\mathscr{F}}(\rho_1), f_{\mathscr{F}}(\rho_2) \Rightarrow f_{\mathscr{F}}(\rho_1 \wedge \rho_2)$	
$(p_{\mathscr{F}}(\rho_i))_{i\in I} \Rightarrow p_{\mathscr{F}}(\wedge_{i\in I}\rho_i)$	
$f_{\mathscr{F}}(\rho) \Rightarrow p_{\mathscr{F}}(\neg\rho)$	
$p_{\mathscr{F}}(\rho) \Rightarrow f_{\mathscr{F}}(\neg\rho)$	$\mathscr{F} \subseteq$ *Ultrafilters*

Corollary

Corollary

In any institution, any sentence which is accessible from the finitary basic sentences by

- *Boolean connectives,*
- *finitary representable quantification, and*
- *projectively representable quantification (assuming that the institution has epi model projections)*

is preserved by ultraproducts and ultrafactors.

Examples includes **FOL**, **PA**, **IPL**, etc.
In **MFOL**, $\mathbf{FOL}_\infty$ sentences preserved only by ultraproducts.

(Semantic) compactness

An institution is

1. *m-compact* when each set of sentences has a model if and only if any of its *finite* subsets has a model.
2. *compact* when $E \models \rho$ implies that there exists *finite* $E_0 \subseteq E$ such that $E_0 \models \rho$.

Proposition

- *Each compact institution having false is m-compact.*
- *Each m-compact institution having negations is compact.*

Compactness by ultraproducts

Corollary

Any institution in which each sentence is preserved by ultraproducts is m-compact.

Examples include **FOL**, **PA**, **IPL**, etc. but also **MFOL**.

Corollary

Let E be a set of sentences preserved by ultraproducts, and let e be a sentence preserved by ultrafactors such that $E \models e$. Then there exists a finite subset $E' \subseteq E$ such that $E' \models e$.