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Outline

1 Internal Logic
   - Boolean and other connectives
   - Quantifiers
   - General substitutions
   - Basic sentences

2 Ultraproducts
   - Categorical ultraproducts
   - Fundamental Ultraproducts Theorem
   - Compactness by ultraproducts
A $\Sigma$-sentence $\rho$ is a *conjunction* $(\rho_1 \land \rho_2)$ of $\Sigma$-sentences $\rho_1$ and $\rho_2$ when

$$\rho^* = \rho_1^* \cap \rho_2^*$$

The *institution has conjunctions* when any $\Sigma$-sentences $\rho_1$ and $\rho_2$ have a conjunction.
A $\Sigma$-sentence $\rho$ is a *disjunction* $(\rho_1 \lor \rho_2)$ of $\Sigma$-sentences $\rho_1$ and $\rho_2$ when

$$\rho^* = \rho_1^* \cup \rho_2^*$$

The *institution has disjunctions* when any $\Sigma$-sentences $\rho_1$ and $\rho_2$ have a disjunction.
A $\Sigma$-sentence $\rho$ is an *implication* $(\rho_1 \Rightarrow \rho_2)$ of $\Sigma$-sentences $\rho_1$ and $\rho_2$ when

$$\rho^* = \overline{\rho_1^*} \cup \rho_2^*$$

The *institution has implications* when any $\Sigma$-sentences $\rho_1$ and $\rho_2$ have a implication.
A $\Sigma$-sentence $\rho$ is a *negation* ($\neg \rho'$) of a $\Sigma$-sentence $\rho'$ when

$$\rho^* = \overline{\rho'^*}$$

The *institution has negations* when any $\Sigma$-sentence $\rho'$ has a negation.
Abstract connectives

A (semantic logical) connective $c$ of arity $n$ consists of a family $(c_\Sigma)_{\Sigma \in \text{Sig}}$ of functions

$$c_\Sigma : \mathcal{P}(|\text{MOD}(\Sigma)|)^n \rightarrow \mathcal{P}(|\text{MOD}(\Sigma)|).$$

- A connective is *Boolean* when it is a (derived) operation of the Boolean algebra $(\mathcal{P}(|\text{MOD}(\Sigma)|), \cap, \cup, \neg, \emptyset)$.
- $\rho$ is a $c$-connection of $\rho_i$, $1 \leq i \leq n$, ($\rho = c(\rho_1, \ldots, \rho_n)$) when $\rho^* = c_\Sigma(\rho_1^*, \ldots, \rho_n^*)$. 
## Examples

<table>
<thead>
<tr>
<th>institution</th>
<th>$\land$</th>
<th>$\neg$</th>
<th>$\lor$</th>
<th>$\Rightarrow$</th>
<th>$\Leftrightarrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FOL, PL, HOL, HNK</strong></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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<td><strong>WPL (Béziau)</strong></td>
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<td><strong>FOL$^+$</strong></td>
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<td><strong>EQL, HCL, MVL</strong></td>
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<td><strong>EQLN</strong></td>
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<tr>
<td><strong>MFOL, MPL</strong></td>
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<tr>
<td><strong>IPL</strong></td>
<td>✓</td>
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</table>
Quantifiers

Given signature morphism $\chi : \Sigma \to \Sigma'$, $\rho \in Sen(\Sigma)$ and $\rho' \in Sen(\Sigma')$,

- $\rho$ is a **universal** $\chi$-quantification of $\rho'$ when

  $$\rho^* = \text{MOD}(\chi)(\rho'^*)$$

- $\rho$ is a **existential** $\chi$-quantification of $\rho'$ when

  $$\rho^* = \text{MOD}(\chi)(\rho'^*)$$

The *institution has universal/existential $\mathcal{D}$-quantifiers* when for each $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$, any $\Sigma'$-sentence $\rho'$ has a universal/existential $\chi$-quantification.
<table>
<thead>
<tr>
<th>institution</th>
<th>$\mathcal{D}$</th>
<th>$\forall$</th>
<th>$\exists$</th>
</tr>
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<tbody>
<tr>
<td>FOL, MVL</td>
<td>fin. inj. sign. ext. with constants</td>
<td>√</td>
<td>√</td>
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<tr>
<td>SOL</td>
<td>fin. inj. sign. ext.</td>
<td>√</td>
<td>√</td>
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<tr>
<td>PA</td>
<td>fin. inj. sign. ext. with total constants</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>EQL, HCL</td>
<td>fin. inj. sign. ext. with constants</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>MFOL</td>
<td>fin. inj. sign. ext. with rigid constants</td>
<td>√</td>
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</tr>
<tr>
<td>HOL, HNK</td>
<td>fin. inj. sign. ext.</td>
<td>√</td>
<td>√</td>
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</table>

fin. inj. sign. ext. = finitary injective signature extension
Representable signature morphisms

Many results depend on the quantification being *first order*.

At the level of abstract institutions this is captured by the condition that the signature morphism $\chi$ is *representable*:

$$
\Sigma' \quad \xrightarrow{\text{MOD}(\Sigma')} \quad M_{\chi} / \text{MOD}(\Sigma) \quad \xleftarrow{(M_{\chi} \xrightarrow{h} M)} \quad \Sigma
$$

$\chi$ is *finitary representable* when $M_{\chi}$ is finitely presented.
A concrete example: MSA first order quantifiers

\[(S, F \cup X) \quad \xrightarrow{\chi} \quad (S, F) \quad \xrightarrow{\mathrm{MOD}} \quad \mathrm{MOD}(S, F) \quad \xrightarrow{\simeq} \quad T_{(S,F)}(X) / \mathrm{MOD}(S, F) \]

\[(S, F) \quad \xrightarrow{\mathrm{MOD}(\chi)} \quad \mathrm{MOD}(S, F) \quad \xrightarrow{(T_{(S,F)}(X) \xrightarrow{h} M)} \quad \mathrm{MOD}(S, F) \quad \xrightarrow{(X \xrightarrow{h_0} M)} \quad M \]

\[(X \xrightarrow{h_0} M) \quad \xrightarrow{M} \quad M \]
A weaker very useful version of representability:

\[ \chi : \Sigma \rightarrow \Sigma' \] is \textit{quasi-representable} if and only if

\[ M'/\text{MOD}(\Sigma') \cong (M'|\chi)/\text{MOD}(\Sigma) \]

**Proposition**

A signature morphism \( \chi : \Sigma \rightarrow \Sigma' \) is representable if and only if it is quasi-representable and \( \text{MOD}(\Sigma') \) has initial models.
## Examples

<table>
<thead>
<tr>
<th>institution</th>
<th>$\chi$</th>
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</thead>
<tbody>
<tr>
<td>FOL</td>
<td>(fin.) inj. sign. ext. with const.</td>
<td>(fin.) rep.</td>
</tr>
<tr>
<td>MVL</td>
<td>(fin.) inj. sign. ext. with const.</td>
<td>(fin.) rep.</td>
</tr>
<tr>
<td>PA</td>
<td>(fin.) inj. sign. ext. with total const.</td>
<td>(fin.) rep.</td>
</tr>
<tr>
<td>$E$(FOL)</td>
<td>(fin.) inj. sign. ext. with const.</td>
<td>(fin.) quasi-rep.</td>
</tr>
<tr>
<td>MFOL</td>
<td>(fin.) inj. sign. ext. with rigid const.</td>
<td>(fin.) quasi-rep.</td>
</tr>
<tr>
<td>HOL</td>
<td>(fin.) inj. sign. ext.</td>
<td>(fin.) quasi-rep.</td>
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</table>
Substitution $\psi : \chi_1 \rightarrow \chi_2$

such that

$\text{Sen}(\Sigma_1) \xrightarrow{\text{Sen}(\psi)} \text{Sen}(\Sigma_2)$

$\text{MOD}(\Sigma_1) \xleftarrow{\text{MOD}(\psi)} \text{MOD}(\Sigma_2)$

$M_2 \models \Sigma_2 \text{Sen}(\psi)(\rho_1)$
Examples

- First order substitutions $\psi : X \rightarrow T_{(S,F)}(Y)$:
  - $\psi : ((S,F) \rightarrow (S,F \cup X)) \rightarrow ((S,F) \rightarrow (S,F \cup Y))$
  - $\text{Sen}(\psi) : \text{Sen}(S,F \cup X) \rightarrow \text{Sen}(S,F \cup Y)$,
  - $\text{MOD}(\psi) : \text{MOD}(S,F \cup Y) \rightarrow \text{MOD}(S,F \cup X)$

  $$\text{MOD}(\psi)(M)_x = M_{\psi(x)}$$

- Second order substitutions mapping operations to terms (such that arity preserved).
- In **HOL, HNK**, higher order substitutions.
Capture abstractly the concept ‘first order’ substitutions.

\[ \psi : \chi_1 \rightarrow \chi_2 \text{ with } \chi_1 \text{ and } \chi_2 \text{ representable.} \]

**Proposition**

Any substitution \( \psi : \chi_1 \rightarrow \chi_2 \) between representable signature morphisms \( \chi_1 : \Sigma \rightarrow \Sigma_1 \) and \( \chi_2 : \Sigma \rightarrow \Sigma_2 \) determines canonically a \( \Sigma \)-model homomorphism \( M_\psi : M_{\chi_1} \rightarrow M_{\chi_2} \).

Moreover, the mapping \( \psi \mapsto M_\psi \) is functorial and faithful [modulo substitution equivalence].
Example:

\[
((S, F) \rightarrow (S, F \cup X)) \xrightarrow{\psi} ((S, F) \rightarrow (S, F \cup Y))
\]

\[
\downarrow
\]

\[
T_{(S, F)}(X) \xrightarrow{M_{\psi}} T_{(S, F)}(Y)
\]

\[
\downarrow
\]

\[
(X \rightarrow T_{(S, F)}(Y))
\]
In MSA, categorical characterization of satisfaction of atoms in the style of ‘satisfaction by injectivity’ (Nemeti, Andreka, ...):

**Proposition**

\[ M \models_{(S,F)} t = t' \text{ if and only if there exists a homomorphism } T_{(S,F)}/=\{t=t'\} \rightarrow M. \]
In any institution, a $\Sigma$-sentence $\rho$ is \textit{(finitary) basic} when there exists a (finitely presented) $\Sigma$-model $M_\rho$ such that for any $\Sigma$-model $M$

\[ M \models \rho \quad \text{if and only if there exists homomorphism} \quad M_\rho \to M \]

In actual institutions atoms are finitary basic, but also:

**Proposition**

\textit{Basic sentences are closed under existential quasi-representable quantification.}
The following rules out some ‘non-atomic’ basic sentences (e.g. $(\exists \chi)\rho$, for atomic $\rho$):

In any institution with initial models of signatures (denoted $0_\Sigma$), a basic sentence $\rho$ is *epic basic* when the unique homomorphism $0_\Sigma \to M_\rho$ is epi.

Epic basic are a tighter capture of ‘atomic’ sentences, yet not a perfect one.
The method of ultraproducts

One of the most powerful model theory methods, much model theory may be developed through this method (see Bell and Slomson classic book).

Applications include:

- (semantic) compactness,
- preservation and axiomatizability,
- Keisler-Shelah Isomorphism Theorem,
- interpolation and definability,
- applications to algebra (fields, algebraic geometry, etc.).
Filters and Ultrafilters

$F \subseteq \mathcal{P}(I)$ is filter over $I$ when

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$,
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

Ultrafilter when in addition, for all $X \subseteq I$, we have that

$$X \in F \text{ if and only if } I \setminus X \notin F$$

If $F$ filter over $I$, and $I' \subseteq I$, then the reduction of $F$ to $I'$:

$$F|_{I'} = \{I' \cap X \mid X \in F\}$$
Concrete filtered products

In MSA: given $F$ filter over $I$, and $(M_i)_{i \in I}$ family of $(S,F)$-algebras, the $F$-filtered product of $(M_i)_{i \in I}$ is defined as

$$\prod_F M_i = (\prod_{i \in I} M_i) / \sim_F$$

where

1. $\prod_{i \in I} M_i$ is direct product of $(M_i)_{i \in I}$, and
2. $\sim_F$ is the congruence defined by

$$m \sim_F m' \text{ if and only if } \{i \in I \mid m_i \sim_F m'_i\} \in F.$$
Categorical filtered products

Co-limit of the diagram of projections \((J, J' \subseteq I)\):

\[
\begin{align*}
\Pi_{i \in J'} M_i & \xrightarrow{p_{J', J}} \Pi_{i \in J} M_i \\
\mu_{J'} & \searrow \mu_J \\
\Pi_F M_i & \end{align*}
\]

Idea much exploited by approaches to categorical model theory (Nemeti, Andreka, Makkai, etc.)
Some examples

<table>
<thead>
<tr>
<th>institution</th>
<th>direct prod.</th>
<th>directed co-lim.</th>
<th>filt.prod.</th>
<th>ultraprod.</th>
</tr>
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<tbody>
<tr>
<td>FOL</td>
<td>√</td>
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Preservation of sentences by filtered factors/products

For a signature $\Sigma$ in an institution, for each filter $F \in \mathcal{F}$ over a set $I$ and for each family $\{A_i\}_{i \in I}$ of $\Sigma$-models, a $\Sigma$-sentence $e$ is

- preserved by $\mathcal{F}$-filtered factors: $f_{\mathcal{F}}(e)$
  
  if $\prod_{F} A_i \models \Sigma e$ implies $\{ i \in I \mid A_i \models \Sigma e \} \in F$,

- preserved by $\mathcal{F}$-filtered products: $p_{\mathcal{F}}(e)$
  
  if $\{ i \in I \mid A_i \models \Sigma e \} \in F$ implies $\prod_{F} A_i \models \Sigma e$.

**Preservation by ultrafactors-ultraproducts** when $\mathcal{F}$ is the class of ultrafilters.
Fundamental Ultraproducts Theorem (Łos)

In any institution:

<table>
<thead>
<tr>
<th>preservation property</th>
<th>condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \mathcal{F}(\text{basic})$</td>
<td></td>
</tr>
<tr>
<td>$f \mathcal{F}(\text{finitary basic})$</td>
<td></td>
</tr>
<tr>
<td>$p \mathcal{F}(\rho) \Rightarrow p \mathcal{F}(\exists \chi \rho)$</td>
<td>$\text{MOD}(\chi)$ pres. $\mathcal{F}$-filtered prod.</td>
</tr>
<tr>
<td>$f \mathcal{F}(\rho) \Rightarrow f \mathcal{F}(\exists \chi \rho)$</td>
<td>$\text{MOD}(\chi)$ lifts $\mathcal{F}$-filtered prod.</td>
</tr>
<tr>
<td>$p \mathcal{F}(\rho_1), p \mathcal{F}(\rho_2) \Rightarrow p \mathcal{F}(\rho_1 \land \rho_2)$</td>
<td></td>
</tr>
<tr>
<td>$f \mathcal{F}(\rho_1), f \mathcal{F}(\rho_2) \Rightarrow f \mathcal{F}(\rho_1 \land \rho_2)$</td>
<td></td>
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<tr>
<td>$(p \mathcal{F}(\rho_i))<em>{i \in I} \Rightarrow p \mathcal{F}(\land</em>{i \in I} \rho_i)$</td>
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<tr>
<td>$f \mathcal{F}(\rho) \Rightarrow p \mathcal{F}(\neg \rho)$</td>
<td></td>
</tr>
<tr>
<td>$p \mathcal{F}(\rho) \Rightarrow f \mathcal{F}(\neg \rho)$</td>
<td>$\mathcal{F} \subseteq \text{Ultrafilters}$</td>
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</table>
Corollary

In any institution, any sentence which is accessible from the finitary basic sentences by

– Boolean connectives,
– finitary representable quantification, and
– projectively representable quantification (assuming that the institution has epi model projections)

is preserved by ultraproducts and ultrafactors.

Examples includes FOL, PA, IPL, etc.
In MFOL, FOL∞ sentences preserved only by ultraproducts.
(Semantic) compactness

An institution is

1. \textit{m-compact} when each set of sentences has a model if and only if any of its \textit{finite} subsets has a model.

2. \textit{compact} when \( E \models \rho \) implies that there exists \textit{finite} \( E_0 \subseteq E \) such that \( E_0 \models \rho \).

Proposition

- Each compact institution having false is m-compact.
- Each m-compact institution having negations is compact.
Corollary

Any institution in which each sentence is preserved by ultraproducts is \( m \)-compact.

Examples include \textbf{FOL}, \textbf{PA}, \textbf{IPL}, etc. but also \textbf{MFOL}.

Corollary

Let \( E \) be a set of sentences preserved by ultraproducts, and let \( e \) be a sentence preserved by ultrafactors such that \( E \models e \). Then there exists a finite subset \( E' \subseteq E \) such that \( E' \models e \).