INTEGRABLE SYSTEMS AND CONNES-KREIMER RENORMALIATION

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Abstract. We study the complete integrability of the Lax pair equations associated to the Connes-Kreimer Birkhoff factorization of the character group of a Hopf algebra.

1. Introduction

Many authors used general group-theoretical methods to construct completely integrable Hamiltonian systems and their solutions (see the survey due to Reyman and Semenov-Tian-Shansky in [6]). In [1], Steven Rosenberg and I constructed a Lax pair equation associated to the Connes-Kreimer-Birkhoff factorization of the character group of a connected graded commutative Hopf algebra and we gave a completely integrable Lax pair equation example (see §8.3.2 in [1]). In this note, I study the complete integrability of finite-dimension truncations of the Lax pair equations introduced in [1]. In section 2, we recall these Lax pair equations and some basic notion in the Connes-Kreimer renormalization. In §3, we use the proof of the Mishchenko-Fomenko conjecture to see that the finite-dimension truncations of the Lax pair equation for the beta-function is completely integrable for polynomial Hamiltonians. In the last section, we present three worked examples for the Lax pair equations of the beta-function.

2. Preliminaries

We now briefly recall the some notion from Connes-Kreimer renormalizations and the Lax pair equations introduced in [1].

Let $\mathcal{H} = (\mathcal{H}, 1, \mu, \Delta, \varepsilon, S)$ be a graded connected Hopf algebra over $\mathbb{C}$. Let $\mathcal{A}$ be a unital commutative algebra with unit $1_{\mathcal{A}}$. Important choices for $\mathcal{A}$ are: (i) $\mathcal{A}$ be the algebra of Laurent series; or (ii) $\mathcal{A} = \mathbb{C}$. (see [5]).

Definition 2.1. (see [4, 5, 1]) (i) The **character group** $G_{\mathcal{A}}$ of the Hopf algebra $\mathcal{H}$ is the set of algebra morphisms $\phi : \mathcal{H} \to \mathcal{A}$ with $\phi(1) = 1_{\mathcal{A}}$. The group law is given by the convolution product

$$(\psi_1 \star \psi_2)(h) = \langle \psi_1 \otimes \psi_2, \Delta h \rangle;$$

the unit element is ε.

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(ii) An $\mathcal{A}$-valued **infinitesimal character** of a Hopf algebra $\mathcal{H}$ is a $\mathbb{C}$-linear map $Z : \mathcal{H} \to \mathcal{A}$ satisfying

$$\langle Z, hk\rangle = \langle Z, h\rangle \varepsilon(k) + \varepsilon(h)\langle Z, k\rangle.$$

The set of infinitesimal characters is denoted by $\mathfrak{g}_{\mathcal{A}}$ and is endowed with a Lie algebra bracket:

$$[Z, Z'] = Z \star Z' - Z' \star Z, \quad \text{for } Z,\ Z' \in \mathfrak{g}_{\mathcal{A}},$$

where $\langle Z \star Z', h\rangle = \langle Z \otimes Z', \Delta(h)\rangle$.

2.1. **Lax pair equations for the truncated Lie algebra of infinitesimal characters.** Let $\mathcal{H} = \oplus_{n\in\mathbb{N}}\mathcal{H}_n$ be a graded connected commutative Hopf algebra of finite type (i.e. each homogeneous component $\mathcal{H}_n$ is a finite dimensional vector space). Let $\mathcal{B} = \{T_i\}_{i\in\mathbb{N}}$ be a minimal set of homogeneous generators of the Hopf algebra H such that $\deg(T_i) \le \deg(T_j)$ if $i < j$ and such that $T_0 = 1$. For $i > 0$, we define the $\mathbb{C}$-valued infinitesimal character Z_i on generators by $Z_i(T_j) = \delta_{ij}$. Let $\mathfrak{g}^{(k)}$ be the vector space generated by $\{Z_i \mid \deg(T_i) \le k\}$. We define $\deg(Z_i) = \deg(T_i)$ and set

$$[Z_i, Z_j]_{\mathfrak{g}^{(k)}} = \begin{cases} [Z_i, Z_j] & \text{if } \deg(Z_i) + \deg(Z_j) \le k \\ 0 & \text{if } \deg(Z_i) + \deg(Z_j) > k \end{cases}$$

We identify $\varphi \in G_{\mathbb{C}}$ with $\{\varphi(T_i)\} \in \mathbb{C}^{\mathbb{N}}$ and on $\mathbb{C}^{\mathbb{N}}$ we set a group law given by $\{\varphi_1(T_i)\} \oplus \{\varphi_2(T_i)\} = \{(\varphi_1 \star \varphi_2)(T_i)\}$. $G^{(k)} = \{\{\varphi(T_i)\}_{\{i\mid \deg(T_i)\le k\}} \mid \varphi \in G_{\mathbb{C}}\}$ is a finite dimensional Lie subgroup of $G_{\mathbb{C}} = (\mathbb{C}^{\mathbb{N}}, \oplus)$ and the Lie algebra of $G^{(k)}$ is $\mathfrak{g}^{(k)}$.

The double Lie algebra $\delta^{(k)}$ of $\mathfrak{g}^{(k)}$ is the Lie algebra on $\mathfrak{g}^{(k)} \oplus \mathfrak{g}^{(k)*}$ with the Lie bracket given by

$$[X, Y]_{\mathfrak{g}^{(k)}\oplus\mathfrak{g}^{(k)*}} = [X, Y], \quad [X^*, Y^*]_{\mathfrak{g}^{(k)}\oplus\mathfrak{g}^{(k)*}} = 0, \quad [X, Y^*] = \mathrm{ad}^*_X(Y^*),$$

for any $X,\ Y \in \mathfrak{g}^{(k)}$, $X^*,\ Y^* \in \mathfrak{g}^{(k)*}$. Let $\tilde{G}^{(k)}$ be the simply connected Lie group with $\mathrm{Lie}(\tilde{G}^{(k)}) = \delta^{(k)}$. Let $L\delta^{(k)}$ be algebra of polynomials in λ and λ^{-1} with coefficients in $\delta^{(k)}$. The natural pairing $\langle\cdot,\cdot\rangle$ on $L\delta^{(k)}$ given by

$$\left\langle \sum_{i=M}^{N} \lambda^i L_i, \sum_{j=M'}^{N'} \lambda^j L'_j \right\rangle = \sum_{i+j=-1} \langle L_i, L'_j\rangle,$$

induces an isomorphism $I : L(\delta^{(k)*}) \to L\delta^{(k)}$.

Theorem 2.2 ([1])**.** *Let $\psi : L\delta^{(k)} \to \mathbb{C}$ be a Casimir function (e.g. $\psi(L) = \psi_{m,n}(L(\lambda)) = \mathrm{Res}_{\lambda=0}(\lambda^m\psi(\lambda^n L(\lambda)))$ with $\psi : \delta^{(k)} \times \delta^{(k)} \to \mathbb{C}$ the natural paring of $\delta^{(k)}$). Set $X = I(d\psi(L_0))$ for $L_0 \in L\delta^{(k)}$. Then the solution in $L\delta^{(k)}$ of*

$$\frac{dL}{dt} = [L, M]_{L\delta^{(k)}}, \quad M = \frac{1}{2}R(I(d\psi(L))) \tag{2.1}$$

with initial condition $L(0) = L_0$ is given by

$$L(t) = \mathrm{Ad}_{L\tilde{G}^{(k)}}\, g_\pm(t) \cdot L_0, \tag{2.2}$$

where $\exp(-tX)$ *has the Connes-Kreimer Birkhoff factorization* $\exp(-tX) = g_-(t)^{-1}g_+(t)$.

2.2. **The Lax pair for the general case.**

Definition 2.3 ([7])**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. A map $f : \mathfrak{g} \to \mathfrak{g}$ is Ad-*covariant* if $\mathrm{Ad}(g)(f(L)) = f(\mathrm{Ad}(g)(L))$ for all $g \in G$, $L \in \mathfrak{g}$.

Let $\pi : \mathcal{A} \to \mathcal{A}$ be a Rota-Baxter projection, which by definition is a linear map with $\pi \circ \pi = \pi$ and satisfying the Rota-Baxter equation :

$$\pi(ab) + \pi(a)\pi(b) = \pi(a\pi(b)) + \pi(\pi(a)b)$$

for $a, b \in \mathcal{A}$. For any Rota-Baxter projection π on $\mathcal{A}$, $\mathcal{A}$ splits into a direct sum of two subalgebras $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ with $\mathcal{A}_- = \mathrm{Im}(\pi)$.. In the next theorem we recall the Lax pair equation corresponding to the Connes-Kreimer-Birkhoff factorization of character group of $\mathcal{H}$ and of a Rota-Baxter projection π.

Theorem 2.4. *(*[1, Theorem 5.9+§5.4]*) Let* $f : \mathfrak{g}_\mathcal{A} \to \mathfrak{g}_\mathcal{A}$ *be an* Ad-*covariant map. Let* $L_0 \in \mathfrak{g}_\mathcal{A}$ *satisfy* $[f(L_0), L_0] = 0$. *Set* $X = f(L_0)$. *Then the solution of*

$$\frac{dL}{dt} = [L, M], \quad M = \frac{1}{2}R(f(L)) \tag{2.3}$$

with initial condition $L(0) = L_0$ *is given by*

$$L(t) = \mathrm{Ad}_G g_\pm(t) \cdot L_0, \tag{2.4}$$

where $\exp(-tX)$ *has the Connes-Kreimer Birkhoff factorization* $\exp(-tX) = g_-(t)^{-1}g_+(t)$.

When $\mathcal{A} = \mathbb{C}[\lambda^{-1}, \lambda]]$, the map $f : \mathfrak{g}_\mathcal{A} \to \mathfrak{g}_\mathcal{A}$, given by $f(L) = 2\lambda^{-n+2m}L$, is Ad-covariant and $[f(L_0), L_0] = 0$.

Corollary 2.5. *(*[1, Corollary 5.11]*) Let* $L_0 \in \mathfrak{g}_\mathcal{A}$ *(with* $\mathcal{A} = \mathbb{C}[\lambda^{-1}, \lambda]]$*)and set* $X = 2\lambda^{-n+2m}L_0$. *Then the solution of*

$$\frac{dL}{dt} = [L, M], \quad M = R(\lambda^{-n+2m}L) \tag{2.5}$$

with initial condition $L(0) = L_0$ *is given by*

$$L(t) = \mathrm{Ad}_{G_\mathcal{A}} g_\pm(t) \cdot L_0, \tag{2.6}$$

where $\exp(-tX)$ *has the Connes-Kreimer Birkhoff factorization* $\exp(-tX) = g_-(t)^{-1}g_+(t)$.

Definition 2.6. For $s \in \mathbb{C}$ and $\varphi \in G_\mathcal{A}$, we define $\varphi^s(x)$ for a homogeneous $x \in H$ by

$$\varphi^s(x)(\lambda) = e^{s\lambda|x|}\varphi(x)(\lambda),$$

for $\lambda \in \mathbb{C}$ where $|x|$ is the degree of x. Let

$$G^\Phi_\mathcal{A} = \{\varphi \in G_\mathcal{A} \mid \frac{d}{ds}(\varphi^s)_- = 0\}, \tag{2.7}$$

be the set of local characters.

Definition 2.7 ([5])**.** Let Y be the biderivation on $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ given on homogeneous elements by

$$Y : \mathcal{H}_n \to \mathcal{H}_n, \qquad Y(x) = nx \quad \text{for } x \in \mathcal{H}_n.$$

We define the bijection $\tilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ by

$$\tilde{R}(\varphi) = \varphi^{-1} \star (\varphi \circ Y).$$

The locality of the Lax pair flow is preserved.

Theorem 2.8 ([1])**.** *Let $\varphi \in G^{\Phi}_{\mathcal{A}}$ and let $L(t)$ be the solution of the Lax pair equation* (2.3) *for any* Ad*-covariant function f, with the initial condition $L_0 = \tilde{R}(\varphi)$. Let φ_t be the flow given by*

$$\varphi_t = \tilde{R}^{-1}(L(t)). \tag{2.8}$$

Then φ_t is a local character for all t.

Definition 2.9 ([1])**.** For $\varphi \in G^{\Phi}_{\mathcal{A}}$, $x \in H$, set

$$\tilde{\beta}_{\varphi}(x)(\lambda) = \frac{d}{ds}\Big|_{s=0} (\varphi^{-1} \star \varphi^{s})(x)(\lambda).$$

Remark 2.10. For any $\varphi \in G^{\Phi}_{\mathcal{A}}$, by [1, Lemma 6.6], $\tilde{\beta}_{\varphi}$ is a holomorphic infinitesimal character (i.e. $\tilde{\beta}_{\varphi}(x) \in \mathcal{A}_+$ for any x) and the relation to the celebrated beta-function of the quantum field theory is given by

$$\beta_{\varphi} = \mathrm{Ad}(\varphi_+(0))(\tilde{\beta}_{\varphi}\big|_{\lambda=0}), \tag{2.9}$$

where $\varphi = \varphi_-^{-1} \star \varphi_+$ is the Birkhoff decomposition of φ.

Assuming that $\mathcal{A} = \mathbb{C}[\lambda^{-1}, \lambda]]$, for φ_t given by (2.8), let the Taylor expansion of $\tilde{\beta}_{\varphi_t}$ be

$$\tilde{\beta}_{\varphi_t} = \sum_{k=0}^{\infty} \tilde{\beta}_k(t)\lambda^k.$$

Theorem 2.11 ([1])**.** *For a local character $\varphi \in G^{\Phi}_{\mathcal{A}}$, let $L(t)$ be the Lax pair flow of Corollary 2.5 with $\mathcal{A} = \mathbb{C}[\lambda^{-1}, \lambda]]$ and with initial condition $L_0 = \tilde{R}(\varphi)$. Let $\varphi_t = \tilde{R}^{-1}(L(t))$. Then*

(i) *for $-n + 2m \geq 1$, $\varphi_t = \varphi$ and hence $\beta_{\varphi_t} = \beta_{\varphi}$ and $\tilde{\beta}_0(t) = \tilde{\beta}_0(0)$ for all t.*
(ii) *for $-n + 2m \leq 0$, $\beta_{\varphi_t} \in \mathfrak{g}_{\mathbb{C}}$ satisfies*

$$\frac{d\tilde{\beta}_0(t)}{dt} = 2[\tilde{\beta}_0(t), \tilde{\beta}_{n-2m+1}(t)]. \tag{2.10}$$

Proposition 2.12 ([1])**.** *In the setup of Theorem 2.4, we have*

$$\frac{d\beta_{\varphi_t}}{dt} = \Big[\frac{d((\varphi_t)_+(0))}{dt}((\varphi_t)_+(0))^{-1} + \mathrm{Ad}((\varphi_t)_+(0))(M_+(0)), \beta_{\varphi_t}\Big], \tag{2.11}$$

where M comes from the Lax pair equation $dL/dt = [L, M]$, and M_+ is the projection of M into $\mathfrak{g}_{\mathcal{A}+}$.

3. Completely Integrable Lax pair equations

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. We denote by $S(\mathfrak{g})$ the space of polynomials on $\mathfrak{g}^*$ endowed with the Poisson-Lie bracket.

Definition 3.1. A commutative set of algebraically independent polynomials $f_1, \cdots, f_k \in S(\mathfrak{g})$ is called complete if $2k = \dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}$.

Since the transcendency degree $trdeg(f_1, \cdots, f_k) = rank(J(f_1, \cdots, f_k))$, we clearly have $df_1, \cdots, df_k \in S(\mathfrak{g})$ are linearly independent.

In 2003, Sadetov proved the Mishchenko-Fomeko conjecture (cf. [2]), which we now recall.

Theorem 3.2 ([2]). *If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic zero then there exists a complete commutative set of polynomials on g^*.*

By theorem 3.2, if the equation (2.11) is Hamiltonian, and if the Hamiltonian belongs to a complete set of polynomials, then the finite-dimensional truncation of (2.11) is completely integrable.

Remark 3.3. By [2], there exists a codimension one filtrations $\mathfrak{g}_1 \subset \cdots \mathfrak{g}_m$ for nilpotent algebras, and one can use the chain of algebras method to explicitly find a complete set of polynomials on g^* (see also [8]).

4. Worked examples for $\tilde{\beta}_0$

Under set up of Theorem 2.11 and we give explicitly the integrals of motion the equation (2.10) on the double Lie algebra δ of certain particular truncated Lie algebras. We always assume that we are in the nontrivial case of Theorem 2.11, namely that $-n + 2m \leq 0$. The gauge transformation $\tilde{\beta}_0(t) \to (\varphi_t)_+(0) \star \tilde{\beta}_0(t) \star ((\varphi_t)_+(0))^{-1}$ changes equation (2.10) into the equation of beta functions.

Let $\mathcal{H}_1$ be the Hopf subalgebra of rooted trees generated by

$$t_0 = 1_{\mathcal{T}}, \quad t_1 = \bullet, \quad t_2 = \overset{\bullet}{\bullet}, \quad t_4 = \bullet\!\wedge\!\bullet,$$

and let G_1 and $\mathfrak{g}_1$ be its group of character with values in $\mathbb{C}$ and respectively its Lie algebra of infinitesimal characters. Identifying $G_1 \equiv \mathbb{C}^3$ via $\varphi \to \varphi(f_i)_{i=1,2,4}$, with $\{f_i\}_{i=1,2,4}$ the normal coordinates ([3]) associated to t_1, t_2, t_4:

$$f_1 = \bullet\,, \quad f_2 = \overset{\bullet}{\bullet} - \frac{1}{2}\bullet\bullet\,, \quad f_4 = \bullet\!\wedge\!\bullet - \bullet\overset{\bullet}{\bullet} + \frac{1}{6}\bullet^3\,,$$

we get G_1 is exactly the 3-dimensional Heisenberg group $(\mathbb{C}^3, \oplus)$, where

$$(x_1, x_2, x_3) \oplus (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2 - x_2 y_1),$$

and its Lie algebra $\mathfrak{g}_1 = \operatorname{Span}(X_1, X_2, X_3)$ is the 2-step nilpotent given by $[X_1, X_2] = 2X_3$ and $[X_i, X_j] = 0$ for any $(i, j) \neq (1, 2)$ and $(i, j) \neq (2, 1)$.

Since there is no Ad-invariant, non-degenerate, symmetric bilinear form on $\mathfrak{g}_1$, we pass to its double Lie algebra $\delta_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_1^*$ which has a natural pairing $\langle \cdot, \cdot \rangle$. The natural Lie-Poisson bracket of δ_1^* rises to a Poisson structure on δ_1 via $\langle \cdot, \cdot \rangle$. The nontrivial Lie brackets of $\delta_1 = \mathrm{Span}(X_1, X_2, X_3, X_1^*, X_2^*, X_3^*)$ are

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3^*] = -2X_2^*, \quad [X_2, X_3^*] = 2X_1^*,$$

while its associated Lie-Poisson bracket on δ_1 is determined by

$$\{x_1, x_2\}(x) = 2x_3^*, \quad \{x_1, x_3^*\}(x) = -2x_2, \quad \{x_2, x_3^*\}(x) = 2x_1,$$

with $x = \sum_{i=1}^{i=3} x_i X_i + \sum_{i=1}^{i=3} x_i^* X_i^*$. Straightforward computations give the following lemma.

Lemma 4.1. *(i) Both $\mathfrak{g}_1$ and δ_1 are 2-step nilpotent Lie algebras, i.e. $[\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] = 0$, $[\delta_1, [\delta_1, \delta_1]] = 0$.*

(ii) The functions

$$H_1(x) = \frac{x_1^2}{2} + \frac{x_2^2}{2}, \quad H_2(x) = \frac{x_3^2}{2}, \quad H_3(x) = \frac{x_1^{*2}}{2}, \quad H_4(x) = \frac{x_2^{*2}}{2}, \quad H_5(x) = \frac{x_3^{*2}}{2} + \frac{x_1^2}{2} + \frac{x_2^2}{2},$$

are in involution with respect to the Lie-Poisson bracket on δ_1 and dH_1, dH_2, dH_3, dH_4, dH_5 are linearly independent on an open dense set in δ_1.

(iii) δ_1 is a Poisson manifold of rank $2r = 2$.

Proof. (iii) The adjoint representation $\mathrm{ad} : \delta_1 \to \mathfrak{gl}(\delta_1)$,

$$\mathrm{ad}_{\delta_1}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2x_2 & 2x_1 & 0 & 0 & 0 & 0 \\ 0 & -2x_3^* & 0 & 0 & 0 & 2x_2 \\ 2x_3^* & 0 & 0 & 0 & 0 & -2x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has the maximal rank 2, thus the rank of the Poisson structure is also $2r = 2$. □

In order to show that equation (2.10) is an integrable system on δ_1 it is sufficient to find a function $H : \delta_1 \to \mathbb{R}$ such $\tilde{\beta}_{n-2m+1}(t) = \nabla H(\tilde{\beta}_0(t))$ (i.e H is a Hamiltonian for (2.10)) and to show that the functions H, H_2, H_3, H_4, H_5 are in involution and linearly independent (in the sense that their differentials are linearly independent on an open dense set). Here $\nabla H(x)$ denotes the gradient of $H(x)$. The idea is to show that the 2-step nilpotency of $\mathfrak{g}_1$ implies that both $\tilde{\beta}_{n-2m+1}(t)$ and $\tilde{\beta}_0(t)$ are linear in the variable t.

Lemma 4.2. *There exists a Hamiltonian function of the form*

$$H(x) = k_1 x_1 + k_2 x_2 + k_3 x_3 + \frac{l_1 x_1^2}{2} + \frac{l_2 x_2^2}{2} + \frac{l_3 x_3^2}{2}$$

such that $\tilde{\beta}_{n-2m+1}(t) = \nabla H(\tilde{\beta}_0(t))$.

Proof. Differentiating (2.10) and then using the 2-step nilpotency of δ_1, we get that

$$\begin{aligned}\frac{d^2\tilde{\beta}_0(t)}{dt^2} &= [[\tilde{\beta}_0(t),\tilde{\beta}_{n-2m+1}(t)],\tilde{\beta}_{n-2m+1}(t)]+[\tilde{\beta}_0(t),\frac{d\tilde{\beta}_{n-2m+1}(t)}{dt}] \\ &= [\tilde{\beta}_0(t),\frac{d\tilde{\beta}_{n-2m+1}(t)}{dt}]. \qquad (4.1)\end{aligned}$$

By the proof of Theorem 7.7, $\tilde{\beta}_{\varphi_t}$ satisfies the equation

$$\frac{d\tilde{\beta}_{\varphi_t}}{dt}=-2\left[\sum_{k=n-2m+1}^{\infty}\tilde{\beta}_k(t)\lambda^{k-n+2m-1},\sum_{j=0}^{n-2m}\tilde{\beta}_j(t)\lambda^j\right],$$

thus the coefficient $\tilde{\beta}_{n-2m+1}(t)$ corresponding to the power λ^{n-2m+1} in the Taylor expansion of $\tilde{\beta}_{\varphi_t}$ satisfies the relation:

$$\frac{d\tilde{\beta}_{n-2m+1}(t)}{dt} = -2\sum_{k=0}^{n-2m}[\tilde{\beta}_{n-2m+2+k}(t),\tilde{\beta}_{n-2m-k}(t)], \qquad (4.2)$$

which implies

$$\begin{aligned}\frac{d^2\tilde{\beta}_{n-2m+1}(t)}{dt^2} &= -2\sum_{k=0}^{n-2m}\left(\frac{d}{dt}[\tilde{\beta}_{n-2m+2+k}(t),\tilde{\beta}_{n-2m-k}(t)]\right) \\ &= -2\sum_{k=0}^{n-2m}\Bigg(-2\sum_{j=0}^{n-2m}[[\tilde{\beta}_{n-2m+3+k+j}(t),\tilde{\beta}_{n-2m-j}(t)],\tilde{\beta}_{n-2m-k}(t)] \\ &\quad -2\sum_{j=0}^{n-2m}[\tilde{\beta}_{n-2m+2+k}(t),[\tilde{\beta}_{n-2m-k+1+j}(t),\tilde{\beta}_{n-2m-k-j}(t)]]\Bigg) \qquad (4.3)\end{aligned}$$

By equations (2.10), (4.1), (4.3), the 2-step nilpotency of $\mathfrak{g}_1$ implies

$$\frac{d^2\tilde{\beta}_0(t)}{dt^2}=0 \text{ and } \frac{d^2\tilde{\beta}_{n-2m+1}(t)}{dt^2}=0,$$

thus $\tilde{\beta}_0(t)=at+b$ and $\tilde{\beta}_{n-2m+1}(t)=ct+d$ for some $a,b,c,d\in\mathfrak{g}_1$. If $H(x)=k_1x_1+k_2x_2+k_3x_3+\frac{l_1x_1^2}{2}+\frac{l_2x_2^2}{2}+\frac{l_3x_3^2}{2}$ then its gradient is $\nabla H(x)=\sum_{k=1}^{3}(k_i+l_ix_k)X_i$. Obviously, there exists a solution $\{k_1,k_2,k_3,l_1,l_2,l_3\}$ of system of equations $c_i+td_i=k_i+l_i(a_i+tb_i),\ i\in\{1,2,3\}$ for any t. □

Lemma 4.3. *The functions H, H_2, H_3, H_4, H_5 are in involution and linearly independent.*

Proof. Since the Poisson bracket between any of the coordinates x_1^*,x_2^*,x_3 and any coordinate x_i or x_i^*, $i\in\{1,2,3\}$ vanishes, it follows that any of H_2, H_3, H_4 is in involution with any of H, H_2, H_3, H_4, H_5. We additionally have

$$\begin{aligned}
\{H, H_5\} &= \left(\frac{\partial H}{\partial x_1}\frac{\partial H_5}{\partial x_2} - \frac{\partial H}{\partial x_1}\frac{\partial H_5}{\partial x_2}\right)\{x_1, x_2\} + \frac{\partial H}{\partial x_1}\frac{\partial H_5}{\partial x_3^*}\{x_1, x_3^*\} + \frac{\partial H}{\partial x_2}\frac{\partial H_5}{\partial x_3^*}\{x_2, x_3^*\} \\
&= \big((k_1 + l_1x_1)x_2 - (k_2 + l_2x_2)x_1\big)(2x_3^*) + (k_1 + l_1x_1)x_3^*(-2x_2) + (k_2 + l_2x_2)x_3^*(2x_1) \\
&= 0.
\end{aligned}$$

Since the Jacobian matrix

$$\frac{\partial(H, H_2, H_3, H_4, H_5)}{\partial(x_1, x_2, x_3, x_1^*, x_2^*, x_3^*)} = \begin{pmatrix} k_1 + l_1x_1 & k_2 + l_2x_2 & k_3 + l_3x_3 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2^* & 0 \\ x_1 & x_2 & 0 & 0 & 0 & x_3^* \end{pmatrix}$$

has the rank 5 on an open dense set, we get that dH, dH_2, dH_3, dH_4, dH_5 are linearly independent on that set. □

By Lemmas 4.1(iii), 4.2, 4.3, we conclude that equation (2.10) is an integrable system on δ_1.

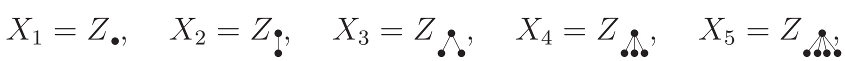
4.1. **3-step and 4-step nilpotent examples.** Similarly to the δ_1 case, we can prove that the equation (2.10) is an integrable system for a 3-step nilpotent Lie algebra and Hamiltonian for a 4-step nilpotent Lie algebra. We consider to the Hopf subalgebra of rooted trees $\mathcal{H}_2$ generated by

$1_{\mathcal{T}}$, 

and the Hopf subalgebra $\mathcal{H}_3$ generated by

$1_{\mathcal{T}}$, 

Let $\mathfrak{g}_2$ and $\mathfrak{g}_3$ be the Lie algebras of infinitesimal characters with values in $\mathbb{C}$ of $\mathcal{H}_2$ and respectively $\mathcal{H}_3$. Let δ_2 and δ_3 the double Lie algebras of $\mathfrak{g}_2$ and $\mathfrak{g}_3$. For a nonempty tree T_1, let Z_{T_1} be the infinitesimal character determined on trees by $Z_{T_1}(T_2) = 1$ if $T_1 = T_2$ and $Z_{T_1}(T_2) = 0$ if $T_1 \neq T_2$. Let



and $X_1^*, X_2^*, X_3^*, X_4^*, X_5$ the dual base of X_1, X_2, X_3, X_4, X_5. The Lie algebras of δ_2 and δ_3 have the following nonzero Lie brackets:

$$[X_1, X_2]_{\delta_2} = 2X_3, \quad [X_1, X_3^*]_{\delta_2} = -2X_2^*, \quad [X_2, X_3^*]_{\delta_2} = 2X_1^*,$$
$$[X_1, X_3]_{\delta_2} = 3X_4, \quad [X_1, X_4^*]_{\delta_2} = -3X_3^*, \quad [X_3, X_4^*]_{\delta_2} = 3X_1^*,$$
$$[X_1, X_2]_{\delta_3} = 2X_3, \quad [X_1, X_3^*]_{\delta_3} = -2X_2^*, \quad [X_2, X_3^*]_{\delta_3} = 2X_1^*,$$
$$[X_1, X_3]_{\delta_3} = 3X_4, \quad [X_1, X_4^*]_{\delta_3} = -3X_3^*, \quad [X_3, X_4^*]_{\delta_3} = 3X_1^*,$$
$$[X_1, X_4]_{\delta_3} = 4X_5, \quad [X_1, X_5^*]_{\delta_3} = -4X_4^*, \quad [X_4, X_5^*]_{\delta_3} = 4X_1^*,$$

Notice that δ_2 is not a Lie subalgebra of δ_3, just because of $[X_1, X_4]_{\delta_2} = 0$ and $[X_1, X_4]_{\delta_3} = 4X_5$, but δ_2 is a truncation of δ_3. Notice also that $\mathfrak{g}_2$ and δ_2 are step 3-nilpotent Lie algebras, while $\mathfrak{g}_3$ and δ_3 are step 4-nilpotent Lie algebras. By an argument similar to the one above, the 3-step nilpotency of $\mathfrak{g}_2$ implies that $\tilde{\beta}_0(t)$ and $\tilde{\beta}_{n-2m+1}(t)$ are quadratic in t when we consider equation (2.10) on $\mathfrak{g}_2$, while the 4-step nilpotency of $\mathfrak{g}_3$ implies that $\tilde{\beta}_0(t)$ and $\tilde{\beta}_{n-2m+1}(t)$ are cubic in t for the Lie algebra $\mathfrak{g}_3$.

The Lie-Poisson brackets are nonzero only for the following pairs:

$$\{x_1, x_2\}_{\delta_2}(x) = 2x_3^*, \quad \{x_1, x_3^*\}_{\delta_2}(x) = -2x_2, \quad \{x_2, x_3^*\}_{\delta_2}(x) = 2x_1,$$
$$\{x_1, x_3\}_{\delta_2}(x) = 3x_4^*, \quad \{x_1, x_4^*\}_{\delta_2}(x) = -3x_3, \quad \{x_3, x_4^*\}_{\delta_2}(x) = 3x_1,$$
$$\{x_1, x_2\}_{\delta_3}(x) = 2x_3^*, \quad \{x_1, x_3^*\}_{\delta_3}(x) = -2x_2, \quad \{x_2, x_3^*\}_{\delta_3}(x) = 2x_1,$$
$$\{x_1, x_3\}_{\delta_3}(x) = 3x_4^*, \quad \{x_1, x_4^*\}_{\delta_3}(x) = -3x_3, \quad \{x_3, x_4^*\}_{\delta_3}(x) = 3x_1,$$
$$\{x_1, x_4\}_{\delta_3}(x) = 4x_5^*, \quad \{x_1, x_5^*\}_{\delta_3}(x) = -4x_4, \quad \{x_4, x_5^*\}_{\delta_3}(x) = 4x_1.$$

Straightforward computations shows that $(\delta_2, \{\cdot,\cdot\}_{\delta_2})$ and $(\delta_3, \{\cdot,\cdot\}_{\delta_3})$ are Poisson manifolds of ranks $2r_2 = 4$ and respectively $2r_2 = 6$. The following functions defined on δ_2 are in involution with respect to $\{\cdot,\cdot\}_{\delta_2}$ and linearly independent:

$$H_1^{\delta_2}(x) = \frac{x_1^2}{2} + \frac{x_3^2}{2}, \quad H_2^{\delta_2}(x) = \frac{x_4^2}{2}, \quad H_3^{\delta_2}(x) = \frac{x_1^{*2}}{2}, \quad H_4^{\delta_2}(x) = \frac{x_2^{*2}}{2},$$
$$H_5^{\delta_2}(x) = \frac{x_4^{*2}}{2} + H_1^{\delta_2}(x), \quad H_6^{\delta_2}(x) = \frac{x_2^2}{2} + \frac{x_3^{*2}}{2} + H_5^{\delta_2}(x).$$

On δ_3, the following functions are in involution with respect to $\{\cdot,\cdot\}_{\delta_3}$ and linearly independent:

$$H_1^{\delta_3}(x) = \frac{x_1^2}{2} + \frac{x_4^2}{2}, \quad H_2^{\delta_3}(x) = \frac{x_5^2}{2}, \quad H_3^{\delta_3}(x) = \frac{x_1^{*2}}{2}, \quad H_4^{\delta_3}(x) = \frac{x_2^{*2}}{2},$$
$$H_5^{\delta_3}(x) = \frac{x_2^2}{2} + \frac{x_3^{*2}}{2}, \quad H_6^{\delta_3}(x) = \frac{x_3^2}{2} + \frac{x_4^{*2}}{2}, \quad H_7^{\delta_3}(x) = \frac{x_5^{*2}}{2}.$$

Now, we show that the equation (2.10) is a Hamiltonian equation both for $\mathfrak{g}_2$, $\mathfrak{g}_3$. Let $H^{\mathfrak{g}_2}$, $H^{\mathfrak{g}_3}$ be quadratic functions on $\mathfrak{g}_2$ and respectively on $\mathfrak{g}_3$. The function

$$H^{\mathfrak{g}_2}(x) = \sum_{i=1}^{4} k_i x_i + \frac{l_i x_i^2}{2} + \sum_{1 \le j < p \le 4} \xi_{j,p} x_j x_p$$

is determined by 14 variables $\{k_i, l_i, \xi_{j,p}\}_{1 \le i \le 4,\ 1 \le j < p \le 4}$, while a quadratic function on $\mathfrak{g}_3$ is determined by 20 variables. In local coordinates, after identifying the coefficients in t and taking into the account that $\tilde{\beta}_0(t)$, $\tilde{\beta}_{n-2m+1}(t)$ are quadratic in t (for the $\mathfrak{g}_2$ case) or cubic (for the $\mathfrak{g}_3$ case), the equation $\nabla H^{\mathfrak{g}_2}(\tilde{\beta}_0(t)) = \tilde{\beta}_{n-2m+1}(t)$ reduces to a systems of 12 linear equations, and the equation $\nabla H^{\mathfrak{g}_3}(\tilde{\beta}_0(t)) = \tilde{\beta}_{n-2m+1}(t)$ reduces to a systems of 20 linear equations, thus in both cases, there exists a Hamiltonian function for (2.10).

For $i \in \{2,3\}$, let $\pi_i : \delta_i = \mathfrak{g}_i \oplus \mathfrak{g}_i^* \to \mathfrak{g}_i$ be the projections onto $\mathfrak{g}_i$ and let $H^{\delta_i} = H^{\mathfrak{g}_i} \circ \pi_i$. The functions H^{δ_2}, $H_2^{\delta_2}$, $H_3^{\delta_2}$, $H_4^{\delta_2}$, $H_5^{\delta_2}$, $H_6^{\delta_2}$ are in involution with respect to $\{\cdot,\cdot\}_{\delta_2}$ and linearly independent. This concludes that equation (2.10) is an integrable system for δ_2.

The functions H^{δ_3}, $H_2^{\delta_3}$, $H_3^{\delta_3}$, $H_4^{\delta_3}$, $H_1^{\delta_3} + H_5^{\delta_3} + H_6^{\delta_3} + H_7^{\delta_3}$ are in involution with respect to $\{\cdot,\cdot\}_{\delta_3}$ and linearly independent. In order to show that equation (2.10) is an integrable system with the Hamiltonian H^{δ_3}, it would be sufficient to find another two functions f_1, f_2 on δ_3 such that H^{δ_3}, $H_2^{\delta_3}$, $H_3^{\delta_3}$, $H_4^{\delta_3}$, $H_1^{\delta_3} + H_5^{\delta_3} + H_6^{\delta_3} + H_7^{\delta_3}$, f_1, f_2 are in involution with respect to $\{\cdot,\cdot\}_{\delta_3}$ and linearly independent.

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