Scientific Report for the project 'Special geometries and associated structures', 2nd stage of the project

December 2018

We describe the scientific activity of the team members of the project 'Special geometries and related structures', project code PN-III-ID-P4-PCE-2016-0019, during the second stage of implementation of the project (1st of January 2018 - 30th of December 2018).

The members of the team are: CSI Dr. Liana David (Director of the project), Conf. Univ. Dr. Monica Aprodu (key person of the project), CS III Dr. Gabriel Baditoiu, CSI Dr. Radu Pantilie, CSII Dr. Costin Vîlcu and master student Ionut-Alexandru Hurjui.

1 Scientific papers

The next papers were ellaborated during this period (all papers mention the financial support of the grant):

M. A. Aprodu: *Harmonic non-minimizing spheres in Hirzebruch surfaces*, sent for publication;

L. David, C. Hertling: *(T)-structures over 2-dimensional F-manifolds: formal classification*, arxiv: 1811.03406 (29 pages), sent for publication;

R. Pantilie: *Projective structures and ρ-connections*, published in **J. Inst. Math. Jussieu**, https://doi.org/10.1017/S1474748018000129, pages 1-9.

R. Pantilie: *Harmonic morphisms and the Penrose-Ward transform*, accepted for publication in **Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.**;

A. Rivière, J. Rouyer, **C. Vîlcu**, T. Zamfirescu: *Double normals of most convex bodies*, published in **Adv. in Math.** vol. 343 (2019), 245-272; arXiv:1804.07015 [math.MG].

2 Research results of the team members

In this section we present the research activity of the team members of the project.

Monica Alice Aprodu: She ellaborated the preprint: "Harmonic non-minimizing spheres in Hirzebruch surfaces". A Hirzebruch surface is a compact complex surface S that is diffeomorphic either with the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ or with H_1, the blowup in a point of the complex projective plane. Moreover, any Hirzebruch surface is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ and its geometry is governed by the so-called n–invariant, where $n \geq 0$ is an integer. Another description of a Hirzebruch surface with invariant n, is to be isomorphic with $\mathbb{P}(L_{-n} \oplus \mathbb{C})$, where L_n is the quotient of the trivial bundle $((\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C})/\{(z, v) \sim (\lambda z, \lambda^n v), \lambda \in \mathbb{C}^*\}$.

A natural problem occurs: to deform Hirzebruch surfaces to one other in a convenient way and to obtain clases of stable harmonic maps from $\mathbb{CP}^1$ to S which are not holomorphic. In this preprint she gives examples of such clases of maps.

Gabriel Baditoiu: He worked on the classification of pseudo-Riemannian Einstein homogeneous metrics on pseudo-hyperbolic spaces.

In a preprint, previous to this grant, Gabriel Baditoiu obtained the classification of pseudo-Riemannian Einstein homogeneous metrics on pseudo-hyperbolic spaces under the hypothesis that

(i) G is a closed connected Lie subgroup of connected component of indefinite special orthogonal group, $SO_0(n-r, r+1)$, and

(ii) G acts completely reducibly on $\mathbb{R}^{n+1}$ on $r < n$.

In the same preprint, it is also obtained a classification of the Lie subgroups acting transitively and effectively on a hyperboloid (also called a pseudo-hyperbolic space) under the hypothesis (i).

Current work in progress, funded by the grant, deals with the classification of pseudo-Riemannian Einstein homogeneous metrics on pseudo-hyperbolic spaces under the assumption (i) and

(iii) G does not acts completely reducibly on $\mathbb{R}^{n+1}$ on $r < n$.

Liana David: Together with Claus Hertling, she ellaborated the preprint '(T)-structures on 2-dimensional F-manifolds: formal classification'. An important class of meromorphic connections are the so called (TE)-structures. They are meromorphic connections defined on holomorphic vector bundles

over products $\mathbb{C} \times M$ (where M is a complex manifold) which admit poles of Poincaré rank one along $\{0\} \times M$. The parameter space M of a (TE)-structure inherits, when the so called unfolding condition is satisfied, a multiplication on TM (which preserves the fibers, is associative, commutative, with unit and satisfies an integrability condition), and a vector field on M (called the Euler field), giving rise to an F-manifold structure with Euler field on M. A (TE)-structure ∇ determines a family of flat connections parameterized by $z \in \mathbb{C} \setminus \{0\}$, defined on vector bundles over M. (This is done by restricting ∇ to the derivatives in the parameter direction). Such a family is known in the literature as a (T)-structure. When the unfolding condition is satisfied, a (T)-structure induces on its parameter space the structure of an F-manifold (without Euler field). In the above preprint the authors determine normal forms for (T)-structures which induce 2-dimensional germs $((M, 0), \circ, e)$ of F-manifolds. They obtain a short list of (T)-structures and they proved that any (T)-structure which induces $((M, 0), \circ, e)$ is formally isomorphic to a (T)-structure from this list.

Radu Pantilie: In the article Projective structures and ρ-connections, *J. Inst. Math. Jussieu* (https://doi.org/10.1017/S1474748018000129), Radu Pantilie extended T. Y. Thomas's approach to projective structures, over the complex analytic category, by involving the ρ-connections. During the implementation of the grant Radu Pantilie improved and extended a previous version of this preprint (which was initially sent for publication in April 2017). In the preprint, a better control of projective flatness was obtained and, consequently, the following application was developed: if the twistor space of a quaternionic manifold P is endowed with a complex projective structure then P can be locally identified, through quaternionic diffeomorphisms, with the quaternionic projective space.

In the article "Harmonic morphisms and the Penrose-Ward transform", *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* (to appear), Radu Pantilie showed that, in quaternionic geometry, the Ward transform is a manifestation of the functoriality of the basic correspondence between the ρ-quaternionic manifolds and their twistor spaces. He applied this fact, together with the Penrose transform, to obtain existence results for quaternionic manifolds and for harmonic morphisms from hyper-Kähler manifolds.

Costin Vîlcu: He ellaborated, with his collaborators A. Rivière, J. Rouyer, and T. Zamfirescu, the paper *Double normals of most convex bodies*, **Adv. in Math.** 343 (2019), 245-272. There they consider a typical (in the sense of Baire categories) convex body K in $\mathbb{R}^{d+1}$. The set of feet of its double normals is a Cantor set, having lower box-counting dimension 0 and

packing dimension d. The set of lengths of those double normals is also a Cantor set of lower box-counting dimension 0. Its packing dimension is equal to $\frac{1}{2}$ if $d = 1$, is at least $\frac{3}{4}$ if $d = 2$, and equals 1 if $d \geq 3$. They also consider the lower and upper curvatures at feet of double normals of K, with a special interest for local maxima of the length function (they are countable and dense in the set of double normals). In particular, a previous result about the metric diameter is improved.

2.1 Activity of the master student

Ionut-Alexandru Hurjui: The master student Ionut-Alexandru Hurjui read the book Riemannian Geometry by Manfredo Perdigao do Carmo (during the months January - September 2018) and from October onwards began to read bibliography in the field of Kähler geometry. In September 2018 he gave a presentation on the material which he studied. He attended regularly the geometry seminars from the host institution IMAR.

3 Participations at workshops

Monica Alice Aprodu will participate at the workshop *Mathematics, Computer science and Applications* University "Dunărea de Jos" of Galaţi, December 2018, and will give a talk with title "Harmonic non-minimizing spheres in Hirzebruch surfaces".

Liana David participated at the workshop *Special Metrics and Symmetries on Complex Manifolds*, which was helt at Universitaty of Regensburg (Germany), during 10-15 September 2018. She gave a 50 minutes presentation with title "Twist, elementary deformation and the KK-correspondence in generalized geometry", where she presented the results she obtained in the first stage of the project (12th of July - 30th of December 2017).

Radu Pantilie was invited to give a 50 minutes presentation at *The Differential Geometry Workshop 2018*, organized by Dipartimento di Matematica e Informatica, Università di Cagliari, Italy, between 12-14 September 2018 (http://people.unica.it/montaldo/dgw2018/). His talk had the title "Harmonic maps and twistorial structures" and was based on a collaboration with G. Deschamps si E. Loubeau, which began during a scientific visit at Laboratoire de Mathématiques de Bretagne Atlantique, Brest, Franta, during the first stage of the project.

4 Scientific visits

Liana David visited Universitaty of Hamburg (Germany), during the period 10-15 September 2018, to work with Prof. Vicente Cortes on generalized complex geometry.

Remark: We mention that all scientific activities described in this report are aimed at the scientific goals of the project.