On the projection of pseudoconvex domains

Cezar Joiţa

1 Introduction

Let $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection $\pi((z_1, z_2, ..., z_{n+1})) = (z_2, , ..., z_{n+1})$. It is obvious that if D is a convex domain in $\mathbb{C}^{n+1}$ then $\pi(D)$ is convex too. Although under certain conditions the projection of a pseudoconvex domain remains pseudoconvex (see in this sense [7],[8]), for general pseudoconvex domains this is not true anymore. A counterexample was given by Peter Pflug in [11].
In the first part of this note we prove that any connected open subset of $\mathbb{C}^n$ is the projection of a connected Runge open subset of $\mathbb{C}^{n+1}$ (Theorem 3).

In the same circle of ideas we study then the following problem: if P_1 and P_2 are closed subsets of $\mathbb{C}^n$, each one having a fundamental system of Runge neighborhoods, does their union have the same property ?
For arbitrary closed subsets P_1 and P_2 this is not true. A thorough discussion about the union of totally real subspaces of $\mathbb{C}^n$ can be found in [12].
Corollary 1 (see also Remark 2) gives sufficient conditions such that if P_1 and P_2 are contained in two analytic subsets of $\mathbb{C}^n$, A_1 and A_2 respectively, then $P_1 \cup P_2$ has a fundamental system of Runge neighborhoods.

Aknowledgements I am very grateful to Professor Mihnea Colţoiu for suggesting me these problems and for his helpful advises.
I would also like to thank Professors Mohan Ramachandran, Terrence Napier and Anca Popa for their comments and Professor Klas Diederich for suggesting me a better version of Proposition 4 and Corollary 1.

2 Preliminaries

Throughout this paper by a closed analytic subset I will understand a closed complex analytic subset.
If A is a closed analytic subset of $\mathbb{C}^n$ and $K \subset A$ is a compact subset, we denote by $\widehat{K}_A = \{z \in A : |f(z)| \leq \sup_K |f| \text{ for any } f \in \mathcal{O}(A)\}$ its holomorphically convex hull. K is called holomorphically convex with respect to A if $K = \widehat{K}_A$.
It is easy to see that $\widehat{K}_A = \widehat{K}_{\mathbb{C}^n}$, therefore a compact subset of A is holomorphically convex with respect to A iff it is holomorphically convex with respect to $\mathbb{C}^n$.

If $D \subset A$ is a Stein open subset, D is said to be Runge in A if the restriction map $\mathcal{O}(A) \to \mathcal{O}(D)$ has dense image.

The following result is proved in [2].

Proposition 1. *Let X be a Stein space, φ be a holomorphic function on X. Let K_1, K_2 be compact subsets of X and let $Re(\varphi) < 0$ on K_1, $Re(\varphi) > 0$ on K_2. Then we have $(K_1 \cup K_2)\widehat{\ } = \widehat{K}_1 \cup \widehat{K}_2$*

Theorem 1 is proved in [5].

Theorem 1. *Let $A \subset \mathbb{C}^n$ be a closed analytic subset, $K \subset \mathbb{C}^n$ a holomorphically convex compact subset and U an open neighborhood of $K \cup A$.Then there exists a C^∞ plurisubharmonic function ψ on $\mathbb{C}^n$ such that $\psi < 0$ on $K \cup A$ and $\psi > 0$ on $\complement U$. In particular, $K \cup A$ has a fundamental system of Runge neighborhoods.*

The next two results are from [4].

Proposition 2. *Let $A \subset \mathbb{C}^n$ be a closed analytic subset, $K \subset \mathbb{C}^n$ a holomorphically convex compact subset and $L \subset A$ a holomorphically convex compact subset with $K \cap A \subset L$.*

Then $K \cup L$ has a fundamental system of Runge neighborhoods, hence it is holomorphically convex.

With a slight modification Theorem 3 in [4] becomes:

Theorem 2. *Let X be a Stein space of finite embedding dimension, $A \subset X$ be a closed analytic subset, $D \subset A$ a Runge open subset $K \subset X$ a holomorphically convex compact subset such that $K \cap A \subset D$ and $V \subset X$ an open subset such that $D \cup K \subset V$. Then there exists a Runge open subset $\widetilde{D}$ in X with $\widetilde{D} \cap A = D$, $K \subset \widetilde{D}$ and $\widetilde{D} \subset V$.*

The only thing we have to change in the proof of Theorem 3 in [4] is to choose V_n such that $V_n \subset V$ and to embed X as a closed subvariety of $\mathbb{C}^N$ for some N.

3 The Results

Theorem 3. *For every D open and connected subset of $\mathbb{C}^n$ there exists a connected Runge open subset $\widetilde{D}$ of $\mathbb{C}^{n+1}$ such that $\pi(\widetilde{D}) = D$. Furthermore, if D is bounded we can find a bounded $\widetilde{D}$ with this property.*

Proof. Let $\{P_m\}_{m\geq 1}$ be a covering of D by compact polydiscs such that $P_m \cap P_{m+1} \neq \emptyset$ and let $x_m \in P_m \cap P_{m+1}$. Such a covering exists since D is connected.
We construct by induction a sequence $\{V_m\}_{m\geq 1}$ of connected open subsets of $\mathbb{C}^{n+1}$ with the following properties:

(1)$\overline{V}_m$ is compact and $\overline{V}_m \subset V_{m+1}$

(2)V_m is Runge in $\mathbb{C}^{n+1}$ and $\overline{V}_m$ is holomorphically convex with respect to $\mathbb{C}^{n+1}$

(3)$\overline{V}_m \subset ((0,1)\times(0,1)) \times D$ and $P_m \subset \pi(V_m)$

We define V_1 as follows: let B_1 be a compact disc in $\mathbb{C}$, $B_1 \subset (0,1)\times(0,1)$ and let $K_1 = B_1 \times P_1$. Then K_1 is a holomorphically convex compact subset of $\mathbb{C}^{n+1}$ so it has a fundamental system of Runge neighborhoods.
Let $U_1 \subset \mathbb{C}^{n+1}$ be a Runge open subset, $K_1 \subset U_1 \subset ((0,1)\times(0,1)) \times D$ and let φ_1 be a C^∞ strongly plurisubharmonic exhaustion function for U_1.
Using Sard Theorem we choose $c \in \mathbb{R}$ a regular value for φ_1 such that $\{\varphi < c\} \supset K_1$. Because c is a regular value $\overline{\{\varphi < c\}} = \{\varphi \leq c\}$.
We choose V_1 to be the connected component of $\{\varphi < c\}$ that contains K_1. Then $\overline{V}_1$ is the connected component of $\{\varphi \leq c\}$ that contains K_1. So V_1 is a Runge open subset of U_1 and $\overline{V}_1$ is holomorphically convex with respect to U_1. Since U_1 is Runge in $\mathbb{C}^{n+1}$ we deduce that V_1 is Runge in $\mathbb{C}^{n+1}$ and $\overline{V}_1$ is holomorphically convex with respect to $\mathbb{C}^{n+1}$.
Because $B_1 \times P_1 \subset V_1$ and $U_1 \subset ((0,1)\times(0,1)) \times D$ we have $\pi(V_1) \supset P_1$ and $\overline{V}_1 \subset ((0,1)\times(0,1)) \times D$. So V_1 has the desired properties.

Now assume that we have constructed $V_1, ..., V_m$ with the required properties and define V_{m+1} as follows: because $\overline{V}_m$ is compact and $\overline{V}_m \subset ((0,1)\times(0,1)) \times D$ there exists a real number α_m, $\alpha_m \in (0,1)$, such that $\overline{V}_m \subset ((0,\alpha_m)\times(0,1)) \times D$. Let B_{m+1} be a compact disc in $\mathbb{C}$,

$B_{m+1} \subset (\alpha_m, 1) \times (0,1)$ and let $K_{m+1} = \overline{V}_m \cup B_{m+1} \times P_{m+1}$.
Since $\overline{V}_m$ and $B_{m+1} \times P_{m+1}$ are holomorphically convex subsets of $\mathbb{C}^{n+1}$ and $Re(z_1 - \alpha_m) < 0$ on $\overline{V}_m$, $Re(z_1 - \alpha_m) > 0$ on $B_{m+1} \times P_{m+1}$ it follows from Proposition 1 (see also the Separation Lemma in [6]) that K_{m+1} is a holomorphically convex compact subset of $\mathbb{C}^{n+1}$.
Let $A_m = \mathbb{C} \times \{x_m\}$. A_m is an analytic subset of $\mathbb{C}^{n+1}$ and by Theorem 1 $K_{m+1} \cup A_m$ has a fundamental system of Runge neighborhoods. Note also that $\pi(K_{m+1} \cup A_m) = \pi(K_{m+1}) \subset\subset D$.
Let then $\widetilde{U}_{m+1}$ be a Runge open subset of $\mathbb{C}^{n+1}$ such that $K_{m+1} \cup A_m \subset \widetilde{U}_{m+1}$ and $\pi(\widetilde{U}_{m+1}) \subset\subset D$ and put $U_{m+1} = ((0,1) \times (0,1)) \times \mathbb{C}^n \cap \widetilde{U}_{m+1}$.
Then U_{m+1} is Runge in $\mathbb{C}^{n+1}$, $K_{m+1} \subset U_{m+1}$ and $U_{m+1} \subset ((0,1) \times (0,1)) \times D$.
Because $\pi(V_m) \supset P_m$ and $x_m \in P_m$ there exists $a_m \in \mathbb{C}$ such that $(a_m, x_m) \in V_m$. But $V_m \subset ((0,1) \times (0,1)) \times D$. So $a_m \in (0,1) \times (0,1)$.
Let $a_{m+1} \in B_{m+1}$ and $\gamma_m : [0,1] \to \mathbb{C}^{m+1}, \gamma_m(t) = (ta_m + (1-t)a_{m+1}, x_m)$ and $\Gamma_m = \gamma_m([0,1])$. Then $\Gamma_m \subset A_m \subset \widetilde{U}_{m+1}$ and $\Gamma_m \subset ((0,1) \times (0,1)) \times D$. So $\Gamma_m \subset U_{m+1}$.
K_{m+1} has two connected components and Γ_m is a path that joins them. It follows that $K_{m+1} \cup \Gamma_m$ is a connected compact subset of U_{m+1}.
Let φ_{m+1} be a C^∞ strongly plurisubharmonic exhaustion function for U_{m+1} and let $c \in \mathbb{R}$ a regular value for φ_{m+1} such that $K_{m+1} \cup \Gamma_m \subset \{\varphi_{m+1} < c\}$. Put V_{m+1} :=the connected component of $\{\varphi_{m+1} < c\}$ that contains $K_{m+1} \cup \Gamma_m$.

As in the construction of V_1, V_{m+1} is Runge in $\mathbb{C}^{n+1}$ and $\overline{V}_{m+1}$ is holomorphically convex with respect to $\mathbb{C}^{n+1}$, $\pi(V_{m+1}) \supset P_{m+1}$, $\overline{V}_{m+1} \subset ((0,1) \times (0,1)) \times D$ and since $\overline{V}_m \subset K_{m+1}$ it follows $\overline{V}_m \subset V_{m+1}$. So V_{m+1} satisfies (1),(2) and (3) and the existence of the sequence $\{V_m\}_{m \geq 1}$ is proved.

Put now $\widetilde{D} = \bigcup_{m=1}^{\infty} V_m$. Because every V_m is a connected Runge open subset of $\mathbb{C}^{n+1}$, $\widetilde{D}$ is connected and Runge. From $\pi(V_m) \supset P_m$ we get that $\pi(\widetilde{D}) \supset \bigcup_{m=1}^{\infty} P_m = D$. On the other hand, since $V_m \subset ((0,1) \times (0,1)) \times D$ it follows $\widetilde{D} \subset ((0,1) \times (0,1)) \times D$, so $\pi(\widetilde{D}) \subset D$. Therefore $\pi(\widetilde{D}) = D$. Moreover, if D is bounded then $((0,1) \times (0,1)) \times D$ is bounded so $\widetilde{D}$ is bounded. □

Remark 1: If we did not ask $\widetilde{D}$ to be connected the construction would be immediate: just take $\widetilde{D} = \bigcup D_k \times Q_k$ for some disjoint open discs $D_k \subset \mathbb{C}$ and $\{Q_k\}$ a covering by open polydiscs for D.

The following proposition shows that if D is an arbitrary relatively compact open and connected subset of $\mathbb{C}^n$ we cannot expect to find $\widetilde{D}$ a relatively compact open Runge subset of $\mathbb{C}^{n+1}$ with smooth boundary such that $\pi(\widetilde{D}) = D$, so the statement of theorem 3 is in this respect maximal.

Proposition 3. *Let Δ be the unit disc in $\mathbb{C}$. There exists $A \subset \Delta$ a countable closed set such that there exists no integer $k > 2$ and no open and bounded subset of $\mathbb{R}^k$ with C^2 boundary whose projection on $\mathbb{C}$ is $\Delta \setminus A$.*

Proof. We will construct A as follows:
For every integer $n \geq 2$ we consider the following four points:
$P_{n,k} = \frac{1}{n} exp(\frac{2\pi k\sqrt{-1}}{4})$, $k = 1, 2, 3, 4$ and put $A_n = \{P_{n,1}, P_{n,2}, P_{n,3}, P_{n,4}\}$
Note that for every open disc Δ_1 of radius $\frac{1}{n}$ if $0 \in \partial\Delta_1$ then $\Delta_1 \cap A_n \neq \varnothing$.

Let $A = \{0\} \cup \bigcup_{n=1}^{\infty} A_n$. Then for every open disc Δ_1 in $\mathbb{C}$ if $0 \in \overline{\Delta}_1$ then $\Delta_1 \cap A \neq \varnothing$.
This is obvious if $0 \in \Delta_1$. If $0 \in \partial\Delta_1$ just take an integer $n \geq \frac{1}{r}$ and an open disc $\Delta_2 \subset \Delta_1$ of radius $\frac{1}{n}$ such that $0 \in \partial\Delta_2$ and then we have $\Delta_2 \cap A_n \neq \varnothing$.

We will prove now that A has the desired properties.
It is obvious that A is closed, 0 being its only accumulation point.
Suppose that there exists $k > 2$ and V an open subset of $\mathbb{R}^k$, V bounded and with C^2 boundary, such that $p(V) = \Delta \setminus A$. Here $p : \mathbb{R}^k \to \mathbb{C}$ is the projection $p(x_1, x_2, ..., x_k) = x_1 + ix_2$.
There exists a point $Q \in \partial V$, $p(Q) = 0$. Indeed: take $\{z_n\}$ a sequence of points in $\Delta \setminus A$ that converges to 0 and $\{x_n\}$ in V with $p(x_n) = z_n$. V being bounded $\{x_n\}$ has a convergent subsequence and and we may take Q to be its limit.

Because V has C^2 boundary we can find $B \subset V$ an open ball such that $Q \in \partial B$. In fact if R is on the inward normal of ∂V at Q and it is sufficiently close to Q then the open ball B with center R and radius $d(R, Q)$ is contained in V and $d(R, Q) = d(R, \partial V)$.
B being a ball, $\Delta_1 := p(B)$ is a disc in $\mathbb{C}$ and $0 \in \partial\Delta_1$. So $A \cap \Delta_1 \neq \varnothing$.
But $B \subset V$ and $p(V) = \Delta \setminus A$. This contradiction proves our proposition.
□

Proposition 4. *Let A_1, A_2 be closed analytic subsets of $\mathbb{C}^n$ and U_1, U_2 be Runge open subsets of A_1 and A_2 respectively. If $A_1 \cap U_2 = A_2 \cap U_1$ then $U_1 \cup U_2$ is a Runge open subset of $A_1 \cup A_2$.*

This proposition is a direct consequence of Corollary 3.8 in [9]. We give here an alternative proof.

Proof. Because $A_1 \cap U_2 = A_2 \cap U_1$ it is easy to see that $U_1 \cup U_2$ is open in $A_1 \cup A_2$.
For $j = 1, 2$ let $\{K_{n,j}\}$ be an exhaustion of U_j with holomorphically convex compact subsets such that:

(α) $K_{n,j} \subset Int(K_{n+1,j})$

(β) $K_{n,1} \cap A_2 \subset K_{n,2}$

(γ) $K_{n,2} \cap A_1 \subset Int(K_{n+1,1})$

Put $F_n = K_{n,1} \cup K_{n,2}$.Then ($\alpha$),($\beta$), ($\gamma$) guarantee that $F_n \subset Int(F_{n+1})$. From (β) and Proposition 2 it follows that F_n is a holomorphically convex compact subset of $A_1 \cup A_2$. Therefore $U_1 \cup U_2 = \bigcup_{n=1}^{\infty} F_n$ is a Runge open subset of $A_1 \cup A_2$. □

Corollary 1. *If A_1 and A_2 are closed analytic subsets of $\mathbb{C}^n$ and $P_1 \subset A_1$, $P_2 \subset A_2$ are closed subsets each one having a fundamental system of Runge neighborhoods, $\{W_{1,n}\}$ and $\{W_{2,n}\}$ respectively, such that $A_1 \cap W_{2,n} = A_2 \cap W_{1,n}$ then $P_1 \cup P_2$ has a fundamental system of Runge neighborhoods.*

Proof. Let V be an open set that contains $P_1 \cup P_2$.
For $j = 1, 2$ let W_j Runge in $\mathbb{C}^n$ with $P_j \subset W_j \subset V$ and $A_1 \cap W_2 = A_2 \cap W_1$. Then $U_1 = A_1 \cap W_1$ and $U_2 = A_2 \cap W_2$ satisfy the conditions of Proposition 4 so $U_1 \cup U_2$ is a Runge open subset of $A_1 \cup A_2$ and $P_1 \cup P_2 \subset U_1 \cup U_2$.
From Theorem 2 we deduce that that there exists a Runge domain $W \subset \mathbb{C}^n$, with $W \subset V$ and $W \cap (A_1 \cup A_2) = U_1 \cup U_2$. So W is Runge and $P_1 \cup P_2 \subset W \subset V$. □

Remark 2. If P_1 and P_2 have fundamental systems of Runge neighborhoods then the conditions of Corollary 1 are fulfiled if:
a) $P_1 \cap P_2 = A_1 \cap A_2$
or if
b) P_1 or P_2 is compact and $A_1 \cap P_2 = A_2 \cap P_1$
a) is straightforward and for b) one has to apply Theorem 2.
Note also that if P_1 and P_2 have fundamental systems of neighborhoods $\{W_{1,n}\}$ and $\{W_{2,n}\}$ such that $A_1 \cap W_{2,n} = A_2 \cap W_{1,n}$ then $A_1 \cap P_2 = A_2 \cap P_1$.

Corollary 2. *Let in $\mathbb{C}^2$: $\mathbb{C} = \{(z_1, z_2) : z_2 = 0\}$, $\mathbb{R} = \{(z_1, z_2) : z_1 = 0, y_2 = 0\}$ where $z_j = x_j + iy_j$.*
Then $\mathbb{R} \cup \mathbb{C}$ has a fundamental system of Runge neighborhoods.

Remark 3. $\mathbb{C} \times \mathbb{R}$ does not have a fundamental system of Runge neighborhoods. In fact it has been proved in [1] that this is the case for every open subset of $\mathbb{C} \times \mathbb{R}$.

As Prof. Colţoiu pointed to me, if A is a closed subset of $\mathbb{C}$ then $\mathbb{C} \times A$ has a fundamental system of Runge neighborhoods iff A is complete pluripolar. If $\mathbb{C} \times A$ has a fundamental system of Runge neighborhoods then one can choose U to be a Runge neighborhood of $\mathbb{C} \times A$ such that U does not contain the fibre $\mathbb{C} \times \{x\}$ if x is outside A and define $\varphi : \{0\} \times \mathbb{C} \cap U \to \mathbb{R} \cup \{-\infty\}$, $\varphi = -\log d$ where d is the distance to the boundary of U in the z_1 direction. It follows then that φ is subharmonic and $A = \{\varphi = -\infty\}$.
If A is complete pluripolar, then $\mathbb{C} \times A$ is complete pluripolar and using Corollary 1 in [10] we note that, for $q = 1$ and X a Stein space, the neighborhoods constructed in the proof of Theorem 2 in [3], are in fact Runge in X. (This argument shows also that $\mathbb{C} \times \mathbb{R}$ does not have a fundamental system of Stein neighborhoods either.)

Remark 4. In all our results $\mathbb{C}^n$ can be replaced by any Stein space X of finite embedding dimension.

References

[1] Andreotti,A;Nacinovich, M.: Analytic convexity. Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4) **7** (1980), no. 2, 287–372.

[2] Andreotti,A; Narasimhan,R.: A topological property of Runge pairs. Ann. of Math.(2) **76** (1962), 499–509.

[3] Colţoiu,M.: Complete locally pluripolar sets. J.reine angew. Math. **412** (1990), 108-112.

[4] Colţoiu,M.:Traces of Runge domains on analytic subsets. Math. Ann. **290** (1991),545-548.

[5] Colţoiu,M.,Mihalache,N.: On the homology groups of Stein spaces and Runge pairs. J.reine angew.Math. **371** (1986), 216-220.

[6] Kallin,E.: Polynomial convexity: the Three Spheres Problem. Proceedings of the Conference on Complex Analysis (Mineapolis,1964) Springer-Verlag, New York, 1965.

[7] Kiselman,C.O.: The partial Legendre transformation for plurisubharmonic functions.Invent. Math. **49** (1978), no. 2, 137–148.

[8] Loeb, J.J.: Action d'une forme réelle d'un groupe de Lie complexe sur les fonctions plurisousharmoniques. Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 59–97.

[9] Mihalache, N.: The Runge theorem on 1-dimensional Stein spaces. Rev. Roumaine Math. Pures Appl. **33** (1988), no. 7, 601–611.

[10] Narasimhan, R: The Levi problem for complex spaces. II. Math. Ann. **146** (1962), 195–216.

[11] Pflug, P.: Ein C^∞-glattes, streng pseudokonvexes Gebiet im $\mathbb{C}^3$ mit nicht holomorph-konvexer Projektion. Special issue dedicated to the seventieth birthday of Erich Kähler. Abh. Math. Sem. Univ. Hamburg **47** (1978), 92–94.

[12] Thomas,P.: Enveloppes polynomiales d'unions de plans réels dans $\mathbb{C}^n$. Ann. Inst. Fourier (Grenoble) **40** (1990), 371-390.