# On the n-concavity of covering spaces with parameters

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## 1 Introduction

We consider the following situation:

$$\begin{array}{ccc} \widetilde{X} & & \\ \sigma \big\downarrow & & \\ X & \xrightarrow{\pi} & T \end{array}$$

where $X$ and $T$ are connected complex manifolds of dimensions $n+m$ and $m$ respectively, $\pi$ is a proper and surjective holomorphic submersion, and $\sigma$ is a covering map. Thus, $\widetilde{X}$ can be regarded as a family of n-dimensional complex manifolds over $T$.

We will use the same definition of $q$-convexity as in [1]. For the precise formulation, see the next section.

R. Green and H. Wu [7] proved that a connected, non-compact complex manifold is $n$-complete. In [5] M. Colţoiu and V. Vâjâitu proved the following:

**Theorem** *In the above situation if for some $t_0$ the fiber $(\pi \circ \sigma)^{-1}(t_0)$ does not have compact components then there exists an open neighborhood $U$ of $t_0$ such that $(\pi \circ \sigma)^{-1}(U)$ is $n$-complete.*

Here we want to prove a similar result in the $n$-concave case. In [3] M. Colţoiu proved the following theorem:

**Theorem 1.** *Let $X$ be a connected complex manifold of dimension $n$. Then $X$ is $n$-concave.*

Using the same technique as in [5] we will prove the following:

**Theorem 2.** *In the above situation if for some $t_0$ the fiber $(\pi \circ \sigma)^{-1}(t_0)$ has at most finitely many compact components then there exists an open neighborhood $U$ of $t_0$ such that $\pi \circ \sigma_{|(\pi\circ\sigma)^{-1}(U)} : (\pi \circ \sigma)^{-1}(U) \to U$ is a $n$-concave morphism.*

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## 2 Preliminaries

**Definition 1.** *Let $X$ be a complex manifold. A function $\phi \in C^\infty(X, \mathbb{R})$ is said to be strictly $q$-convex if its Levi form*

$$L_\phi(z, \xi) = \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \bar{\xi}_j,\ \xi \in T_z X,$$

*has at least $n - q + 1$ positive eigenvalues for every $z \in X$.*

**Definition 2.** *Let $X$ be a complex manifold. $X$ is said to be $q$-convex if there exists a compact set $K \subset X$ and a smooth function $\phi : X \to \mathbb{R}$ such that $\phi$ is strictly $q$-convex on $X \setminus K$ and for every real number $\alpha$ the level set $\{\phi < \alpha\}$ is relatively compact in $X$. If we can choose $K = \emptyset$, then $X$ is said to be $q$-complete.*

*$X$ is said to $q$-concave if there exists a compact set $K \subset X$ and a smooth function $\phi : X \to (0, \infty)$ such that $\phi$ is strictly $q$-convex on $X \setminus K$ and for every positive real number $\alpha$ the level set $\{\phi > \alpha\}$ is relatively compact in $X$.*

**Definition 3.** *Let $X$ be a complex manifold and $Y$ a $C^\infty$-manifold. $\pi \in C^\infty(X, Y)$ is said to be $q$-concave if there exists $\phi \in C^\infty(X, \mathbb{R}_+)$ and $F \subset X$, a closed subset, such that*
*1)$\pi|_F$ is proper*
*2) $\phi|_{X \setminus F}$ is strictly $q$-convex*
*3)For every $\epsilon > 0$, $\pi|_{\{\phi \geq \epsilon\}}$ is proper.*

One can state similar definitions for $q$-convex and $q$-complete morphisms. As mentioned in the Introduction the definition of $q$-convexity that we use here is the one in [1]. For Definition 3 see [8]. Some authors include one more condition in this definition. Namely they require that $F \subset \{\phi > \alpha\}$ for some positive number $\alpha$. This is however inconsequential for the conclusion of Theorem 2 since once we found a neighborhood of a point $t_0 \in T$ we can shrink it and then this extra condition will be satisfied.

Definitions 4 and 5, Lemma 1 and Proposition 1 are due to M. Peternell [11]. We consider $X$ a complex manifold and $W$ an open subset of $X$. We denote by $TX$ the holomorphic tangent bundle of $X$.

**Definition 4.** *i) A subset $\mathcal{M} \subset TX$ is said to be a linear set over $X$ if for every point $x \in X$, $\mathcal{M}_x := \mathcal{M} \cap T_xX \subset T_xX$ is a complex vector subspace.*
*ii) If $\mathcal{M}$ is a linear set over $X$ we define $\mathcal{M}|_W$ as $(\mathcal{M}|_W)_x = \mathcal{M}_x$ for every $x \in W$ and we put $codim_W \mathcal{M} = \sup_{x\in W} codim\ \mathcal{M}_x$.*
*iii) If $Z$ and $X$ are complex manifolds and $\pi : Z \to X$ is a holomorphic map we set*

$$\pi^*\mathcal{M} := \bigcup_{z\in Z} (\pi_{*,z})^{-1}(\mathcal{M}_{\pi(z)})$$

**Definition 5.** *Let $X$ be a complex manifold, $W$ open in $X$, $\mathcal{M}$ a linear set over $W$, and $\phi \in C^\infty(W, \mathbb{R})$.*

*(a) Let $x \in W$. We say that $\phi$ is weakly 1-convex with respect to $\mathcal{M}_x$ if there is a local chart $(z_1, ..., z_n)$ around $x$ such that $L_\phi(x, \xi) \geq 0$ for every $\xi \in \mathcal{M}_y$.*

*We say that $\phi$ is weakly 1-convex with respect to $\mathcal{M}$ if $\phi$ is weakly 1-convex with respect to $\mathcal{M}_x$ for every $x \in W$.*

*(b) The function $\phi$ is said to be strictly 1-convex with respect to $\mathcal{M}$ if every point of $W$ admits an open neighborhood $U \subset W$ such that there exists a strictly 1-convex function $\theta$ on $U$ with $\phi - \theta$ weakly 1-convex with respect to $\mathcal{M}_{|_U}$.*

**Lemma 1.** *Let $Z$ be a complex manifold, $H$ a hermitian metric on $Z$, and $\mathcal{M}$ a linear set over $Z$. Then a function $\phi \in C^\infty(Z, \mathbb{R})$ is strictly 1-convex with respect to $\mathcal{M}$ if and only if for every compact set $K \subset Z$ there is $\delta > 0$ such that*

$$L_\phi(z, \xi) \geq \delta \|\xi\|^2$$

*for every $z \in K$, $\xi \in \mathcal{M}_z$. ($\|\cdot\|$ denotes the norm induced by $H$.)*

**Proposition 1.** *Let $X$ be a complex manifold and $\phi \in C^\infty(X, \mathbb{R})$ a strictly $q$-convex function. Then there is a linear set $\mathcal{M}$ over $X$ of codimension $\leq q-1$ such that $\phi$ is strictly 1-convex with respect to $\mathcal{M}$.*

Definition 7 and Lemmas 2 and 3, and Proposition 2 are due to M. Colţoiu and V. Vâjâitu [4],[5], and [12]. The proofs of Proposition 2 and Lemma 2 are based on the ideas developed in [6].

**Definition 6.** *Let $Y$ be a complex manifold and $\mathcal{M}$ a linear set over $Y$. We denote by $\mathcal{B}(Y, \mathcal{M})$ the set of all $\phi \in C^o(Y, \mathbb{R})$ such that every point of $Y$ admits an open neighborhood $D$ on which there are functions $f_1, \ldots, f_k \in C^\infty(D, \mathbb{R})$ which are strictly 1-convex with respect to $\mathcal{M}_{|_D}$ and*

$$\phi_{|_D} = \max(f_1, \ldots, f_k).$$

**Proposition 2.** *Let $\mathcal{M}$ be a linear set over a complex manifold $Y$ and $f \in \mathcal{B}(Y, \mathcal{M})$. Then for every $\eta \in C^o(Y, \mathbb{R})$, $\eta > 0$, there exists $\widetilde{\phi} \in C^\infty(Y, \mathbb{R})$ which is strictly 1-convex with respect to $\mathcal{M}$ and*

$$\phi \leq \widetilde{\phi} < \phi + \eta.$$

*In particular, if codim $\mathcal{M} \leq q-1$, then $\widetilde{\phi}$ is $q$-convex.*

**Lemma 2.** *Let $X$ be a complex manifold and $\{W_i\}_{i \in I}$ a locally finite open covering of $X$. Suppose $\mathcal{M}_i$ are linear sets over $W_i$, $i \in I$. Then there is a linear set $\mathcal{M}$ over $X$ with the following properties:*

*a) $codim_X \mathcal{M} \leq \sup_{i \in I} codim_{W_i} \mathcal{M}_i$.*

*b) If $\{G_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary family of open subsets of $X$ and $f_\alpha \in C^\infty(G_\alpha, \mathbf{R})$ are such that $f_\alpha|_{G_\alpha \cap W_i}$ are strictly 1-convex with respect to $\mathcal{M}_i$ over $G_\alpha \cap W_i$, then $f_\alpha$ are strictly 1-convex with respect to $\mathcal{M}$ over $G_\alpha$.*

**Lemma 3.** *Let $X$ be a complex manifold. Let $\{V_i\}_{i \in \mathbf{N}}$ and $\{W_i\}_{i \in \mathbf{N}}$ be two families of open subsets of $X$ such that:*
*1) $\{V_i\}_{i \in \mathbf{N}}$ is a locally finite open covering of $X$ with relatively compact connected sets,*
*2) $\emptyset \neq W_i \subset V_i$ and $W_i \cap V_j = \emptyset$ if $i \neq j$.*
*Then for every discrete subset $A \subset X$ there exists a diffeomorphism $\Phi : X \to X$ with $\Phi(A) \subset \cup_{i \in N} W_i$, and $\Phi$ is biholomorphic near $A$.*

The next theorem is Theorem 2.3 in [10]. See also [9].

**Theorem 3.** *Let $X$ and $T$ be complex manifolds and $\pi : X \to T$ be a holomorphic submersion which is proper and surjective.*

*Then for every $t_o \in T$ and every finitely many points $p_1, \dots, p_s \in X(t_o) := \pi^{-1}(t_o)$ there is an open neighborhood $U$ of $t_o$ and a smooth diffeomorphism $S : U \times X(t_o) \to X(U)$, where $X(U) := \pi^{-1}(U)$, with the following properties:*

1) *$S(t, X(t_o)) = X(t) := \pi^{-1}(t)$ for every $t \in U$.*
2) *The mappings from $U$ into $X(U)$ given by $t \mapsto S(t, x_o)$, $x_o \in X(t_o)$, are holomorphic sections of $\pi : X(U) \to U$ for every $x_o \in X(t_o)$ and $X(U)$ is the disjoint union of their images $\{S(U, x_o)\}_{x_o \in X(t_o)}$.*
3) *The map $r : X(U) \to X(t_o)$ given by $S(\pi(x), r(x)) = x, x \in X(U)$, is a $C^\infty$ retraction of $X(U)$ onto $X(t_o)$ such that there is an open neighborhood $V$ of $\{p_1, \dots, p_s\}$ with $r|_{r^{-1}(V)}$ is holomorphic.*

# 3 The Results

**Proposition 3.** *Let $X$ be a complex manifold, $Y$ a $C^\infty$ manifold and $\pi \in C^\infty(X, Y)$. Also let $\{X_n\}$ be a sequence of open subsets of $X$ and $F \subset X_1$ a closed subset of $X$ such that $\pi|_F$ is proper, $\cup X_n = X$ and for every $n \geq 1$, $\overline{X_n} \subset X_{n+1}$. We consider $\mathcal{M}$ a linear set over $X \setminus F$. We suppose that for every $n \in \mathbb{N}$ there exists $\phi_n \in C^\infty(X_n, \mathbb{R}_+)$ with the following properties:*
*1)$\phi_n|_{X_n \setminus F}$ is strictly 1-convex with respect to $\mathcal{M}|_{X_n \setminus F}$.*
*2) For every $\epsilon > 0$ $\pi|_{\{x \in X_n : \phi_n(x) \geq \epsilon\}}$ is proper.*
*Then there exists $\phi \in C^\infty(X, \mathbb{R}_+)$ a strictly 1-convex function with respect to $\mathcal{M}$ on $X \setminus F$ and such that $\pi|_{\{x \in X : \phi(x) \geq \epsilon\}}$ is proper for every $\epsilon > 0$.*

*Proof.* Let $\{U_n\}$ be a sequence of open subsets of $X$ such that $U_n \subset\subset U_{n+1}$, $U_n \subset\subset X_n$ and $\cup U_n = X$. Let also $\{Y_n\}$ be a sequence of compact subsets of $Y$ such that $\cup Y_n = Y$ and $Y_n \subset Int(Y_{n+1})$.
We will construct inductively a sequence of functions $\psi_n \in C^\infty(X_n, \mathbb{R}_+)$ with the following properties:
1)$\psi_n \in \mathcal{B}(\mathcal{M}, X_n \setminus F)$
2)$\psi_n = \psi_{n-1}$ on $\overline{U}_{n-1}$
3)$\pi|_{\{x \in X_n : \psi_n(x) \geq \epsilon\}}$ is proper for every $\epsilon > 0$.

4)$\psi_n \le \frac{1}{n}$ on $\pi^{-1}(Y_n) \cap (X_n \setminus X_{n-1})$
5)If $x \in \pi^{-1}(Y_n) \cap X_n$ and $\psi_n(x) > \frac{1}{n}$ then $\psi_n(x) = \psi_{n-1}(x)$

Multiplying by a constant we can suppose that $\phi_1 < 1$ on $\pi^{-1}(Y_1) \cap X_1$. Then we put $\psi_1 = \phi_1$.
Suppose now that we have defined $\psi_1, ..., \psi_{n-1}$ and we construct $\psi_n$.
Multiplying $\phi_n$ by a constant we can suppose that for every $x \in \overline{U}_{n-1}$, $\phi_n(x) < min\{\psi_{n-1}(y) : y \in \overline{U}_{n-1}\}$ and for every $x \in \pi^{-1}(Y_n)$, $\phi_n(x) < \frac{1}{n}$.
We define:

$$\psi_n(x) = \begin{cases} max\{\phi_n(x), \psi_{n-1}(x)\} & \text{on} \quad X_{n-1}, \\ \phi_n(x) & \text{on} \quad X_n \setminus X_{n-1} \end{cases} .$$

There exists $W$ a neighborhood of $\partial X_{n-1}$ such that for every $x \in W$, $\phi_n(x) > \psi_{n-1}(x)$. Indeed:
For every $x_0 \in \partial X_{n-1}$ we take $V$ an open, relatively compact neighborhood. Then $\{x \in X_{n-1} : \psi_{n-1}(x) \ge \phi_n(x_0)\} \cap \pi^{-1}(\pi(\overline{V}))$ is a compact subset of $X_{n-1}$ and it does not contain $x_0$. Let $V_1 \subset V$ an open neighborhood of $x_0$ such that $\overline{V}_1 \cap \{x \in X_{n-1} : \psi_{n-1}(x) \ge \phi_n(x_0)\} \cap \pi^{-1}(\pi(\overline{V})) = \emptyset$. Thus $\overline{V}_1 \cap \{x \in X_{n-1} : \psi_{n-1}(x) \ge \phi_n(x_0)\} = \emptyset$. Therefore on $\overline{V}_1$, $\phi_n(x_0) > \psi_{n-1}(x)$. It follows that $\psi_n \in \mathcal{B}(\mathcal{M}, X_n \setminus F)$. On the other hand $\{x \in X_n : \psi_n(x) \ge \epsilon\} \subset \{x \in X_n : \phi_n(x) \ge \epsilon\} \cup \{x \in X_{n-1} : \psi_{n-1}(x) \ge \epsilon\}$. Thus $\pi|_{\{x \in X_n : \psi_n(x) \ge \epsilon\}}$ is proper.

Therefore $\psi_n$ satisfies 1)−5).

We define now $\widetilde{\phi} = \lim \psi_n$.
$\widetilde{\phi} \in \mathcal{B}(\mathcal{M}, X \setminus F)$ because $\psi_n$ is stationary on compacts.
Let $K \subset Y$ be a compact subset and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $K \subset Y_n$ and $\epsilon > \frac{1}{n}$.
Then $\pi^{-1}(K) \cap \{x \in X : \widetilde{\phi}(x) \ge \epsilon\} \subset \pi^{-1}(Y_n) \cap \{x \in X : \widetilde{\phi}(x) \ge \frac{1}{n}\}$.
4) and 5) $\Longrightarrow \psi_k \le \frac{1}{n}$ on $\pi^{-1}(Y_n) \cap (X_k \setminus X_{n-1})$ for every $k \ge n$. And then $\pi^{-1}(Y_n) \cap \{x \in X : \widetilde{\phi}(x) \ge \frac{1}{n}\} = \pi^{-1}(Y_n) \cap \{x \in X_n : \widetilde{\phi}(x) \ge \frac{1}{n}\}$.
Using again 5) we obtain that
$\pi^{-1}(Y_n) \cap \{x \in X_n : \widetilde{\phi}(x) \ge \frac{1}{n}\} = \pi^{-1}(Y_n) \cap \{x \in X_n : \psi_n(x) \ge \frac{1}{n}\}$ and this set is compact. The conclusion follows now from Proposition 2.

□

If $Y$ is a point, one can improve the previous proposition as follows:

**Proposition 4.** *Let $X$ be a complex manifold, $\mathcal{M}$ a linear set over $X$ and $\{X_n\}$ a sequence of open sets such that $X_n \subset\subset X_{n+1}$ and $\cup X_n = X$. We* *suppose that for every $n \geq 1$ there exists a compact set $K_n \subset X_n$ and $\phi_n \in C^\infty(X_n, \mathbb{R}_+)$ such that: $K_n \subset X_{n-1}$, $\phi_n$ is strictly 1-convex with respect to $\mathcal{M}$ on $X_n \setminus K_n$ and for every $\epsilon > 0$, $\{\phi_n > \epsilon\} \subset\subset X_n$. Then there exists $\phi \in C^\infty(X, \mathbb{R}_+)$ such that $\phi$ is strictly 1- convex with respect to $\mathcal{M}$ on $X \setminus K_1$ and for every $\epsilon > 0$, $\{\phi > \epsilon\} \subset\subset X$.*

*Proof.* The only thing that we have to change in the proof of Proposition 3 is to choose $U_n$ such that $K_n \subset U_n$. □

**Lemma 4.** *Let $X$ and $Y$ be $C^\infty$ manifolds, $h, g : X \to (0, \infty)$, $\pi : X \to Y$ be $C^\infty$ functions such that $|g(x)| \leq 1$, and $p$ a positive integer. Then there exists a unique $C^\infty$ function $\psi : X \to (0, \infty)$ such that :*

$$\frac{h^2}{\psi} + \frac{g^p}{1+\psi} = 1$$

*Moreover if $h$ has the property that $\pi|_{\{x \in X : h(x) \geq \epsilon\}}$ is proper for every $\epsilon > 0$ then $\psi$ has the same property.*

*Proof.* The above equation is in fact a quadratic equation in $\psi$. This equation has a unique positive solution, namely:

$$\psi = \frac{h^2 + g^p - 1 + \sqrt{(h^2 + g^p - 1)^2 + 4h^2}}{2}$$

It follows then that $\psi$ is $C^\infty$.
If for some $x \in X$ $\psi(x) \geq \epsilon$, since $g(x) \leq 1$, we have $\frac{g(x)^p}{1+\psi(x)} \leq \frac{1}{1+\epsilon}$. It follows then that $\frac{h^2(x)}{\psi(x)} \geq 1 - \frac{1}{1+\epsilon} = \frac{\epsilon}{1+\epsilon}$. Thus $h(x) \geq \frac{\epsilon\psi(x)}{1+\epsilon} \geq \frac{\epsilon^2}{1+\epsilon}$.
Therefore $\{\psi \geq \epsilon\} \subset \{h \geq \frac{\epsilon}{\sqrt{1+\epsilon}}\}$. □

**Proposition 5.** *Let $X$ be a complex manifold, $Y$ a $C^\infty$-manifold, $\pi \in C^\infty(X, Y)$ and $H$ a hermitian metric on $X$. Suppose that there exist:*
*a) $\psi \in C^\infty(X, \mathbb{R}_+)$*
*b) $F$ and $F_1$ two closed subsets of $X$, $U$ a relatively compact open subset of $Y$, $V_1$ and $V_2$ open subsets of $X$ such that $F_1 \subset Int(F)$, $\pi|_F$ is proper and $V_1 \cup V_2 = X$*
*c) $\mathcal{M}_1$ a linear set over $V_1 \setminus F_1$, a $\mathcal{M}_2$ linear set over $V_2 \setminus F_1$*
*with the following properties:*
*1) for every real number $\epsilon > 0$, $\pi|_{\{\psi \geq \epsilon\}}$ is proper,*
*2) $\psi$ is strictly 1-convex with respect to $\mathcal{M}_1$ on $V_1 \setminus F_1$*

*3) if for every $x \in V_2 \setminus F_1$ we set $\mathcal{K}_x := \{\xi \in \mathcal{M}_{2,x} : \langle \partial\psi, \xi \rangle = 0\}$ then $\mathcal{K}_x \neq \mathcal{M}_{2,x}$ and $\psi$ is strictly 1-convex with respect to $\mathcal{K}$ on $V_2 \setminus F_1$*
*Let $\mathcal{M}$ be the linear set given by the Lemma 2 applied to $\mathcal{M}_1$ and $\mathcal{M}_2$.*

*Then there exists $\phi \in C^\infty(\pi^{-1}(U), \mathbb{R}_+)$ such that $\phi$ is strictly 1-convex with respect to $\mathcal{M}$ on $\pi^{-1}(U) \setminus F$ and $\{\phi \geq \epsilon\} \cap \pi^{-1}(K)$ is compact for every $\epsilon > 0$ and every compact $K \subset U$.*

*Proof.* Set $\mathcal{L}_x := \mathcal{M}_{2,x} \cap \{\xi \in T_xX : \langle \partial\psi, \xi \rangle = 0\}^\perp$
Let $V_3$ be an open subset of $X$ such that $V_1 \cup V_3 = X$ and $\overline{V}_3 \subset V_2$. We will use Proposition 3.
Since $U \subset\subset Y$, multiplying by a constant we can suppose that $|\psi(x)| \leq 1$ for $x \in \pi^{-1}(U)$. There exists also $p \in \mathbb{N}$ such that $\psi(x) > \frac{1}{p}$ for every $x \in F \cap \pi^{-1}(U)$.
Let $X_n = \pi^{-1}(U) \cap \{x \in X : \psi(x) > \frac{1}{n}\}$, $n \geq p$. Note that $X_n$ is a relatively compact subset of $X$.
Because $\overline{V}_3 \subset V_2$ , $F_1 \subset Int(F)$ there exist four constants $C_1, C_2$, $C_3$ and $C_4$ such that:

$$\left.\begin{array}{l} |\langle \partial\psi_x, \xi \rangle| \geq C_1 \|\xi''\|, \\ L_\psi(z, \xi') \geq C_2 \|\xi'\|^2, \\ Re(L_\psi(z, \xi', \xi'')) \geq -C_3 \|\xi'\| \|\xi''\| \\ L_\psi(z, \xi'') \geq -C_4 \|\xi''\|^2 \end{array}\right\} (1)$$

for every $x \in (V_3 \cap X_n) \setminus F$, $\xi' \in \mathcal{K}_x$, $\xi'' \in \mathcal{L}_x$. Here $\|\cdot\|$ is the norm induced by $H$.
Let $\phi_n = (\psi - \frac{1}{n})^k$ where k is a positive integer.Then:
$L_{\phi_n}(z, \xi) = k(\psi - \frac{1}{n})^{k-1} L_\psi(z, \xi) + k(k-1)(\psi - \frac{1}{n})^{k-2} |\langle \partial\psi, \xi \rangle|^2 \quad (2)$

Because $\psi$ is strictly 1-convex with respect to $\mathcal{M}_1$ on $V_1 \setminus F$, from (2) and Lemma 1 it follows that $\phi_n$ is strictly 1-convex with respect to $\mathcal{M}_1$ on $(V_1 \cap X_n) \setminus F$.
Let $x \in (V_3 \cap X_n) \setminus F$ and $\xi = \xi' + \xi'' \in \mathcal{M}_{2,x} = \mathcal{K}_x \oplus \mathcal{L}_x$.
From (1) and (2) we obtain:
$L_{\phi_n}(x, \xi) \geq k(\psi - \frac{1}{n})^{k-2} \{(\psi - \frac{1}{n}) C_2 \|\xi'\|^2 - 2(\psi - \frac{1}{n}) C_3 \|\xi'\| \cdot \|\xi''\| + ((k-1)C_1^2 - (\psi - \frac{1}{n}) C_4) \|\xi''\|^2\}$
If $k$ is large enough
$\frac{1}{2}(\psi - \frac{1}{n}) C_2 \|\xi'\|^2 - 2(\psi - \frac{1}{n}) C_3 \|\xi'\| \cdot \|\xi''\| + (\frac{k-1}{2} C_1^2 - (\psi - \frac{1}{n}) C_4) \|\xi''\|^2 \geq 0.$
Then we have $L_{\phi_n}(x, \xi) \geq \frac{1}{2} C_2 k (\psi - \frac{1}{n})^{k-1} \|\xi'\|^2 + \frac{k(k-1)}{2} C_1^2 (\psi - \frac{1}{n})^{k-2} \|\xi''\|^2$.

And since $\xi'$ and $\xi''$ are orthogonal:

$$L_{\phi_n}(x,\xi) \geq \frac{k}{2}(\psi - \frac{1}{n})^{k-2} \cdot min\{C_2(\psi - \frac{1}{n}), (k-1)C_1^2\}\|\xi\|^2.$$

Therefore Lemma 1 and Lemma 2 imply that $\phi_n$ is strictly 1-convex with respect to $\mathcal{M}$ on $X_n \setminus F$.

In the same time $\{x \in X_n : \phi_n(x) \geq \epsilon\} = \{x \in X_n : \psi(x) \geq \frac{1}{n} + \sqrt[k]{\epsilon}\} = \{x \in \pi^{-1}(U) : \psi(x) \geq \frac{1}{n} + \sqrt[k]{\epsilon}\}$ so for every compact $K \subset U$ we have $\{x \in X_n : \phi_n(x) \geq \epsilon\} \cap \pi^{-1}(K) = \{x \in X : \psi(x) \geq \frac{1}{n} + \sqrt[k]{\epsilon}\} \cap \pi^{-1}(K)$ which is compact.

□

We will now begin to prove Theorem 2. We will proceed as in [5] and we will consider a covering $\{V_1, ..., V_s\}$ of $X(t_0) = \pi^{-1}(t_0)$ by local charts, each $V_i$ biholomorphic to an open ball in $\mathbb{C}^n$, and a set of points $\{p_1, ..., p_s\}$ such that $p_i \in V_i$ and $p_i \notin \overline{V}_j$ for $i \neq j$. Let $W_i \subset V_i$ be open neighborhoods of $p_i$ biholomorphic to open balls in $\mathbb{C}^n$ such that $W_i \cap V_j = \emptyset$ and the retraction $r$ in theorem 3 is holomorphic on $r^{-1}(\cup_{i \leq s} W_i)$.

**Lemma 5.** *There exists a Morse function $h_0 : \widetilde{X}(t_0) \to \mathbb{R}_+$ and $K \subset \widetilde{X}(t_0)$ a compact subset such that*
*a) $\{h_0 \geq \epsilon\}$ is compact for every $\epsilon > 0$*
*b) $A := \{x : x$ is a critical point for $h_0\} \setminus K$ is a subset of $\sigma^{-1}(\cup_{i \leq s} W_i)$*
*c) $h_0$ is strictly $n$-convex on a neighborhood of $A$.*

*Proof.* Since $\widetilde{X}(t_0)$ has at most finitely many compact components there is, by Theorem 1, a compact subset $K_1 \subset \widetilde{X}(t_0)$ and a $C^\infty$ function $h_1 : \widetilde{X}(t_0) \to \mathbb{R}_+$ such that $\{h_1 \geq \epsilon\}$ is compact for every $\epsilon > 0$ and $h_1$ is strictly $n$-convex on $\widetilde{X}(t_0) \setminus K_1$ . We may also suppose that $h_1$ is a Morse function. See in this sense [2]. Let $A_1$ be the set of its critical points that are not in $K_1$ (which is a discrete set).

We put $\sigma^{-1}(V_i) = \cup_{j \in \mathbb{N}} M_{i,j}$ and $\sigma^{-1}(W_i) = \cup_{j \in \mathbb{N}} N_{i,j}$ for their decompositions into connected components. Then $\{M_{i,j}\}$ and $\{N_{i,j}\}$ satisfy the conditions of Lemma 3. Let $\Phi : \widetilde{X}(t_0) \to \widetilde{X}(t_0)$ a diffeomorphism such that $\Phi(A_1) \subset \sigma^{-1}(\cup_{i \leq s} W_i)$ and $\Phi$ is holomorphic on a neighborhood of $A$. Then $h_0 = h_1 \circ \Phi$ has the required properties. □

We choose now a simply connected neighborhood of $t_0$ and we lift the map $S$, given by Theorem 3, to $\widetilde{X}$. We observe then that, in order to complete the proof of Theorem 2, it suffices to prove the following:

**Proposition 6.** *Let $X$ be a complex manifold and $\pi : X \to T$ a holomorphic submersion, where $X$ has dimension $n+m$, and $T = \{t \in \mathbb{C}^m\,;\ |t| < 1\}$. Set $X_t := \pi^{-1}(t), t \in T$. Assume that there exists a diffeomorphism $S : T \times X_0 \to X$ with the following properties:*

1) *$S(t, X_0) = X_t$ for every $t \in T$.*

2) *The map $s_{x_o} : T \to X$ given by $s_{x_0}(t) = S(t, x_0)$ is a holomorphic section of $\pi$ for every $x_0 \in X_0$ and $X$ is the disjoint union of $\{s_{x_0}(T)\}_{x_0 \in X_0}$.*

3) *The map $r : X \to X_0$ given by $S(\pi(x), r(x)) = x, x \in X$, defines a $C^\infty$ retraction of $X$ onto $X_0$. Moreover there is a Morse function $h_0 \in C^\infty(X_0, \mathbb{R}_+)$, $K \subset X_0$ a compact set, $V_0 \subset X_0$ an open set, $V_0 \supset A := \{x \in X_0 \setminus K : x$ is a critical point for $h_0\}$, such that $h_0|_{V_o}$ is $n$-convex, $\{h_0 \geq \epsilon\}$ is compact for every $\epsilon > 0$ and $r|_{r^{-1}(V_0)}$ is holomorphic.*

*Then for every $U$ an open neighborhood of $0$, $U \subset\subset T$, $\pi|_{\pi^{-1}(U)} : U \to U$ is a $n$-concave morphism.*

*Proof.* Let $g : X \to (0, \infty)$, $g(x) = \frac{|\pi(x)|^2+1}{2}$.
For $x \in X$ let $\Sigma_x = \{S(t, r(x))\,;\ t \in T\}$ and $\Phi_x = \pi^{-1}(\pi(x))$. $\Sigma_x$ and $\Phi_x$ are closed submanifolds of $X$.

Following [5] we will use:

**Definition 7.** *A hermitian metric $H$ on $X$ is called "special" if for any point $x \in X$ the complex vector subspaces $T_x(\Sigma_x)$ and $T_x(\Phi_x)$ of $T_xX$ are orthogonal with respect to $H$.*

**Lemma 6.** *There exists a special hermitian metric $H$ on $X$.*

Let $h = h_0 \circ r$ and $F_2 = h^{-1}(K)$. Choose $V_0'$ an open subset of $X_0$ such that $V_0' \supset A$ and $\overline{V_0'} \subset V_0$ and put $V_1 = r^{-1}(V_0)$ and $V_2 = X \setminus r^{-1}(\overline{V_0'})$.
Using Proposition 1 we choose $\mathcal{N}$ a linear set of codimension $\leq n-1$ such that $h_0$ is strictly 1-convex with respect to $\mathcal{N}$ over $V_0$ and put $\mathcal{M}_1 = r^*(\mathcal{N})$. Since $h$ does not have critical points in $V_2 \setminus F_2$, at any point $x \in V_2 \setminus F_2$ we have an orthogonal decomposition with respect to $H$: $T_xX = \Gamma_x' \oplus \Gamma_x''$ where $\Gamma_x'$ is the holomorphic tangent space at $x$ to the real hypersurface $\{h = h(x)\}$ and $\Gamma_x''$ is its orthogonal complement. Thus $\Gamma_x''$ is a 1-dimensional complex vector space and $T_x(\Sigma_x) \subset \Gamma_x'$, so $\Gamma_x''$ and $T_x(\Sigma_x)$ are orthogonal (with respect

to $H$). Therefore $\Gamma''_x \subset T_x(\Phi_x)$.
We set $\mathcal{M}_2 = \mathcal{M}'_2 \oplus \mathcal{M}''_2$ where $\mathcal{M}'_2$ and $\mathcal{M}''_2$ are linear sets over $V_2 \setminus F_2$ given by $\mathcal{M}'_{2,x} = T_x(\Sigma_x)$ and $\mathcal{M}''_{2,x} = \Gamma''_x$.
Let $\mathcal{M}$ be the linear set given by Lemma 2 applied to $\mathcal{M}_1$ and $\mathcal{M}_2$.
Since $K$ is compact there exists $p \in \mathbb{N}$ such that for every $x \in K$, $h_0(x) > \frac{1}{p}$.

Let $X_n = \{x \in X : h(x) > \frac{1}{n}\}$, $n \geq p$. Then $F_2 \subset X_n$. Let $F$ and $F_1$ be two closed $X$ such that $F \subset X_p$, $F_1 \subset Int(F)$, $F_2 \subset Int(F_1)$ and $\pi|_F$ is proper.
We will find for each $n \geq p$ a function $\phi_n \in C^\infty(X_n, \mathbb{R}_+)$ such that $\phi_n|_{X_n \setminus F}$ is strictly 1-convex with respect to $\mathcal{M}|_{X_n \setminus F}$ and $\{x \in X_n \cap \pi^{-1}(U) : \phi_n(x) \geq \epsilon\} \cap \pi^{-1}(L)$ is compact for every real number $\epsilon > 0$ and every compact $L \subset U$. The conclusion of the proposition will follow then from Proposition 3.

To obtain $\phi_n$ we will use Proposition 5.
Let $Y$ be an open subset of $T$ such that $U \subset\subset Y \subset\subset T$.
$(\overline{\pi^{-1}(Y)} \cap \overline{V}_2 \cap \overline{X}_n) \setminus Int(F_1)$ is a compact subset of $X$ and for every $x \in (\overline{\pi^{-1}(Y)} \cap \overline{V}_2 \cap \overline{X}_n) \setminus Int(F_1)$ and $\xi'' \in \Gamma''_x$, $\langle \partial\ h_x, \xi'' \rangle \neq 0$.
Since $\Gamma''_x$ depends continuously on $x$ it follows that there exists $C > 0$ such that $|\langle \partial\ h_x, \xi'' \rangle| \geq C \|\xi''\|$ for every $x \in (\overline{\pi^{-1}(Y)} \cap \overline{V}_2 \cap \overline{X}_n) \setminus Int(F_1)$ and every $\xi'' \in \Gamma''_x$

Let $h_n = e^{\alpha_n(h - \frac{1}{n})} - 1$ where $\alpha_n$ is a positive real number.
$L_{h_n}(x, \xi) = \alpha_n e^{\alpha_n(h-\frac{1}{n})}(L_h(x,\xi) + \alpha_n |\langle \partial h, \xi \rangle|^2)$. Choose $\alpha_n$ large enough such that on $(\pi^{-1}(Y) \cap V_2 \cap X_n) \setminus Int(F_1)$, $L_{h_n}(\xi'') \geq 0$ for every $\xi'' \in \mathcal{M}''_2$.
Note that because $\langle \partial h, \xi'' \rangle \neq 0$ we have also $\langle \partial h_n, \xi'' \rangle \neq 0$
If $\xi = \xi' + \xi'' \in \mathcal{M}'_2 \oplus \mathcal{M}''_2$ since $h$ is constant on $\Sigma_x$ we get $\langle \partial h, \xi' \rangle = 0$ and $L_h(\xi') = 0$. It follows then that $\langle \partial h_n, \xi' \rangle = 0$ and $L_{h_n}(\xi') = 0$.

A direct computation shows that $L_g(\xi) = L_g(\xi') \geq C_1 \|\xi'\|^2$ for some $C_1 > 0$ (see also Lemma 8 in [5]).
Let $C_2, C_3$ be positive constants such that on $(\pi^{-1}(Y) \cap V_2 \cap X_n) \setminus Int(F_1)$ we have:
$|\langle \partial h_n, \xi'' \rangle| \geq C_2 \|\xi''\|$ and $2Re(L_{h_n}(\xi', \xi'')) \geq -C_3 \|\xi'\| \|\xi''\|$
and choose a positive integer $p$ such that $p \geq 3$ and

$$\frac{p-1}{4} \frac{C_1 C_2^2}{g(x)} \geq C_3^2 \quad (3)$$

for every $x \in (\pi^{-1}(Y) \cap V_2 \cap X_n) \setminus Int(F_1)$. (Notice that $\frac{1}{2} \leq g(x) \leq 1$.)

Let $\psi_n \in C^\infty(X_n, \mathbb{R}_+)$ such that

$$\frac{h_n^2}{\psi_n} + \frac{g^p}{1+\psi_n} = 1 \quad (4)$$

There exists such $\psi_n$ by Lemma 4 and $\pi|_{\{x\in X_n : \psi_n(x)\geq \epsilon\}}$ is proper (the level sets for $h_n$ are level sets for $h$).
For $x \in (\pi^{-1}(Y) \cap V_2 \cap X_n) \setminus F$ let $\mathcal{K}_x := \{\xi \in \mathcal{M}_{2,x} : \langle \partial\psi_n, \xi\rangle = 0\}$.
Differentiating (4) once we obtain:

$$(\frac{h_n^2}{\psi_n^2} + \frac{g^p}{(1+\psi_n)^2})\langle \partial\psi_n, \xi\rangle = \frac{2h_n}{\psi_n}\langle \partial h_n, \xi''\rangle + \frac{pg^{p-1}}{1+\psi_n}\langle \partial g, \xi'\rangle$$

Since $\langle \partial h_n, \xi''\rangle \neq 0$ and $\langle \partial g, \xi'\rangle \neq 0$ for $\xi' \neq 0$ and $\xi'' \neq 0$ it follows that $\mathcal{K}_x \neq \mathcal{M}_{2,x}$.
Also for $\xi \in \mathcal{K}_x$ we obtain: $\langle \partial g, \xi'\rangle = -\frac{1+\psi_n}{pg^{p-1}}\frac{2h_n}{\psi_n}\langle \partial h_n, \xi''\rangle$ (5)

Differentiating (4) twice we obtain:

$$\rho L_{\psi_n}(\xi) = \frac{2h_n}{\psi_n} L_{h_n}(\xi) + \frac{pg^{p-1}}{1+\psi_n} L_g(\xi) + A(\xi) + B(\xi) + \frac{p(p-1)g^{p-2}}{4(1+\psi_n)}|\langle \partial g, \xi\rangle|^2$$

where $\rho$, $A(\xi)$ and $B(\xi)$ are given by:

$$\rho = (\frac{h_n^2}{\psi_n^2} + \frac{g^p}{(1+\psi_n)^2})$$

$$A(\xi) = \frac{2}{\psi_n}\{|\langle \partial h_n, \xi\rangle|^2 - \frac{2h_n}{\psi_n} Re(\langle \partial h_n, \xi\rangle \overline{\langle \partial \psi_n, \xi\rangle}) + \frac{h_n^2}{\psi_n^2}|\langle \partial\psi_n, \xi\rangle|^2\}$$

$$B(\xi) = \frac{g^{p-2}}{1+\psi_n}\{\frac{3}{4}p(p-1)|\langle \partial g, \xi\rangle|^2 - \frac{2pg}{(1+\psi_n)} Re(\langle \partial g, \xi\rangle \overline{\langle \partial\psi_n, \xi\rangle}) + \frac{2g^2}{(1+\psi_n)^2}|\langle \partial\psi_n, \xi\rangle|^2\}$$

Notice that $A(\xi) \geq 0$ and $B(\xi) \geq 0$.
For $\xi \in \mathcal{K}_x$ using the previous inequalities we obtain that:

$$\rho L_{\psi_n}(\xi) \geq -\frac{2h_n}{\psi_n} C_3\|\xi'\|\|\xi''\| + \frac{pg^{p-1}}{1+\psi_n} C_1\|\xi'\|^2 + \frac{p(p-1)g^{p-2}}{4(1+\psi_n)}|\langle \partial g, \xi'\rangle|^2$$

And (5) implies that

$$\rho L_{\psi_n}(\xi) \geq -\frac{2h_n}{\psi_n} C_3 \|\xi'\| \|\xi''\| + \frac{pg^{p-1}}{1+\psi_n} C_1 \|\xi'\|^2 + \frac{(p-1)(1+\psi_n)}{pg^p} \frac{h_n^2}{\psi_n^2} C_2^2 \|\xi''\|^2$$

But (3) implies that:

$$\frac{pg^{p-1}}{2(1+\psi_n)} C_1 \|\xi'\|^2 - \frac{2h_n}{\psi_n} C_3 \|\xi'\| \|\xi''\| + \frac{(p-1)(1+\psi_n)}{2pg^p} \frac{h_n^2}{\psi_n^2} C_2^2 \|\xi''\|^2 \geq 0$$

and therefore

$$\rho L_{\psi_n}(\xi) \geq \frac{pg^{p-1}}{2(1+\psi_n)} C_1 \|\xi'\|^2 + \frac{(p-1)(1+\psi_n)}{2pg^p} \frac{h_n^2}{\psi_n^2} C_2^2 \|\xi''\|^2$$

Since $\xi'$ and $\xi''$ are orthogonal this last inequality implies that $\psi_n$ is strictly 1-convex with respect to $\mathcal{K}$.

Because $r$ is holomorphic on $V_1$ it follows that $L_h(\xi) = L_h(\xi'')$ and we deduce that $\psi_n$ is strictly 1-convex with respect to $\mathcal{M}_1$ on $V_1$. Thus all the conditions of Proposition 5 are fulfilled. □

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