Minors in Weighted Graphs *

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Abstract

We define a notion of minor for weighted graphs. We prove that with this minor relation, the set of weighted graphs is directed. We also prove that given any two weights on a connected graph with the same total weight, we can transform one into the other using a sequence of edge subdivisions and edge contractions.

1 Introduction

The notion of minor is a central one in Graph Theory. Of course, the study of minor-closed classes of graphs culminated with the proof by Robertson and Seymour [2] of Wagner's conjecture. For an excellent survey, see L. Lovász's paper [1].

In this paper we define the notion of minor for weighted graphs. Since given any connected network it is not desirable to disconnect it, we will work only with connected graphs. And since when defining a weight, one has to choose a unit, we can restrict ourselves to graphs of total weight 1. This is the same as identifying two weights if one of them is a multiple of the other. The two operations used to define a minor, for weighted graphs, are the two standard ones: edge contraction and edge deletion. One has to define what happens to the flow through a deleted or contracted edge. The definition that we adopt here is, we think, the most natural one. Namely, the flow is distributed proportionally to the adjacent edges. It can be seen easily that with this definition, Wagner's conjecture does not hold.

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Our first theorem states that with this minor relation, the set of weighted graphs is directed. This is not obvious since a subgraph is not anymore a minor. In fact, we prove that for any two weighted graphs, we can find another one which has them both as minors and subgraphs.

In the second part of the paper we show that given any two weights on a connected graph with the same total weight we can transform one into the other using a sequence of edge subdivisions and edge contractions. However, in general, we cannot perform all the edge subdivisions at the beginning and then the edge contractions. All proofs are constructive.

2 Results

In what follows we will work with simple (without loops or multiple edges), undirected and connected graphs. By a weighted graph, we understand a pair (G, c) where G is a graph, whose set of vertices and edges will be denoted by $V(G)$ and $E(G)$ respectively, and $c : E(G) \to (0, \infty)$ is the weight function. If H is a subgraph of G, we denote by $c(H)$ the total weight of H, i.e. $c(H) = \sum_{e \in E(H)} c(e)$. If A is a vertex of G, we denote by $c(A)$ the sum of the weights of all edges at A.

Definition 1. *Given two weighted graphs (G_1, c_1) and (G_2, c_2) we say that they are equivalent and we write $(G_1, c_1) \sim (G_2, c_2)$, if $(G_1, \frac{1}{c_1(G_1)}c_1)$ and $(G_2, \frac{1}{c_2(G_2)}c_2)$ are isomorphic. We denote by $[(G, c)]$ the equivalence class of (G, c).*

Definition 2.
For a weighted graph (G, c) we define the following two operations:
(1) Deleting an edge. This operation is allowed only if the resulting graph is connected, i.e. the edge is not a bridge. The weight of the deleted edge is redistributed proportionally to the adjacent edges. This means: if the deleted edge is e and its adjacent edges are $e_1, e_2, , ..., e_k$ then their new weights will be $c_e(e_j) = c(e_j) + \frac{c(e)c(e_j)}{c(e_1)+c(e_2)+...+c(e_k)}$. The weight of an edge not adjacent with e remains unchanged. The resulted graph is denoted by $(G - e, c_e)$.
(2) Contracting an edge e. The new weights are defined as follows:

a) The weight of an edge not adjacent with the one that is contracted remains unchanged.

b) The weight of the contracted edge is redistributed proportionally to the adjacent edges.

c) If the contraction gives rise to multiple edges, they are identified and their weights are added together.
If e *is the edge that is contracted the resulting graph is denoted by* $(G/e, c/e)$.

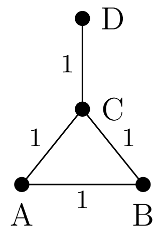
A weighted graph (G_1, c_1) *is called a minor of* (G, c) *if* (G_1, c_1) *can be obtained from* (G, c) *after a sequence of deleting or contracting edges.*

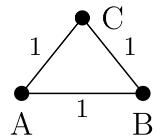
Remarks:
1. If (G, c) is a weighted graph and e is an edge which is not a bridge then $c_e(G - e) = c(G)$.
2. If (G, c) is a weighted graph and e is an edge then $c/e(G/e) = c(G)$.
3. If G is a graph, e is an edge and c and $\tilde{c}$ are two weight functions such that $(G, c) \sim (G, \tilde{c})$ then $(G - e, c_e) \sim (G - e, \tilde{c}_e)$ and $(G/e, c/e) \sim (G/e, \tilde{c}/e)$.

This last property allows us to define the notion of minor for $\sim$-equivalence classes. The subgraph relation for equivalence classes is obviously well-defined.

Definition 3. *a) Given two weighted graphs* (G_1, c_1) *and* (G_2, c_2)*, we say that* $[(G_1, c_1)]$ *is a minor of* $[(G_2, c_2)]$ *if* (G_1, c_1) *is equivalent to a minor of* (G_2, c_2).
b) Given two weighted graphs (G_1, c_1) *and* (G_2, c_2) *we say that* $[(G_1, c_1)]$ *is a subgraph of* $[(G_2, c_2)]$ *if* (G_1, c_1) *is equivalent to a subgraph of* (G_2, c_2).

Other remarks:
5. The minor relation is an order relation on the set of equivalence classes of weighted graphs.
6. The minor relation is *not* a well-quasi-ordering on the set of equivalence classes of weighted graphs: e.g. let $G = K_3$, denote by A, B, C the vertices of G and for $k \in \mathbb{N}^*$ define $c_k : E(G) \to (0, \infty)$ by $c_k(\{A, B\}) = c_k(\{A, C\}) = 1$, $c_k(\{B, C\}) = k$. Then $\{[(G, c_k)]\}$ is an infinite antichain.
7. The equivalence class of a weighted subgraph is not necessarily a minor of the equivalence class of the weighted graph. E.g. if (G, c) and (G_1, c_1) are the graphs below then $[(G_1, c_1)]$ is a subgraph of $[(G, c)]$ but not a minor.





8. If (G, c) is a weighted graph, e is an edge which is not a bridge and e_1 and e_2 are edges adjacent with e then $\dfrac{c_e(e_1)}{c_e(e_2)} = \dfrac{c(e_1)}{c(e_2)}$.
9. If (G, c) is a weighted graph, e is an edge which will not produce multiple edges by contraction and e_1 and e_2 are edges adjacent with e then $\dfrac{c/e(e_1)}{c/e(e_2)} = \dfrac{c(e_1)}{c(e_2)}$.

Proposition 1. *Suppose that (G, c) is a weighted graph. Let $\hat{G}$ be the cone on G, i.e. $V(\hat{G}) = V(G) \cup \{X\}$ where $X \notin V(G)$ and $E(\hat{G}) = E(G) \cup \bigcup_{A \in V(G)} \{\{A, X\}\}$. Then there exists $\hat{c} : E(\hat{G}) \to (0, \infty)$ a weight function for $\hat{G}$ such that*
a) $[(G, c)]$ is a subgraph of $[(\hat{G}, \hat{c})]$,
b) $[(G, c)]$ is a minor of $[(\hat{G}, \hat{c})]$,
c) $\hat{c}(\hat{G}) = 9\hat{c}(G)$.

Proof. Assume that the vertices of G are $A_1, A_2, ..., A_n$. We define the graphs $G_1, G_2, ..., G_n$ as follows. For $1 \leq j \leq n$: $V(G_j) = V(\hat{G}) = \{X, A_1, ..., A_n\}$ and $E(G_j) = E(G) \cup \{\{X, A_1\}, ..., \{X, A_j\}\}$. Hence $G_n = \hat{G}$. For $j = 1, 2 \ldots, n$ we will define inductively the weight c_j on G_j such that G is a minor of G_1 and G_j is a minor of G_{j+1} for $j = 1, 2..., n-1$. We define first c_1:
- we set $c_1(\{X, A_1\}) = \frac{2}{3} c(A_1)$
- if e is an edge of G such that A_1 is not an end of e, we set $c_1(e) = c(e)$
- if A_1 is an end of e, we set $c_1(e) = \frac{1}{3} c(e)$.

Contracting the edge $\{X, A_1\}$ of G_1 one obtains the graph G. The weight of an edge e which is not an edge at A_1 remains unchanged i.e. its weight is

$c_1(e) = c(e)$ and the weight of an edge e at A_1 will be

$$c_1(e)+\frac{c_1(e)c_1(\{X,A_1\})}{c_1(A_1)-c_1(\{X,A_1\})} = \frac{1}{3}c(e)+\frac{\frac{1}{3}c(e)\frac{2}{3}c(A_1)}{c(A_1)-\frac{2}{3}c(A_1)} = \frac{1}{3}c(e)+\frac{2}{3}c(e) = c(e)$$

This means that (G, c) is a minor of (G_1, c_1).

We assume now that we have defined the weights $c_1, ..., c_j$ and we will define c_{j+1}:
- $c_{j+1}(\{X, A_{j+1}\}) = \frac{2}{3}(c_j(A_{j+1} + c_j(X))$
- if the edge e is not adjacent with $\{X, A_{j+1}\}$ then $c_{j+1}(e) = c_j(e)$
- if e is adjacent with $\{X, A_{j+1}\}$ then $c_{j+1}(e) = \frac{1}{3}c_j(e)$.

A computation similar to the one above shows that deleting $\{X, A_{j+1}\}$ one obtains (G_j, c_j) i.e (G_j, c_j) is a minor of (G_{j+1}, c_{j+1}).

We set $\hat{c} = c_n$. It follows that $[(G, c)]$ is a minor of $[(\hat{G}, \hat{c})]$.

It remains to note that when we define c_j, the weight of an edge of G which is not an edge at A_j remains constant and the weight of an edge at A_j is multiplied by $\frac{1}{3}$. Since an edge has two vertices, its weight in G_n will be $\frac{1}{9}$ of its initial weight. In other words, for each $e \in E(G)$, $\hat{c}(e) = \frac{1}{9}c(e)$. This means that $[(G, c)]$ is a sugraph of $[(\hat{G}, \hat{c})]$. At the same time $\hat{c}(\hat{G}) = c(G)$, hence $\hat{c}(\hat{G}) = 9\hat{c}(G)$. □

Theorem 1. *For any two connected weighted graphs (G_1, c_1) and (G_2, c_2) there exists a connected weighted graph (G, c) such that $[(G_1, c_1)]$ and $[(G_2, c_2)]$ are both minors and subgraphs of $[(G, c)]$.*

Proof. Multiplying each weight by a constant (which will not change their equivalence class), we can assume that $c_1(G_1) = c_2(G_2) = 9$. Let's assume that $V(G_1) = \{A_1, ..., A_s\}$ and $V(G_2) = \{B_1, ..., B_p\}$. We consider the cones on G_1 and G_2, $V(\hat{G}_1) = \{X_1, A_1, ..., A_s\}$ and $V(\hat{G}_2) = \{X_2, B_1, ..., B_p\}$ respectively. We define the weights $\hat{c}_1$ and $\hat{c}_2$ on $\hat{G}_1$ and $\hat{G}_2$ as in the previous proposition. Hence $\hat{c}_1(G_1) = \hat{c}_2(G_2) = 1$, $\hat{c}_1(\hat{G}_1) = \hat{c}_2(\hat{G}_2) = 9$. Consider the graph G given by:
- $V(G) = V(\hat{G}_1) \cup V(\hat{G}_2)$
- $E(G) = E(\hat{G}_1) \cup E(\hat{G}_2) \cup \{\{X_1, X_2\}\}$

We define a weight c on G as follows:
- if $e_1 \in E(G_1)$, we set $c(e_1) := \hat{c}_1(e_1)$ and if $e_2 \in E(G_2)$, $c(e_2) := \hat{c}_2(e_2)$
- $c(\{X_1, A_i\}) := \frac{1}{8}\hat{c}_1(\{X_1, A_i\})$ and $c(\{X_2, B_j\}) := \frac{1}{8}\hat{c}_2(\{X_2, B_j\})$
- $c(\{X_1, X_2\}) := 5$.

Note that $c(\hat{G}_1) = c(\hat{G}_2) = 2$ and hence $c(G) = 9$. It is clear that $[(G_1, c_1)]$ and $[(G_2, c_2)]$ are subgraphs of $[(G, c)]$. It remains to be checked that they are also minors. We will show it only for G_1, the proof for G_2 being obviously the same.

We contract one by one all edges of $\hat{G}_2$. This will have no effect on the weight of $\hat{G}_1$. As the total weight remains unchanged, at the end of the process, the weight of $\{X_1, X_2\}$ will be $5 + c(\hat{G}_2) = 7$. Then, if we contract the edge $\{X_1, X_2\}$ we obtain $\hat{G}_1$. The weight of an edge of G_1 does not change. Since the sum of the weights of all edges adjacent with $\{X_1, X_2\}$ is $\frac{1}{8}\hat{c}_1(X_1) = 1$, the weight of an edge $\{X_1, A_j\}$ will be

$$c(\{X_1, A_j\}) + \frac{c(\{X_1, A_j\})c(\{X_1, X_2\})}{1} = \frac{1}{8}c_1(\{X_1, A_j\}) + \frac{\frac{1}{8}c_1(\{X_1, A_j\}) \cdot 7}{1}$$

$$= c_1(\{X_1, A_j\})$$

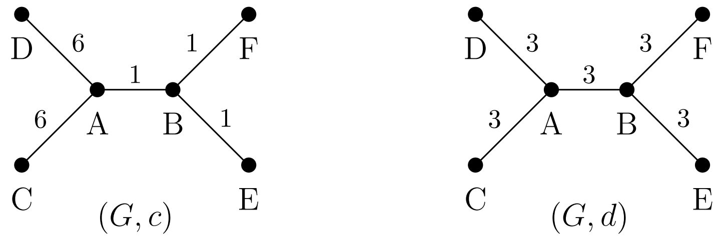
Therefore, contracting $\{X_1, X_2\}$ we obtain exactly $(\hat{G}_1, \hat{c}_1)$. In other words, $(\hat{G}_1, \hat{c}_1)$ is a minor of (G, c). As (G_1, c_1) is a minor of $(\hat{G}_1, \hat{c}_1)$, it follows that (G_1, c_1) is a minor of (G, c). □

Next we will introduce another operation for weighted graphs. As before, this operation is a familiar one for (non-weighted) graphs.

(3) Subdividing an edge by a new node. This means that given a weighted graph (G, c), an edge $\{A, B\}$ of G and a number $k \in (0, 1)$, we define the weighted graph $(G_{A,B,k}, c_{A,B,k})$ by: $V(G_{A,B,k}) = V(G) \cup \{X\}$ where $X \notin V(G)$, $E(G_{A,B,k}) = (E(G) \setminus \{\{A, B\}\}) \cup \{\{A, X\}, \{X, B\}\}$, $c_{A,B,k}(e) = c(e)$ if $e \in E(G) \setminus \{\{A, B\}\}$ and $c_{A,B,k}(\{A, X\}) = kc(\{A, B\})$, $c_{A,B,k}(\{B, X\} = (1 - k)c(\{A, B\})$.

A natural question is then the following: Given a graph G and c and d, two weights on G, is it possible to perform a sequence of operations of type *(3)* on (G, c) such that, denoting by $(\tilde{G}, \tilde{c})$ the resulting graph, $[(G, d)]$ is a minor of $[(\tilde{G}, \tilde{c})]$? And, if the answer is yes, what is the minimum number of edge-subdivisions that is needed?

It turns out that the answer is no, as the following example shows:



(G, c) (G, d)

Indeed, given a vertex A of a graph G let us call an *A-straight path to a terminal vertex* a path of the form $AA_1 \dots A_n$ where A_n is a terminal vertex and each A_j, for $1 \leq j \leq n-1$ has degree 2. If (G, c) is a weighted graph by $sdeg_c(A)$ we denote the sum of total weights of all A-straight paths to a terminal vertex and all edges at A that are not part of a A-straight path to a terminal vertex. Note that if e is a edge of G then $sdeg_c(A) \leq sdeg_{c/e}(A)$. (If e is of the form $\{A, B\}$, the vertex obtained by the identification of A and B is still denoted by A).

For the above example, suppose that $(\tilde{G}, \tilde{c})$ is a graph obtained from (G, c) using a finite number of edge-subdivisions. It is clear that $sdeg_{\tilde{c}}(A) \geq 12$.

If H is any tree and f is an edge of H then H/f is also a tree. Since $\tilde{G}$ is a tree, when defining minors of $(\tilde{G}, \tilde{c})$, deleting an edge is not an allowed operation (according to our definition). Suppose that (G, c_1) is a minor of $(\tilde{G}, \tilde{c})$. It follows that (G, c_1) is obtained from $(\tilde{G}, \tilde{c})$ using a sequence of edge-contractions. Note that the only two vertices of $\tilde{G}$ of degree ≥ 3 are A and B. This implies that none of these contractions will identify A and B. It follows that $sdeg_{c_1}(A) \geq 12 > 9 = sdeg_d(A)$. In other words, (G, d) cannot be a minor of $(\tilde{G}, \tilde{c})$.

On the positive side, we can show that there exists a sequence of edge-subdivisions and edge-contractions that will transform (G, c) in (G, d) provided that $c(G) = d(G)$. Moreover, this can be done using at most $|E(G)| - 1$ edge-subdivisions.

Notation: For a graph G, we denote by $rd(G)$ the number $2|E(G)| - |V(G)| = \sum_{v \in V(G)} (deg(v) - 1)$.

By a *terminal vertex* we understand a vertex of degree 1.

Definition 4. *Suppose that G is a graph, $A \in V(G)$ and $\{A, B_1\}, \dots, \{A, B_k\}$ are the edges at A. We call the blow-up of G at A, the graph $\tilde{G}_A$ defined as*

follows:
- $V(\tilde{G}_A) = (V(G) \setminus \{A\}) \cup \{A_1, ..., A_k\}$ *where* $A_1, A_2, ..., A_k \notin V(G)$
- $E(\tilde{G}_A) = (E(G) \setminus \{\{A, B_1\}, ..., \{A, B_k\}\}) \cup \{\{A_1, B_1\}, ..., \{A_k, B_k\}\}$.

Remarks:
If A is a vertex of G with $deg(A) = k$, then $rd(\tilde{G}_A) = rd(G) - (k - 1)$.
2) A weight on G induces a weight on $\tilde{G}_A$ and viceversa (simply put $c(\{A_j, B_j\}) = c(\{A, B_j\})$).

Lemma 1. *Suppose that* G *is a graph such that* $V(G) = \{X, A_1, ..., A_n\}$ *and* $E(G) = \{\{X, A_1\}, ..., \{X, A_n\}\}$ *and* c *and* d *are two weight functions on* G *such that* $c(G) = d(G)$*. Then there exists a sequence of edge-subdivisions and edge-contractions that will transform* (G, c) *in* (G, d)*. Moreover, this can be done using at most* $n-1$ *edge-subdivisions and the edge-contractions will not involve the terminal vertices* $A_1, A_2, ..., A_n$.

Proof. We introduce first a notation. If H is a graph with $E(H) = \{e_1, \ldots, e_p\}$ and u and v two weight functions on H such that $\frac{u(e_1)}{v(e_1)} = \cdots = \frac{u(e_k)}{v(e_k)} < \frac{u(e_j)}{v(e_j)}$ for every $j > k$, we set $m(u, v) := n - k$. Note that $0 \leq m(u, v) \leq n - 1$.

We will prove by induction on $m(c, d)$ that we can transform c into d using at most $m(c, d)$ edge-subdivisions and the contractions do not involve terminal vertices.

If $m(c, d) = 0$, i.e. all ratios $\frac{c_j}{d_j}$ are equal, since $\sum c_j = \sum d_j$ it follows that $c_j = d_j$ for all j and there is nothing to prove.

We assume that the statement is true for $m(c, d) = p - 1$ and we prove it for $m(c, d) = p$. Let $k = n - p$, put $c(\{X, A_i\}) = c_i$ and $d(\{X, A_i\}) = d_i$ and assume that $\frac{c_1}{d_1} = \frac{c_2}{d_2} = \cdots = \frac{c_k}{d_k} < \frac{c_{k+1}}{d_{k+1}} \leq \frac{c_{k+2}}{d_{k+2}} \leq \frac{c_n}{d_n}$.

We add the vertex P_{k+1} on the edge $\{X, A_{k+1}\}$ such that:
$c(\{A_{k+1}, P_{k+1}\}) = \dfrac{c_1 \cdot d_{k+1}}{d_1}$, $c(\{X, P_{k+1}\}) = c_{k+1} - \dfrac{c_1 \cdot d_{k+1}}{d_1}$

Note that since $\frac{c_1}{d_1} < \frac{c_{k+1}}{d_{k+1}}$ the weight of $\{X, P_{k+1}\}$ will be positive. We denote by $\tilde{c}$ the weight obtained after the contraction of $\{X, P_j\}$. Since this contraction will not create multiple edges, the ratios of those edges that are adjacent to $\{X, P_{k+1}\}$ will not change. The vertex obtained by identifying X and P_{k+1} will be denoted by X as well. Hence we have:
- for $j \leq k$, $\frac{\tilde{c}(\{X,A_j\})}{\tilde{c}(\{X,A_1\})} = \frac{c_j}{c_1} = \frac{d_j}{d_1}$
- $\frac{\tilde{c}(\{X,A_{k+1}\})}{\tilde{c}(\{X,A_1\})} = \frac{c(\{A_1,P_1\})}{c_1} = \frac{c_1 \cdot d_{k+1}}{d_1} \cdot \frac{1}{c_1} = \frac{d_{k+1}}{d_1}$

- for $j > s > k+1$ from $\frac{\tilde{c}(\{X,A_j\})}{\tilde{c}(\{X,A_1\})} = \frac{c_j}{c_1}$ and $\frac{\tilde{c}(\{X,A_s\})}{\tilde{c}(\{X,A_1\})} = \frac{c_s}{c_1}$ we deduce that $\frac{\tilde{c}(\{X,A_j\})}{d_j} \geq \frac{\tilde{c}(\{X,A_s\})}{d_s} > \frac{\tilde{c}(\{X,A_1\})}{d_1}$

This means that if $\tilde{c}_j := \tilde{c}(\{X, A_j\})$ then

$$\frac{\tilde{c}_1}{d_1} = \cdots = \frac{\tilde{c}_{k+1}}{d_{k+1}} < \frac{\tilde{c}_{k+2}}{d_{k+2}} \leq \frac{\tilde{c}_{k+3}}{d_{k+3}} \leq \cdots \leq \frac{\tilde{c}_n}{d_n}.$$

This shows that $m(\tilde{c}, d) = p - 1$ and we can apply the induction hypothesis. As we used only one edge-subdivision to transform c into $\tilde{c}$, the proof is complete.

□

Remark: In general we cannot transform c into d using $n-2$, or less, edge-subdivisions. For example, if $d_i = 1$ for all i and $c_i \neq c_j$ for all i, j with $i \neq j$ and if we use at most $n-2$ edge-subdivisions then (at least) two edges are not subdivided. They cannot be contracted either since this will decrease the degree of X. It follows that the quotient of their weights will remain constant and it cannot be transformed into 1.

Theorem 2. *If G is a graph with n edges and c and d are two weights on G such that $c(G) = d(G)$ then there exists a sequence of edge-subdivisions and edge-contractions that will transform (G, c) in (G, d). Moreover, this can be done using at most $n-1$ edge-subdivisions.*

Proof. For technical reason, we will prove that in fact it is possible to transform c in d such that no contraction will affect a terminal vertex. The proof will be by induction on $rd(G)$.

If $rd(G) = 0$, as G is connected, G is just a single edge and there is nothing to prove.

Assume that the statement is true for $rd(G) \leq k$ and we will prove it for $rd(G) = k+1$. We choose a vertex X of G whose degree is at least 2. Let $A_1, ..., A_m$ be the neighbors of X. Let $\tilde{G}_X$ be the blow-up of G at X and let $X_1, ..., X_m$ be the new vertices introduced by the blow-up.

Let $G_1, G_2, .., G_s$ be the connected components of $\tilde{G}_X$ and let n_j be the number of edges of G_j (it follows that $\sum n_j = n$). Note that $rd(G_j) \leq k$. Assume that $c(G_j) > d(G_j)$ for $j \leq q$ and $c(G_j) \leq d(G_j)$ for $q+1 < j \leq s$. We will do the construction in three steps:

Step 1.

Let $j \leq q$ be a fixed index and let $\{X_{j,1}, ..., X_{j,r}\} = V(G_j) \cap \{X_1, \ldots, X_m\}$.

We define the following weight on G_j:
- if e is an edge such that $e \cap \{X_{j,1}, ..., X_{j,r}\} = \emptyset$ then $d_j(e) = d(e)$,
- if e such that $e \cap \{X_{j,1}, ..., X_{j,r}\} \neq \emptyset$ then $d_j(e) = d(e) + \frac{c(G_j)-d(G_j)}{r}$.

Note that since $c(G_j) > d(G_j)$, the weights $d_j(e)$ defined as such are positive and $d_j(G_j) = c(G_j)$. We apply the induction hypothesis for G_j and the weights c and d_j and we deduce that we can transform the weight c into d_j by a sequence of edge-subdivisions and edge-contractions, using at most $n_j - 1$ edge-subdivisions and such that the contractions will not involve the terminal vertices of G_j. In particular, they will not involve $X_{j,1}, ..., X_{j,r}$.

This last condition guarantees that we can perform all these operations in the original graph G (with $\{A_j, X\}$ instead of $\{A_j, X_j\}$) without changing the weight of an edge that is not in the subgraph corresponding to G_j.

Step 2.

After applying the transformations from Step 1, for $j = 1, \dots, q$, on (G, c) we denote by $\tilde{c}$ the new weight. For each graph G_j with $q+1 \leq j \leq s$ we choose an edge $\{A_{l_j}, X_j\}$. We partition the edges of G at X in three subsets:
- $\mathcal{U} = \cup_{j=1}^{q} \{\{A_l, X\} : A_l \in V(G_j)\}$
- $\mathcal{V} = \{\{A_{l_j}, X_j\} : j = q+1, \dots, s\}$
- $\mathcal{W} = \{\{X, A_1\}, \{X, A_2\}, \dots, \{X, A_m\}\} \setminus (\mathcal{U} \cup \mathcal{V})$

Note that at this moment

$$\sum_{e \in \mathcal{U}} \tilde{c}(e) = \sum_{e \in \mathcal{U}} d(e) + \sum_{j=1}^{q} (c(G_j) - d(G_j)) = \sum_{e \in \mathcal{U}} d(e) + \sum_{j=q+1}^{s} (d(G_j) - c(G_j)).$$

We consider the graph H given by: $V(H) = \{X, A_1, ..., A_m\}$, $E(H) = \{\{X, A_1\}, ..., \{X, A_m\}\}$ and on H we consider two weights, $\tilde{c}$ and c_1 where c_1 is defined as follows:
- $c_1(e) = d(e)$ if $e \in \mathcal{U}$
- $c_1(e) = \tilde{c}(e)$ if $e \in \mathcal{W}$
- $c_1(\{A_{l_j}, X\}) = \tilde{c}(\{A_{l_j}, X\}) + d(G_j) - c(G_j)$ for every j, $q+1 \leq j \leq s$.

Note that all these weights are positive numbers and

$$c_1(H) = \sum_{e \in \mathcal{U}} d(e) + \sum_{e \in \mathcal{V} \cup \mathcal{W}} \tilde{c}(e) + \sum_{j=q+1}^{s} (d(G_j) - c(G_j)) =$$

$$\sum_{e \in \mathcal{U}} \tilde{c}(e) + \sum_{e \in \mathcal{V} \cup \mathcal{W}} \tilde{c}(e) = \tilde{c}(H)$$

Hence, we can apply Lemma 1 and after a sequence of edge-subdivisions and edge-contractions we will transform $\tilde{c}$ into c_1. We do this using at most $m-1$ edge-subdivisions and the contractions will not involve $A_1, ..., A_m$. As before, this guarantees that we can do the same operations in G without changing anything in the remaining of the graph.

After this step the weight function of each G_j for $j \leq q$ will be exactly d and for every $j > q$, the total weight of G_j will be $d(G_j)$.
Step 3.
We apply the induction hypothesis for each G_j, $q+1 \leq j \leq s$.

What is left to be done now is to notice that during the entire process the contractions did not involve terminal vertices of G (they were terminal vertices of G_j as well) and to count the number edge-subdivisions that were used.
- At Step 1 we used, for each $j \leq q$ at most $n_j - 1$ edge-subdivisions, hence altogether $\sum_{j=1}^{q}(n_j - 1)$.
- At Step 2 we used at most $s-1$ edge-subdivisions.
- At Step 3 we used, for each $j > q$ at most $n_j - 1$ edge-subdivisions, hence altogether $\sum_{j=q+1}^{s}(n_j - 1)$.

Adding everything together we used at most:
$\sum_{j=1}^{s}(n_j - 1) + s - 1 = \sum_{j=1}^{s} n_j - 1 = n - 1$ edge-subdivisions. □

References

[1] L. Lovász: Graph minor theory. Bull. Amer. Math. Soc. (N.S.) 43 (2006), no. 1, 75–86

[2] N. Robertson; P. D. Seymour: Graph minors. XX. Wagner's conjecture. J. Combin. Theory Ser. B 92 (2004), no. 2, 325–357.

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