ON RUNGE NEIGHBORHOODS OF CLOSURES OF DOMAINS BIHOLOMORPHIC TO A BALL

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Abstract. We give an example of a domain W in $\mathbb{C}^3$, biholomorphic to a ball, such that W is not Runge in any Stein neighborhood of $\overline{W}$.

1. Introduction

During the conference "Geometric Function Theory in Higher Dimension", held in Cortona in September 2016, Filippo Bracci asked the following question: suppose that W is a domain in $\mathbb{C}^n$, biholomorphic to a ball. Does there exist a Fatou-Bieberbach domain U such that $W \subset U \subset \mathbb{C}^n$ and W is Runge in U?

A related (and natural) question is the following: suppose that W is a domain in $\mathbb{C}^n$ which is biholomorphic to a ball. Does there exist a Stein domain U such that $\overline{W} \subset U \subset \mathbb{C}^n$ and W is Runge in U? The purpose of this note is to show that the answer to the second question is negative, by constructing a counter-example. As it can be seen from our construction, this does not answer Filippo Bracci's question since we construct a domain W with a "bad" point x in the boundary ∂W and, according to the statement of our problem, this point must be in U.

For the basic notions regarding pseudoconvexity we refer, for example, to [1]. For a complex manifold M we denote by $\mathcal{O}(M)$ the ring of holomorphic functions and, if K is a compact subset of M, $\widehat{K}^M$ stands for the holomorphically convex hull of K in M, $\widehat{K}^M = \{x \in M : |f(x)| \leq \|f\|_K,\ \forall f \in \mathcal{O}(M)\}$. If M is a Stein manifold and D is a Stein open subset of M, then D is called Runge in M if the restriction map $\mathcal{O}(M) \to \mathcal{O}(D)$ has a dense image. This is equivalent to the fact that for every compact set $K \subset D$ we have $\widehat{K}^M = \widehat{K}^D$. It is also a standard fact that if M is a Stein manifold, D is a Stein open subset of M which is Runge in M and N is a closed complex submanifold of M, then $N \cap D$ is Runge in N.

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2. The example

The following map was defined by J. Wermer, [2] and [3]: $f : \mathbb{C}^3 \to \mathbb{C}^3$, $f(z,w,t) = (z, zw + t - 1, zw^2 - w + 2wt)$. A direct computation shows that $f_{|\mathbb{C}\times\mathbb{C}\times\{t:\mathrm{Re}(t)<\frac{1}{2}\}}$ is a biholomorphism onto its image.

Let $0 < p < \frac{1}{4}$ be a fixed real number and let

$$B := \left\{(z,w,t) \in \mathbb{C}^3 : p|z|^2 + p|w|^2 + |t|^2 < \frac{1}{4}\right\}.$$

Then B is biholomorphic to a ball and $B \subset \mathbb{C} \times \mathbb{C} \times \{t : |t| < \frac{1}{2}\}$. Hence $f(B)$ is also biholomorphic to a ball. We would like to show that $f(B)$ is the required example.

Suppose that U is a pseudoconvex neighborhood of $\overline{f(B)}$. Note that $\left(0, -\dfrac{1}{2}, 0\right) = f\left(0, 0, \dfrac{1}{2}\right) \in \partial f(B)$. Hence, for a sufficiently small $r > 0$, we have that

$$\left\{(\xi, \eta, \theta) \in \mathbb{C}^3 : |\xi| \le r, |\eta + \frac{1}{2}| \le r, |\theta| \le r\right\} \subset U.$$

We make the following claim:

Claim: If r is small enough, there exists $\alpha \in \mathbb{R}$, $-\dfrac{1}{2} < \alpha < -\dfrac{1}{2} + r$, such that

$$\{\xi \in \mathbb{C} : |\xi| = r\} \times \{\alpha\} \times \{0\} \subset f(B).$$

Proof of the claim:
Note that $f\left(\xi, \dfrac{2\alpha+1}{\xi}, -\alpha\right) = (\xi, \alpha, 0)$, for every $\xi \in \mathbb{C}\backslash\{0\}$, $\alpha \in \mathbb{R}$. Hence it suffices to show that if $r > 0$ is sufficiently small, there exists $\alpha \in \mathbb{R}$, $-\dfrac{1}{2} < \alpha < -\dfrac{1}{2} + r$, such that $\left(\xi, \dfrac{2\alpha+1}{\xi}, -\alpha\right) \in B$ for every ξ with $|\xi| = r$.

In other words, we would like to show that if $r > 0$ is small enough, there exists $\alpha \in \mathbb{R}$, $-\dfrac{1}{2} < \alpha < -\dfrac{1}{2} + r$, such that

$$pr^2 + p\left(\frac{(2\alpha+1)^2}{r^2}\right) + \alpha^2 < \frac{1}{4},$$

or:

$$g_r(\alpha) := \left(\frac{4p}{r^2}+1\right)\alpha^2 + \frac{4p}{r^2}\alpha + pr^2 + \frac{p}{r^2} < \frac{1}{4}.$$

Let $\alpha_0 := -\dfrac{2p}{4p+r^2} > -\dfrac{2p}{4p} = -\dfrac{1}{2}$. We have:

$$\alpha_0 < -\frac{1}{2} + r \Longleftrightarrow \frac{r^2}{4p+r^2} < 2r \Longleftrightarrow r < 2(4p+r^2),$$

which is obviously true for r small enough.
Moreover:

$$g_r(\alpha_0) = \frac{4p+r^2}{r^2}\frac{4p^2}{(4p+r^2)^2} - \frac{8p^2}{r^2(4p+r^2)} + \frac{pr^4+p}{r^2}.$$

Hence:

$$\begin{aligned} g_r(\alpha_0) < \frac{1}{4} \quad &\Longleftrightarrow \quad \frac{-4p^2 + (4p+r^2)(pr^4+p)}{r^2(4p+r^2)} < \frac{1}{4} \\ &\Longleftrightarrow \quad pr^6 + 4p^2r^4 + pr^2 < pr^2 + \frac{r^4}{4} \\ &\Longleftrightarrow \quad pr^2 < \frac{1}{4} - 4p^2. \end{aligned}$$

Since $p < \dfrac{1}{4}$, we have that $\dfrac{1}{4} - 4p^2 > 0$ and therefore the last inequality is true for r small enough. Hence our claim is proved. □

We remark now that $(0,\alpha,0) \notin f(B)$ for $\alpha > -\dfrac{1}{2}$. Indeed

$$(0,\alpha,0) = f(z,w,t) \Longleftrightarrow (z = w = 0,\ t = 1+\alpha)\,, \text{ and then } t^2 > \frac{1}{4}.$$

To summarize, we found $r > 0$, sufficiently small, and α such that:

$$\begin{cases} \{\xi \in \mathbb{C} : |\xi| \le r\} \times \{\alpha\} \times \{0\} \subset U, \\ \{\xi \in \mathbb{C} : |\xi| = r\} \times \{\alpha\} \times \{0\} \subset f(B), \\ (0,\alpha,0) \notin f(B). \end{cases}$$

This shows that $(\mathbb{C} \times \{\alpha\} \times \{0\}) \cap f(B)$ is not Runge in $(\mathbb{C} \times \{\alpha\} \times \{0\}) \cap U$, and therefore $f(B)$ is not Runge in U.

Remarks. 1. The above example is relatively compact in $\mathbb{C}^3$. One can construct an unbouded example as follows: again we let p be a fixed positive real number, $p < \frac{1}{4}$, and we set:

$$S = \left\{ (z, w, t) \in \mathbb{C}^3 : \mathrm{Re}(t) < -p|z|^2 - p|w|^2 + \frac{1}{2} \right\}.$$

Then S is unbounded, biholomorphic to a ball, and $S \subset \mathbb{C} \times \mathbb{C} \times \{t \in \mathbb{C} : \mathrm{Re}(t) < \frac{1}{2}\}$. A completely similar argument shows that if U is a Stein neighborhood of $\overline{f(S)}$, then $f(S)$ is not Runge in U.

It may be possible to construct a counter-example in $\mathbb{C}^2$, using the same procedure, following a construction of J.Wermer in [4].

2. The following interesting question was raised by the referee. Is there a natural number k such that a biholomorphic image of the ball which is $\mathcal{C}^k$ smooth has a Stein neighbourhood basis in which the domain is Runge ?

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