Some problems related to the Levi problem for Riemann domains over Stein spaces *

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Abstract

We consider a more general form of the Levi problem for Stein spaces, namely the Levi problem for unbranched Riemann domains spread over Stein spaces. We raise four problems related to this generalization and we remark that in this case the L^2 methods cannot be applied.

1 Introduction and preliminaries

Let X be a Stein space. An unbranched Riemann domain over X is a pair (Y, π) where Y is a complex space and $\pi : Y \to X$ is a locally biholomorpic map. We will call it pseudoconvex if π is a Stein morphism, i.e. X has a Stein open covering $\{V_i\}_{i\in I}$ such that $\pi^{-1}(V_i)$ is Stein for each $i \in I$. If Y is an open subset of X and π is the inclusion then $\pi : Y \to X$ is pseudoconvex if and only if Y is a locally Stein open subset of X.

We recall in this context the classical open problem:

Levi Problem. (LP) *Let X be a Stein space and $Y \subset X$ a locally Stein open subset. Does it follow that Y is Stein?*

A more general problem is the Levi problem for unbranched Riemann domains:

Generalized Levi Problem. (GLP) *Let X be a Stein space and $\pi : Y \to X$ a pseudoconvex unbranched Riemann domain. Does it follow that Y is Stein?*

In [6] it was shown that the (GLP) has a positive answer for complex spaces with isolated singularities. By normalization it follows that (GLP) has a positive answer in dimension 2.

The aim of this short note is to introduce four problems related to the (GLP), especially for infinitely sheeted unbranched Riemann domains. It seems to us that in this case the L^2 methods cannot be applied and it is not clear which could be the substitute. This is due to the lack of vertical plurisubharmonic exhaustion functions (along the fibers) which could be used to define L^2-weights. For other surveys or lists with open problems, see [1], [2], [3].

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2 The problems

Let $\pi : Y \to X$ be a pseudoconvex unbranched Riemann domain, where X and Y are complex spaces. As in [7] we may define ∂Y, the boundary of Y (relative to π). If $Y \subset X$ is an open subset, then ∂Y is just the set of accessible points of the usual boundary.

We can now state:

Problem 1. *Let $\pi : Y \to X$ be a pseudoconvex unbranched Riemann domain where X is a normal Stein space. Then for every point $x_0 \in \partial Y$ such that $\pi(x_0) \in Reg(X)$ and every sequence of points $\{x_n\}_{n\geq 1}$, $x_n \in Y$, $x_n \to x_0$, there exists a holomorphic function $f \in \mathcal{O}(Y)$ such that $\{f(x_n)\}_{n\geq 1}$ is unbounded.*

Remark 1. If π is the inclusion map, i.e. $Y \subset X$ is a locally Stein open subset, then this was proved in [8] under the additional condition that Y is relatively compact in X. The relative compacity assumption has been removed in [11]. The proof depends on the L^2 estimates of Skoda [12]. In the setting of Problem 1, due to the lack of plurisubharmonic vertical exhaustion functions (along the infinite fibers), it seems that it is not possible to use L^2 techniques.

We recall that for complex space Y the global holomorphic functions separate points if for any $y_1 \neq y_2$ there exists $f \in \mathcal{O}(Y)$ such that $f(y_1) \neq f(y_2)$. We state then the following question.

Problem 2. *Let $\pi : Y \to X$ be a pseudoconvex unbranched Riemann domain such that X is Stein. Does it follow that the global holomorphic functions on Y separate the points of Y?*

We will recall now a few facts about envelopes of holomorphy.

Let $\pi : Y \to X$ be an unbranched Riemann domain over the Stein space X. The pair $(\tilde{Y}, p)$ is called an envelope of holomorphy of π if

1. $p : \tilde{Y} \to X$ is a locally biholomorphic map (so it is an unbranched Riemann domain),
2. $\tilde{Y}$ is Stein,
3. there is a locally biholomorphic map $\tau : Y \to \tilde{Y}$ over X (i.e. $p \circ \tau = \pi$),
4. for every $f \in \mathcal{O}(Y)$ there exists a unique $\tilde{f} \in \mathcal{O}(\tilde{Y})$ such that $\tilde{f} \circ \tau = f$.

It follows that, if it exists, the envelope of holomorphy is unique. In general, in the singular case, the envelope of holomorphy might not exist even if X is an open subset of a normal Stein space with isolated singularities, see [9] or [5]. On the other hand, in the smooth case, for open subsets of Stein manifolds, it is well-known that envelopes of holomorphy always exist. In this context we have the following:

Problem 3. *Let $\pi : Y \to X$ be an unbranched Riemann domain over the Stein space X. We assume that π is pseudoconvex and has envelope of holomorphy. Does it follow that Y is Stein?*

Remark 2. When Y is a relatively compact open subset of of a Stein space this was proved in [5] and without the relative compactness condition in [11]. The difficulty consists in proving the separation of points of Y by global holomorphic functions. If this separation is fulfilled and $\pi : Y \to X$ has an envelope of holomorphy $p : \tilde{Y} \to X$ then Y is isomorphic to an open subset Y' of $\tilde{Y}$ and Y' is pseudoconvex in $\tilde{Y}$ (i.e. locally Stein). This implies that Y' is Stein (hence equal to $\tilde{Y}$).

Finally we want to discuss the problem of Stein hypersections in the context of unbranched Riemann domains. We recall a result from [4]: there exists a Stein normal space of dimension 3 (in fact with with one isolated singularity) and an open subset $D \subset X$ such that

1. $D \cap H$ is Stein for every hypersurface $H \subset X$,
2. D is not Stein.

Remark 3. It is possible to show (however we do not need this fact here) that the example D constructed in [4] has a Stein covering, therefore it satisfies the Kontinuitätssätz.

In the context of unbranched Riemann domains we ask the following question:

Problem 4. *Let X be a normal Stein space, $\dim(X) \geq 3$, and $\pi : Y \to X$ an unbranched Riemann domain. We assume that for every hypersurface $H \subset X$ we have that $\pi^{-1}(H)$ is Stein. Does it follow that the global holomorphic functions on Y separate the points of Y?*

Remark 4. If X is a Stein manifold (i.e. smooth) it is well known that Y is also Stein, see [10].

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