On the separation of the cohomology of universal coverings of 1-convex surfaces *

Mihnea Colţoiu, Cezar Joiţa

Abstract

We construct a 1-convex surface X such that its universal covering $\tilde{X}$ has the property that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated.

1 Introduction

The Hausdorff property of the cohomology of a complex space with values in a coherent sheaf appears naturally when one tries to prove duality results. At the same time, A. Markoe [15] and A. Silva [19] proved that an increasing union of Stein open subsets of a complex space is Stein if and only if its first cohomology group with values in the structure sheaf is separated and B. Jennane [10] proved that if X is a Stein complex space and Ω is a locally Stein open subset then Ω is Stein if and only if $H^1(\Omega, \mathcal{O}_\Omega)$ is separated. More recently M. Brumberg and J. Leiterer [2] proved that for a smooth 2-dimensional 1-corona if the first cohomology group with values in the structure sheaf is separated then the concave end can be filled in.

In [6] it is provided an example of a normal Stein space X of dimension 3, with only one singular point, and a closed analytic subset $A \subset X$ of codimension 1 of X such that $H^1(X \setminus A, \mathcal{O})$ is not separated. On the other hand G. Trautmann [20] proved that if X is a normal Stein space, $A \subset X$ is a closed complex analytic subset with codim A ≥ 2, and $\mathcal{F}$ is torsion free coherent sheaf on X then $H^1(X \setminus A, \mathcal{F})$ is separated. However, to our knowledge, a complete answer to the following question raised by C. Bănică

*Mathematics Subject Classification (2010): 32C35, 32F10, 32F32.
Key words: 1-convex manifolds, universal covering, separated cohomology groups

is not known: suppose that X is a Stein manifold, $A \subset X$ is a closed analytic subset and $\mathcal{F}$ is a locally free sheaf on X. Does it follow that $H^q(X \setminus A, \mathcal{F})$ is separated for every $q \geq 2$? If $\mathcal{F}$ has torsion, this is not the case - see [6].

If X is a 1-convex complex space, that is a proper modification of a Stein space at a finite set, then for every coherent analytic sheaf $\mathcal{F}$ on X the cohomology groups $H^q(X, \mathcal{F})$ are finite dimensional for every $q \geq 1$. In particular they are separated in their canonical QFS topology. If X is holomorphically convex then $H^q(X, \mathcal{F})$ is still separated. On the other hand B. Malgrange [14] gave an example of a weakly pseudoconvex manifold, which is an open subset of an algebraic 2-dimensional torus, that does not have separated cohomology for the structure sheaf. Malgrange's example builds on an example of Grauert (see [16]) of a weakly pseudoconvex manifold on which all global holomorphic functions are constant.

Suppose now that X is a 1-convex complex surface and $p : \tilde{X} \to X$ is the universal covering of X. In general $\tilde{X}$ is not holomorphically convex and not even weakly pseudoconvex (i.e. $\tilde{X}$ does not carry a continuous plurisubharmonic exhaustion function). The only geometric property of $\tilde{X}$ is that it is p_3-convex in the sense of Grauert and Docquier [9] (i.e. it can be exhausted by a sequence of strongly pseudoconvex domains). A natural question in this context is to decide if $\tilde{X}$ has separated cohomology for the structure sheaf. The purpose of this paper is to show that this is not always the case. Namely we will prove the following theorem:

Theorem. *There exists a 1-convex complex surface X such that its universal covering $\tilde{X}$ has the property that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated.*

Our main ingredients for the proof are the following:

- the construction in [8] of a 1-convex surface X such that its universal covering does not satisfy the discrete Kontinuitätssatz,
- the p_3-convexity of $\tilde{X}$,
- the Serre duality [18].

We note that H. Kazama and S. Takayama [11] constructed a smooth projective surface Y and a covering $\tilde{Y}$ of Y such that $\tilde{Y}$ has the property that $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ is not separated. From their example one can construct easily (see Remark 3 at the end of the paper) coverings $\tilde{X} \to X$ of 1-convex manifolds X with $\dim X \geq 3$ such that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated. Our main contribution is that we can produce such examples for $\dim X = 2$ and $\tilde{X}$ the universal covering of X.

In connection with Shafarevich conjecture it would be interesting to know if the universal covering of a smooth projective manifold has separated cohomology for the structure sheaf.

2 Preliminaries

Let X be a 1-convex manifold. This means that X is a proper modification of a normal Stein space at finitely many points. Hence we have a normal Stein space Y, a proper holomorphic map $p : X \to Y$ and a finite set $B \subset Y$ such that $p : X \setminus p^{-1}(B) \to Y \setminus B$ is a biholomorphism. $A := p^{-1}(B)$ is called the exceptional set of X.

The following result is Theorem 2 in [4].

Theorem 1. *Suppose that X is a 1-convex manifold and that the exceptional set of X has dimension 1. Let $\tilde{X}$ be a covering of X. Then $\tilde{X}$ has an exhaustion with relatively compact strongly pseudoconvex open subsets.*

We will use this result for $\dim X = 2$ and therefore the exceptional set must have dimension 1.

Suppose that X is a 1-convex surface, A is its exceptional set and $p : \tilde{X} \to X$ is a covering. Let $\tilde{A} := p^{-1}(A)$. If $\tilde{A}$ is holomorphically convex then $\tilde{X}$ is also holomorphically convex. Therefore the most interesting case is when $\tilde{A}$ is an infinite Nori string, i.e. $\tilde{A}$ is a connected non-compact complex space such that all its irreducible components are compact. If $\tilde{X}$ does not contain an infinite Nori string of rational curves then it was proved in [7] that $\tilde{X}$ satisfies the discrete disk property in the following sense: we denote by Δ_r, $r > 0$, the disc in the complex plane centered at the origin and of radius r and we set $\Delta := \Delta_1$.

Definition 2. Suppose that X is a complex space. We say that X satisfies the discrete disk property if whenever $f_n : U \to X$ is a sequence of holomorphic functions defined on an open neighborhood U of $\overline{\Delta}$ for which there exist an $\epsilon > 0$ and a continuous function $\gamma : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to X$ such that $\Delta_{1+\epsilon} \subset U$, $\bigcup_{n\geq 1} f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in X and $f_{n|S^1}$ converges uniformly to γ we have that $\bigcup_{n\geq 1} f_n(\overline{\Delta})$ is relatively compact in X.

However if $\tilde{A}$ is a Nori string of rational curves this might not be the case. More precisely, according to [8], one has:

Theorem 3. *There exists a 1-convex complex surface X such that its universal covering $\tilde{X}$ does not satisfy the discrete disk property.*

Remark 1. In particular $\tilde{X}$ is not p_5 convex in the sense of [9], i.e. there exists a family of holomorphic disks $f_\nu : \overline{\Delta} \to \tilde{X}$ such that $\bigcup f_\nu(\partial\Delta) \Subset \tilde{X}$ but $\bigcup f_\nu(\overline{\Delta})$ is not relatively compact in $\tilde{X}$. From our construction in [8], each disk is contained in a closed analytic curve of $\tilde{X}$.

Definition 4. Let $(I, \leq)$ be a directed set and $\{G_i, g_{i,j}\}$ be a direct system of abelian groups and let $g_i : G_i \to \varinjlim G_k$ be the canonical morphisms. The inductive limit $\varinjlim G_k$ is called essentially injective if for every $i \in I$ there exists $j \in I$ such that for every $x \in G_i$ if $g_i(x) = 0$ then $g_{i,j}(x) = 0$.

Notation: For a Fréchet space F we denote by F^* its topological dual.

Definition 5. Suppose that X is a complex space and $\{X_j\}_{j\in\mathbb{N}}$ is an increasing sequence of open subsets of X. For $i \leq j$ positive integers we denote by $\rho_{i,j} : \mathcal{O}(X_j) \to \mathcal{O}(X_i)$ the restriction morphism. We say that $\{X_j\}$ is a Runge family if for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$, $j \geq i$, such that for every $l \geq j$ we have that $\overline{\rho_{i,j}(\mathcal{O}(X_j))} = \overline{\rho_{i,l}(\mathcal{O}(X_l))}$, where the closures are taken in the topology of uniform convergence on compacts in X_i.

The following proposition was proved in [3]. There the Runge family condition was called the closed Mittag-Leffler condition.

Proposition 6. *Let X be a complex space and $\{X_j\}_{j\in\mathbb{N}}$ be an increasing sequence of open subsets of X such that $\bigcup X_j = X$, $\{X_j\}_{j\in\mathbb{N}}$ is a Runge family, and $H^1(X_j, \mathcal{O}_{X_j})$ is separated for every j. Then $H^1(X, \mathcal{O}_X)$ is separated.*

3 The Results

Proposition 7. *Suppose that X is a smooth complex manifold and $\{X_j\}_{j\in\mathbb{N}}$ are open subsets of X such that $X_j \Subset X_{j+1}$ and $\bigcup X_j = X$. We assume that $H^1(X_j, \mathcal{O})$ is separated for each $j \in \mathbb{N}$. Then $H^1(X, \mathcal{O})$ is separated if and only if $\{X_j\}$ is a Runge family.*

Proof. The "if" part follows from Proposition 6. We will prove the "only if" statement. The first part of the proof of this statement is completely similar to the proof of Lemma 2.1 in [5]. We give it here for the reader's convenience. Let $n = \dim X$.

It follows from [12], see also [1], that $H^1(X,\mathcal{O})$ is separated if and only if $H^n_c(X,K_X)$ is separated. At the same time $H^n_c(X,K_X)=\varinjlim H^n_c(X_j,K_X)$. By Theorem 4.1 in [17], we have that $H^n_c(X,K_X)$ is separated if and only if the inductive limit $\varinjlim H^n_c(X_j,K_X)$ is essentially injective (here we use the fact that $X_j \Subset X_{j+1}$). We assumed that each $H^1(X_j,\mathcal{O})$ is separated and therefore $H^n_c(X_j,K_X)=\mathcal{O}(X_j)^*$. Therefore $H^n_c(X,K_X)$ is separated if and only if the inductive limit $\varinjlim \mathcal{O}(X_j)^*$ is essentially injective.

For $i \leq j$ positive integers we denote by $\rho_{i,j}:\mathcal{O}(X_j)\rightarrow \mathcal{O}(X_i)$ the restriction morphism and $\rho^*_{i,j}:\mathcal{O}(X_i)^*\rightarrow \mathcal{O}(X_j)^*$ the induced morphism. Let also $\rho^*_i:\mathcal{O}(X_i)^*\rightarrow \varinjlim \mathcal{O}(X_j)^*$ be the canonical morphism.

Let $i\in\mathbb{N}$ be a fixed index and let $j>i$ be such that for every $\mu\in\mathcal{O}(X_i)^*$ if $\rho^*_i(\mu)=0$ then $\rho^*_{i,j}(\mu)=0$. We claim that for every $l\geq j$ we have that $\rho_{i,j}(\mathcal{O}(X_j))$ and $\rho_{i,l}(\mathcal{O}(X_l))$ have the same closure in $\mathcal{O}(X_i)$. Indeed, let's denote by S_1 the closure of $\rho_{i,j}(\mathcal{O}(X_j))$ in $\mathcal{O}(X_i)$ and by S_2 the closure of $\rho_{i,l}(\mathcal{O}(X_l))$.

Note that $S_2\subset S_1$. Because S_2 is closed, to show that $S_1\subset S_2$ it suffices to show that $\rho_{i,j}(\mathcal{O}(X_j))\subset S_2$. Let's suppose that this is not the case and let $f\in\rho_{i,j}(\mathcal{O}(X_j))\setminus S_2$. By Hahn-Banach theorem there exists $\mu\in\mathcal{O}(X_i)^*$ such that $\mu_{|S_2}=0$ and $\mu(f)\neq 0$. Because $\mu_{|S_2}=0$ we have that $\rho^*_{i,l}(\mu)=0$. In particular we have that $\rho^*_i(\mu)=0$. By our choice of j we have that $\rho^*_{i,j}(\mu)=0$ and in particular $\mu(f)=0$, which, of course, is a contradiction.

□

Theorem 8. *There exists a 1-convex surface X such that $H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ is not separated where $\tilde{X}$ is the universal covering of X.*

Proof. We will describe briefly the example constructed in [8]. We start with $\Omega_0:=\mathbb{C}^2$ and we blow-up at the origin $(0,0)=a_0\in\mathbb{C}^2$. Let Ω_1 be this blow-up. We choose a point a_1 on the exceptional divisor of Ω_1. To be precise, if (z_1,z_2) are the coordinate functions on $\mathbb{C}^2$ then $a_1=(0,0,[0:1])\in\mathbb{C}^2\times\mathbb{P}^1$. This means that a_1 is the intersection between the proper transform of $z_1=0$ and the exceptional divisor. We blow-up Ω_1 at a_1 and we obtain Ω_2. We let a_2 be the intersection between the proper transform of $z_1=0$ (more precisely the proper transform of the proper transform of $z_1=0$) and the exceptional divisor of Ω_2 and we blow-up Ω_2 at a_2. In this way we obtain a sequence $\{\Omega_k\}_{k\geq 0}$ of complex manifolds and a sequence of points $a_k\in\Omega_k$ such that $\Omega_k\setminus\{a_k\}\subset\Omega_{k+1}\setminus\{a_{k+1}\}$. We denote by X_0 the union of $\Omega_k\setminus\{a_k\}$ and we call it the infinite blow-up of $\mathbb{C}^2$. Starting with X_0 and going backwards one can construct an increasing sequence of complex surfaces $X_0 \subset X_{-1} \subset X_{-2} \subset \cdots$. Their union $\bigcup_{k=0}^{-\infty} X_k$ is a complex surface that contains an infinite Nori string of $\mathbb{P}^1$ indexed over $\mathbb{Z}$. On each $\mathbb{P}^1$ we choose a suitable point (in appropriately chosen coordinates this point is $[1 : 1]$) and we blow-up each of these points. We obtain in this way a smooth complex surface that contains also an infinite Nori string of $\mathbb{P}^1$. An appropriately chosen neighborhood $\hat{X}$ of the Nori string covers a 1-convex surface X whose exceptional set consists of two $\mathbb{P}^1$ meeting only at two points and the intersection matrix is

$$\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$$

To show that $\hat{X}$ does not satisfy the discrete disc property we defined a sequence of polynomial mappings $\mathbb{C} \to \mathbb{C}^2$ and then we took the proper transform of their images after all the blow-ups described above.

The holomorphic discs were the restriction of these holomorphic functions to $\Delta_{1+\epsilon}$ for some $\epsilon > 0$. (One difficulty of this construction was to show that the images of $\Delta_{1+\epsilon}$ are in $\hat{X}$.)

Note that every non-constant polynomial function $\mathbb{C} \to \mathbb{C}^2$ is proper and therefore its image is a closed analytic subset of $\mathbb{C}^2$. Moreover, the first component of each of the polynomial functions used in our construction is not identically equal to 0. And hence after finitely many blow-ups, its proper transform will not contain the a_k (the center of the blow-up). We deduce that for each of these holomorphic discs f_n there exists A_n, a closed 1-dimensional analytic subset of $\hat{X}$ such that A_n contains the image of f_n.

The last step of our construction was to show that the universal covering $\tilde{X}$ of $\hat{X}$ (and hence of X) does not satisfy the discrete disc property. To do that we noticed that there exists a simply connected open subset U (in [8], U was denoted by $\tilde{W}_r^\rho$ and, in fact, U is a covering of a neighborhood of the exceptional set of X) of $\hat{X}$ which contains the images of all holomorphic discs. Hence each f_n gives us a holomorphic disk in $\tilde{X}$. Taking the preimages of A_n in $\tilde{X}$ we get that the image of each holomorphic disc $f_n : \Delta_{1+\epsilon} \to \tilde{X}$ is included in some closed 1-dimensional analytic subset of $\tilde{X}$.

Hence we have constructed:
• X a 1-convex smooth surface and $p : \tilde{X} \to X$ the universal covering map,
• a sequence of closed 1-dimensional analytic subsets A_n of $\tilde{X}$,
• a sequence of holomorphic discs $f_n : \Delta_{1+\epsilon} \to \tilde{X}$, $n \in \mathbb{N}$, for some $\epsilon > 0$,
such that:

a) $K := \overline{\bigcup f_n(\partial\Delta_1)}$ is compact,
b) $\overline{\bigcup f_n(\overline{\Delta}_1)}$ is not compact,
c) $f_n(\Delta_{1+\epsilon}) \subset A_n$.

Claim: $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated.

Proof of the claim: We assume, by reductio ad absurdum, that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is separated.

Let L be the exceptional set of X and $\tilde{L} = p^{-1}(L)$. By Theorem 1, there exist $Y_1 \Subset Y_2 \Subset \cdots$ an exhaustion of $\tilde{X}$ with strictly pseudoconvex domains with smooth boundaries. By Proposition 7 there exists $j = j(K) \geq 1$ such that for every $l \geq j$ we have that every holomorphic function in $\mathcal{O}(Y_j)$ can be approximated uniformly on K with functions in $\mathcal{O}(Y_l)$.

Let $f : \Delta_{1+\epsilon} \to \tilde{X}$ be a holomorphic disc, $\epsilon > 0$, A a 1-dimensional closed analytic subset of $\tilde{X}$ and $\lambda_0 \in \Delta_1$ such that
- $f(\partial\Delta_1) \subset K$,
- $y_0 := f(\lambda_0) \notin \tilde{L} \cup Y_j$,
- $f(\Delta_{1+\epsilon}) \subset A$.

We choose $l > j$ such that $f(\overline{\Delta}_{1+\epsilon/2}) \subset Y_l$.

We let $p_l : Y_l \to X_l$ and $p_j : Y_j \to X_j$ be the Remmert reductions of Y_l and of Y_j respectively and we set $A_l := p_l(A \cap Y_l)$, $A_j := p_j(A \cap Y_j)$ which are 1-dimensional closed analytic subsets of X_l and X_j. Let also $x_0 := p_l(y_0) \in A_l$.

We choose now a holomorphic function $h : A_l \to \mathbb{C}$ such that $\{x \in A_l : h(x) = 0\} = \{x_0\}$. Because X_l is Stein and A_l is a closed analytic subset of X_l we can extend h to a holomorphic function on X_l, which we denote also by h. Hence we obtain a holomorphic function $h \in \mathcal{O}(X_l)$ such that $Z(h) \cap A_l = \{x_0\}$ where $Z(h)$ is the zero set of h, $Z(h) = \{x \in X_l : h(x) = 0\}$.

We let $\tilde{h} = h \circ p_l \in \mathcal{O}(Y_l)$. Because $y_0 \notin \tilde{L}$ we have that $p^{-1}(x_0) = \{y_0\}$. It follows that $Z(\tilde{h}) \cap A = \{y_0\}$.

Because p_j is the Remmert reduction we have that $p_{j*}\mathcal{O}_{Y_j} = \mathcal{O}_{X_j}$. Hence there exists $h_1 \in \mathcal{O}(X_j)$ such that $h_1 \circ p_j = \tilde{h}_{|Y_j}$. Since $y_0 \notin \tilde{L} \cup Y_j$, it follows that $Z(h_1) \cap A_j = \emptyset$. In particular $\frac{1}{h_1} : A_j \to \mathbb{C}$ is a holomorphic function on A_j. Because X_j is Stein and A_j is a closed analytic subset it follows that there exists $g \in \mathcal{O}(X_j)$ such that $g_{|A_j} = \frac{1}{h_1}$. We set $\tilde{g} := g \circ p_j \in \mathcal{O}(Y_j)$. It follows that $\tilde{g}\tilde{h} \equiv 1$ on $A \cap Y_j$.

We recall that every holomorphic function in $\mathcal{O}(Y_j)$ can be approximated uniformly on K with functions in $\mathcal{O}(Y_l)$. Hence we can find a sequence of functions $\{\phi_n\}_n$, $\phi_n \in \mathcal{O}(Y_l)$, that converges uniformly on K to $\tilde{g}$. In particular it follows that $\phi_n \to \tilde{g}$ uniformly on $f(\partial\Delta_1)$. Therefore $\phi_n\tilde{h} \to \tilde{g}\tilde{h}$ uniformly on $f(\partial\Delta_1)$.

Because $f(\partial\Delta_1) \subset A$ we get that $\phi_n\tilde{h} \to 1$ uniformly on $f(\partial\Delta_1)$ and hence $(\phi_n \circ f)(\tilde{h} \circ f) - 1 \to 0$ uniformly on $\partial\Delta_1$. Note that $\phi_n \circ f$ and $\tilde{h} \circ f$ are holomorphic functions defined on $\Delta_{1+\epsilon/2}$ because we have chosen l such that $f(\overline{\Delta}_{1+\epsilon/2}) \subset Y_l$. We deduce that $(\phi_n \circ f)(\tilde{h} \circ f) - 1 \to 0$ uniformly on $\overline{\Delta}_1$. In particular $(\phi_n \circ f)(\lambda_0)(\tilde{h} \circ f)(\lambda_0) - 1 \to 0$. However we have that $(\tilde{h} \circ f)(\lambda_0) = \tilde{h}(y_0) = 0$ and therefore we obtain a contradiction.

$\square$

Remark 2. Let X_0 be the infinite blow-up defined at the beginning of this section. The arguments given in [8] show the existence of $\{A_n\}$ and $\{f_n\}$ satisfying a), b), and c). At the same time we have that $X_0 = \bigcup_{k\geq 0} \Omega_k \setminus \{a_k\}$. Each Ω_k is 1-convex and hence $H^1(\Omega_k \setminus \{a_k\}, \mathcal{O})$ is separated. At the same time $\mathcal{O}(\Omega_k \setminus \{a_k\}) = \mathcal{O}(\Omega_k) = \mathcal{O}(\Omega_0)$ and the restriction map $\mathcal{O}(\Omega_k \setminus \{a_k\}) \to \mathcal{O}(\Omega_j \setminus \{a_j\})$ is bijective. In particular $\{\Omega_k \setminus \{a_k\}\}_k$ is a Runge family. By Proposition 6 we have that $H^1(X_0, \mathcal{O}_{X_0})$ is separated. In this case X_0 is not p_3-convex.

Remark 3. As we mentioned in the Introduction, in [11], example 4.4, a smooth projective surface Y was constructed, together with a covering $p : \tilde{Y} \to Y$ of Y such that $\tilde{Y}$ has the property that $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ is not separated. Let $F \to Y$ be a negative line bundle over Y. Therefore $X := F$ is a 1-convex manifold and the null-section of F is its exceptional set. Let $\tilde{X} := p^*F$ be the pull-back of F. We have then a covering map $\tilde{X} \to X$ and a line bundle $\rho : \tilde{X} \to \tilde{Y}$. Identifying the zero-section of this line bundle with $\tilde{Y}$, we have also an embedding $i : \tilde{Y} \hookrightarrow \tilde{X}$ and $\rho \circ i = id_{\tilde{Y}}$. We claim that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated. Indeed, we have

$$H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \xrightarrow{\rho^*} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \xrightarrow{i^*} H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$$

and $i^* \circ \rho^*$ is the identity. It follows that ρ^* is one-to-one (and, obviously, continuous). Since $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ is not separated, it follows that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated as well.

Remark 4. If X is any 1-dimensional infinite Nori string then by Proposition 6 it follows that $H^1(X, \mathcal{O}_X)$ is separated (since X is obviously an increasing union of strongly pseudoconvex domains X_n and the family $\{X_n\}$ is Runge).

Remark 5. It was shown in [13] that if $\{X_n\}$ is an increasing sequence of open sets contained in a Stein manifold X, then $\{X_n\}$ is a Runge family.

It would be interesting to know what happens if we allow X to have singularities.

Acknowledgments: *Both authors were supported by CNCS grant PN-II-ID-PCE-2011-3-0269.*

References

[1] C. Bănică; O. Stănăşilă: Méthodes algébriques dans la théorie globale des espaces complexes. Troisième édition. Gauthier-Villars, Paris, 1977.

[2] M. Brumberg; J. Leiterer: On the compactification of concave ends. *Math. Ann.* **347** (2010), 235–244.

[3] A. Cassa: The cohomology of an exhaustible complex analytic space. *Boll. Un. Mat. Ital. B (6)* **4** (1985), no. 2, 321–341.

[4] M. Colţoiu: Coverings of 1-convex manifolds with 1-dimensional exceptional set. *Comment. Math. Helv.* **68** (1993), 469–479.

[5] M. Colţoiu: On the separation of cohomology groups of increasing unions of (1,1) convex-concave manifolds. *J. Math. Kyoto Univ.* **45** (2005), 405–409.

[6] M. Colţoiu; K. Diederich: A remark on non-Hausdorff cohomology groups of analytic complements. *Math. Ann.* **323** (2002), 485–489.

[7] M. Colţoiu; C. Joiţa: The disk property of coverings of 1-convex surfaces. *Proc. Amer. Math. Soc.* **140** (2012), 575–580.

[8] M. Colţoiu; C. Joiţa: Convexity properties of coverings of 1-convex surfaces. *Math. Z.* **275** (2013), 781–792.

[9] F. Docquier; H. Grauert: Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten. *Math. Ann.* **140** (1960), 94–123.

[10] B. Jennane: Rémarques sur les ouverts localement de Stein. *Math. Ann.* **263** (1983), 371–375.

[11] H. Kazama; S. Takayama: $\partial\overline{\partial}$-problem on weakly 1-complete Kähler manifolds. *Nagoya Math. J.* **155** (1999), 81–94.

[12] C. Laurent-Thiébaut; J. Leiterer: On Serre duality. *Bull. Sci. Math.* **124** (2000), no. 2, 93–106.

[13] H. Laufer: On Serre duality and envelopes of holomorphy. *Trans. Amer. Math. Soc.* **128** (1967), 414–436.

[14] B. Malgrange: La cohomologie d'une variété analytique complexe à bord pseudo-convexe n'est pas nécessairement séparée. *C. R. Acad. Sci. Paris Sér. A-B* **280** (1975), 93–95.

[15] A. Markoe: Runge families and inductive limits of Stein spaces. *Ann. Inst. Fourier* **27** (1977), 117–127.

[16] R. Narasimhan: The Levi problem in the theory of functions of several complex variables. Proc. Internat. Congr. Mathematicians (Stockholm, 1962), 385–388.

[17] J. P. Ramis; G. Ruget; J. L. Verdier: Dualité relative en géométrie analytique complexe. *Invent. Math.* **13** (1971), 261–283.

[18] J.-P. Serre: Un théorème de dualité. *Comment. Math. Helv.* **29** (1955), 9–26.

[19] A. Silva: Rungescher Satz and a condition for Steinness for the limit of an increasing sequence of Stein spaces. *Ann. Inst. Fourier* **28** (1978), 187–200.

[20] G. Trautmann: Abgeschlossenheit von Corandmoduln und Fortsetzbarkeit kohärenter analytischer Garben. *Invent. Math.* **5** (1968) 216–230.

Mihnea Colţoiu
Institute of Mathematics of the Romanian Academy
Research Unit 3
P.O. Box 1-764, Bucharest 014700, ROMANIA
E-mail address: Mihnea.Coltoiu@imar.ro

Cezar Joiţa
Institute of Mathematics of the Romanian Academy
Research Unit 3
P.O. Box 1-764, Bucharest 014700, ROMANIA
E-mail address: Cezar.Joita@imar.ro