On the parametrization of germs of two-dimensional singularities *

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Abstract

We consider a germ of a 2-dimensional complex singularity (X, x_0), irreducible at x_0 and F the exceptional divisor of a desingularization. We prove that if there exists a normal isolated singularity (Z, z_0) with simply connected link and a surjective holomorphic map $f : (Z, z_0) \to (X, x_0)$ then all irreducible components of F are rational and if all irreducible components of F are rational then there exists a surjective holomorphic map $f : (\mathbb{C}^2, 0) \to (X, x_0)$.

1 Introduction

Let (X, x_0) be a germ of a 2-dimensional complex singularity which is irreducible at x_0. Our purpose in this paper is to study the existence of a holomorphic surjective parametrizations of (X, x_0). J.E. Fornæss and R. Narasimhan [5] studied the Fermat surface $F_n = \{(x, y, z) \in \mathbb{C}^3 : x^n + y^n + z^n = 0\}$, $n \geq 3$, which has an isolated singularity at 0 and proved that there is no surjective map of germs $(Y, y_0) \to (F_n, 0)$ where $\mathcal{O}_{Y,y_0}$ (the local ring at y_0) is factorial. Note that the singularity $(F_n, 0)$ is obtained by the analytic contraction of a curve of genus > 0 to a point.

On the other hand, D. Prill (in [14]) studied quotient isolated singularities and showed that a quotient isolated singularity (X, x_0) corresponds precisely to a holomorphic surjective germ map $h : (\mathbb{C}^n, 0) \to (X, x_0)$, $n = \dim_{x_0} X$, such that h is finite. In this paper we drop the condition that h is finite, so we allow h to have positive dimensional fibers, and we prove, if $\dim X = 2$, the following result:

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*Mathematics Subject Classification (2010): 32B10, 32B30, 32C45.

Key words: two dimensional singularity, quotient singularity, proper modifications

Theorem. *Let (X, x_0) be a germ of a 2-dimensional singularity which is irreducible at x_0 and let F be the exceptional divisor of the desingularization of X.*
1) If there exists a normal isolated singularity (Z, z_0) with simply connected link and a surjective holomorphic map $f : (Z, z_0) \to (X, x_0)$, then all irreducible components of F are rational.
2) If all irreducible components of F are rational then there exists a surjective holomorphic map $f : (\mathbb{C}^2, 0) \to (X, x_0)$.

In the statement of the theorem we do not assume that (X, x_0) is a normal singularity (not even isolated), however passing from arbitrary singularities that are irreducible at x_0 (not even isolated) to normal ones is immediate by lifting the surjective parametrization to the normalization. Note that we do not assume that Z is 2-dimensional.

We would like to emphasize that we are working in the analytic category and we consider only the complex topology and not the Zariski one.

We would like to remark that in our result is essential to work with surjective morphisms of *germs* of complex spaces. Otherwise the conclusion does not hold as the following example shows: let C be a smooth compact complex curve of genus at least 1 and let $\pi : \mathcal{L} \to C$ be a negative line bundle on C. We let $p : \Omega \to C$ to be the universal covering space (hence Ω is either $\mathbb{C}$ or the unit disk in $\mathbb{C}$) and we are pulling back $\mathcal{L}$ through p. We obtain the trivial line bundle on Ω. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \Omega \times \mathbb{C} & \xrightarrow{\phi} & \mathcal{L} \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Omega & \xrightarrow{p} & C \end{array}$$

where ϕ is the natural map from the pull-back of $\mathcal{L}$ to $\mathcal{L}$.

We blow-down the zero section of $\mathcal{L}$ and we obtain a complex surface X, with only one singular normal point, and a holomorphic contraction map $g : \mathcal{L} \to X$. Then $f := g \circ \phi : \Omega \times \mathbb{C} \to X$ is a surjective holomorphic map.

According to Brieskorn [2], the only non-regular normal and factorial two-dimensional singularity is the germ at 0 of the surface $\{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + z^5 = 0\}$. This singularity can be resolved by 8 successive blow-ups and is in fact a quotient singularity. Thus, according to the above Theorem, any germ of a two-dimensional singularity (X, x_0), irreducible at x_0, and whose exceptional divisor contains a curve of genus > 0, cannot be parametrized surjectively by a two-dimensional normal factorial singularity.

2 Preliminaries

Throughout this paper all complex spaces are assumed to be reduced. By "open set" we understand open in the analytic topology.

Definition 1. Suppose that X and Y are complex spaces. A proper holomorphic map $f : X \to Y$ is called a proper modification if there exist thin analytic subsets $X_1 \subset X$ and $Y_1 \subset Y$ such that $f : X \setminus X_1 \to Y \setminus Y_1$ is biholomorphic.

For the proof of the following proposition see, for example, [1], page 98.

Proposition 2. *If X and Y are smooth complex surfaces and $f : X \to Y$ is a proper modification then f is a locally finite (with respect to Y) sequence of blow-ups at reduced points.*

For the following Proposition, see [7].

Proposition 3. *Let X and Y be complex spaces such that Y is normal and $f : X \to Y$ be a proper modification. Then there exists an analytic subset A of Y such that* $\operatorname{codim}_Y(A) \geq 2$ *and $f : X \setminus f^{-1}(A) \to Y \setminus A$ is a biholomorphism.*

The following theorem is Corollary 4.2 in [4].

Theorem 4. *Let $\pi : X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of π at t_0. Assume that* $\dim X_{t_0} = 1$. *Let $\sigma : \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is Stein, then there exists an open neighborhood Ω of t_0 such that $(\pi \circ \sigma)^{-1}(\Omega)$ is Stein.*

The following proposition shows that a smooth exceptional rational curve in a complex surface, has a neighborhood which is biholomorphic to a neighborhood of the zero section in the normal bundle. For the proof see [6], page 365 or, for a recent survey, see [3].

Proposition 5. *Suppose that X is a smooth complex surface and $A \subset X$ a complex curve such that A is biholomorphic to $\mathbb{P}^1$ and A has negative self-intersection, $A \cdot A = -k$, where $k \geq 1$. Let $\mathcal{O}(-k)$ be the degree $-k$ line* *bundle on $\mathbb{P}^1$. Then there exists an open neighborhood U of the zero section of $\mathcal{O}(-k)$, an open neighborhood V of A and a biholomorphism $f : U \to V$ which is the identity on A.*

Definition 6. Suppose that (X, x_0) is the germ of an isolated singularity embedded in $\mathbb{C}^n$ for some $n \geq 1$. We denote by $B_\varepsilon \subset \mathbb{C}^n$ the ball of radius ε centered at z_0, where $\varepsilon > 0$ is small enough. Then $K := \partial B_\varepsilon \cap X$ is called the link of the singularity (X, x_0).

Remark. The fundamental group of K, $\pi_1(K)$, does not depend on the local embedding and, if $\pi_1(K) = 0$, $B_\epsilon \cap X$ provides a fundamental system of open neighborhoods of x_0 such that $(B_\epsilon \cap X) \setminus \{x_0\}$ is simply connected.

3 The results

Let (X, x_0) be a germ of two-dimensional singularity, irreducible at x_0. As we have already noticed, we may assume that (X, x_0) is normal. Let $\pi : \tilde{X} \to X$ be a desingularization of X and $F = \pi^{-1}(x_0)$ be the exceptional divisor of $\tilde{X}$ such that F is with normal crossings (see [12]). This means that
a) all irreducible components are smooth
b) the intersection of any two irreducible components is transversal.

After performing a finite number of blow-ups we may assume that F satisfies also:
c) through any point pass at most two irreducible components
d) the intersection of any two irreducible components is at most one point

We say in this case that $\tilde{X}$ is a very good resolution.

As we mentioned in the introduction, our main result is the following:

Theorem 7.
1) If there exists a normal isolated singularity (Z, z_0) (not necessarily 2-dimensional) with simply connected link and a surjective holomorphic map $f : (Z, z_0) \to (X, x_0)$, then all irreducible components of F are rational.
2) If all irreducible components of F are rational then there exists a surjective holomorphic map $f : (\mathbb{C}^2, 0) \to (X, x_0)$.

Remark. In dimension ≥ 3 there is a large class of examples of normal isolated singularity with simply connected link. For example:
- set-theoretic complete intersection in $\mathbb{C}^n$ of dimension ≥ 3, see [8] and [9].

- singularities obtained by contracting an exceptional $\mathbb{P}^1$ in manifolds of dimension ≥ 3 (for examples of embeddings of $\mathbb{P}^1$ as an exceptional set, see [11]). Because the codimension is ≥ 2, all these singularities are not factorial. Indeed, since $\mathbb{P}^1$ is 1-dimensional, it has a neighborhood that is embeddable in $\mathbb{C}^n \times \mathbb{P}^m$, see [13]. Therefore there are hypersurfaces near $\mathbb{P}^1$ that meet $\mathbb{P}^1$ at finitely many points. The projections of these hypersurfaces give the non-factoriality of the local ring obtained after contraction.
- normal isolated singularities (Z, z_0) such that the dimension of the Zariski tangent space of Z at z_0, is less than $2\dim Z - 1$, see [9].

For infinitely many examples of non-factorial isolated hypersurface singularities of dimension 3, see [15].

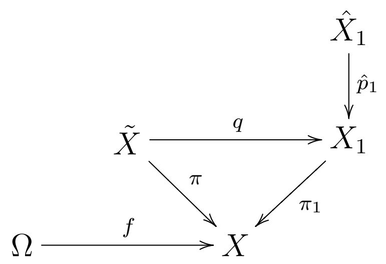
Proof of 1). Let $F = \bigcup_{j=1}^n F_j$ be the decomposition of F into irreducible components. We assume that F_1 has genus at least 1 and we put $L_1 := \bigcup_{j=2}^n F_j$.

Let X_1 be the complex space obtained from $\tilde{X}$ by blowing-down L_1 and $G_1 \subset X_1$ be the image of F_1 through the contraction map $q : \tilde{X} \to X_1$. We have then a proper modification map $\pi_1 : X_1 \to X$ and $(\pi_1)^{-1}(x_0) = G_1$. Let $p : \hat{G}_1 \to G_1$ be the universal covering of G_1.

Since F_1 has genus greater than or equal to 1 it follows that $\hat{G}_1$ is Stein.

We consider an open neighborhood $W \subset X_1$ of G_1 that has a continuous retract $r : W \to G_1$ onto G_1 and the fiber product of p and r. In this way we obtain a complex surface $\hat{X}_1$ that contains $\hat{G}_1$ as a closed subspace and a covering map $\hat{p} : \hat{X}_1 \to W$ that extends p. Using Theorem 4 we deduce that, after shrinking W, we may assume that $\hat{X}_1$ is Stein. As we work at the level of germs we may suppose that $X_1 = W$.

Let's assume now, by reductio ad absurdum, that we have a surjective holomorphic map $f : (Z, z_0) \to (X, x_0)$ where (Z, z_0) is a normal singularity with simply connected link, $\dim Z \geq 2$. We assume that f is defined on a neighborhood Ω of $z_0 \in Z$. Hence we have the following diagram:



We consider now the fiber product $\Omega \times_X X_1$ and the corresponding projections $P_1 : \Omega \times_X X_1 \to \Omega$, $P_2 : \Omega \times_X X_1 \to X_1$. Since π_1 is proper, we have that P_1 is proper. We denote by S the dominant irreducible component of $\Omega \times_X X_1$ relative to P_1. We consider the restrictions of P_1 and P_2 to S and we denote them by P_1 and P_2 as well. Then $P_1 : S \to \Omega$ and $P_2 : S \to X_1$ are holomorphic maps and P_1 is proper and biholomorphic over $\Omega \setminus f^{-1}(x_0)$. (It turns out that one can show that P_2 is surjective and maps surjectively the exceptional set of S onto the exceptional set of X_1; the details are completely similar to those given in the Remark bellow.)

We apply now Proposition 3 and we deduce that we can find a closed analytic subset A of of Ω, $z_0 \in A \subset f^{-1}(x_0)$, such that $\mathrm{codim}(A) \geq 2$ and $P_1 : S \setminus P_1^{-1}(A) \to \Omega \setminus A$ is a biholomorphism. We define the holomorphic map $g : \Omega \setminus A \to X_1$ by $g = P_2 \circ (P_1)^{-1}$. Then we have $\pi_1 \circ g = f$. We assumed that the link of (Z, z_0) is simply connected. As $\mathrm{codim}(A) \geq 2$ it follows that $\Omega \setminus A$ is simply connected and therefore g lifts to a holomorphic map $\hat{g} : \Omega \setminus A \to \hat{X}_1$. However Ω is normal and we have seen that $\hat{X}_1$ is Stein. This implies, by Riemann's second extension theorem, that $\hat{g}$ extends holomorphically to Ω, which implies in turn that g extends holomorphically to Ω. In particular we obtain that g extends continuously at z_0.

The surjectivity of $f : (Z, z_0) \to (X, x_0)$ (i.e. at the level of germ) implies that for every neighborhood Ω_1 of $z_0 \in Z$, its image $f(\Omega_1)$ is a neighborhood of $x_0 \in X$. It follows then that for every neighborhood Ω_1 of $z_0 \in Z$ we have that $g(\Omega_1)$ is a neighborhood of $\pi_1^{-1}(x_0)$. As $\pi_1^{-1}(x_0)$ has pure dimension 1 we obtain a contradiction with the fact that g extends continuously at z_0. $\square$

Remark. If $\dim Z = 2$ (and hence Z is smooth, by Mumford's theorem, see [12]) the proof of this implication can be simplified and we do not need to construct the Stein covering space $\hat{X}_1$. The proof goes as follows: we consider the fiber product $\Omega \times_X \tilde{X}$ and the corresponding projections $Q_1 : \Omega \times_X \tilde{X} \to \Omega$, $Q_2 : \Omega \times_X \tilde{X} \to \tilde{X}$. We denote by $\mathcal{S}$ the irreducible component of $\Omega \times_X \tilde{X}$ that dominates Ω relative to Q_1. We consider the restrictions of Q_1 and Q_2 to $\mathcal{S}$ and we denote them by Q_1 and Q_2 as well. Clearly Q_1 is a proper modification map. We notice now that Q_2 is surjective. Indeed, if y is a point in F we choose C an irreducible germ of curve in $\tilde{X}$ such that $C \cap F = \{y\}$. Let C_1 be the germ of an irreducible component through z_0 of $f^{-1}(\pi(C))$ which is not contained in $f^{-1}(x_0)$ and $C_2 \subset \mathcal{S}$ be the proper transform of C_1. Then $Q_2(C_2) = C$ and therefore $y \in Q_2(Q_1^{-1}(z_0))$. Let $\rho : \tilde{\mathcal{S}} \to \mathcal{S}$ be a desingularization. Because $\tilde{\mathcal{S}}$ and Z are smooth, by Proposition 2, we have that $Q_1 \circ \rho : \tilde{\mathcal{S}} \to \Omega$ is a finite sequence of blow-ups. In particular all irreducible components of $(Q_1 \circ \rho)^{-1}(z_0)$ are rational. As Q_2 maps $(Q_1 \circ \rho)^{-1}(z_0)$ onto F it follows that all irreducible components of F are rational as well.

We now move to the proof of *2)* in statement of Theorem 7. We need some preliminary results.

The following Lemma follows from the fact that one can construct a connected path inside the graph of irreducible components of F that visits each such component.

Lemma 8. *If F is a connected 1-dimensional complex space and $F = L_1 \cup L_2 \cup \cdots \cup L_n$ is its decomposition into irreducible components then there exists a word $L_{j_1} L_{j_2} \cdots L_{j_p}$ (i.e. a finite ordered sequence of elements of the set $\{L_1, L_2, \cdots, L_n\}$; this set is called the alphabet) such that*
1) for every $k \in \{1, 2, \ldots, n\}$ there exists $q \in \{1, 2, \ldots, p\}$ such that $j_q = k$,
2) $L_{j_q} \cap L_{j_{q+1}} \neq \emptyset$ for every $q \in \{1, 2, \ldots, p-1\}$,
3) $L_{j_q} \neq L_{j_{q+1}}$ for every $q \in \{1, 2, \ldots, p-1\}$.

Definition 9. A connected 1-dimensional complex space Z is called a string of $\mathbb{P}^1$ if its decomposition into irreducible components can be written as $Z = L_1 \cup L_2 \cup \cdots \cup L_n$ such that:
- each L_j is biholomorphic to $\mathbb{P}^1$
- $\# L_j \cap L_{j+1} = 1$
- $L_j \cap L_k = \emptyset$ for $|j - k| \geq 2$.

For the next definition, see [1], page 91.

Definition 10. Suppose that Y is a smooth complex surface and $Z \subset Y$ is a 1-dimensional complex subspace. Z is called a Hirzebruch-Jung string if Z is a string of $\mathbb{P}^1$ and, given $Z = L_1 \cup L_2 \cdots L_n$ its decomposition into irreducible components as above we have:
- $L_i \cdot L_i \leq -2$ for every $i \in \{1, 2 \ldots, n\}$,
- $L_i \cdot L_{i+1} = 1$ for every $i \in \{1, 2 \ldots, n-1\}$.

For Proposition 11, see [1], [2], or [10].

Proposition 11. *Suppose that Y is a smooth complex surface and $Z \subset Y$ is a Hirzebruch-Jung string. Then Z is exceptional in Y and the singularity*

(X,x_0) obtained by blowing-down Z is a cyclic quotient-type singularity. In particular there exists a proper surjective morphism of germs $f:(\mathbb{C}^2,0)\to(X,x_0)$.

For a complete classification of 2-dimensional quotient-type singularities, see [2].

We consider now $\psi:\mathbb{P}^1\to\mathbb{P}^1$, defined by $\psi([z_0:z_1])=[z_0^2:z_1^2]$. Then ψ is a ramified covering with two sheets and the ramification sets is $\{[0:1],[1:0]\}$. The pull-back of $\mathcal{O}(-k)$ through ψ is $\mathcal{O}(-2k)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}(-2k) & \xrightarrow{\phi} & \mathcal{O}(-k) \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \end{array} \qquad (*)$$

Remarks. 1) ϕ is also a ramified covering with two sheets and the ramification sets is the union of the fibers over $[0:1]$ and $[1:0]$.

2) For every open neighborhood U of the zero-section of $\mathcal{O}(-2k)$ we have that $\phi(U)$ is an open neighborhood of the zero section of $\mathcal{O}(-k)$.

The proof of the following proposition is based on the plumbing construction.

Proposition 12. *Suppose that X is a smooth complex surface and $F\subset X$ is a compact connected 1-dimensional complex subspace such that all its irreducible components are rational and have negative self-intersection. Then there exist a smooth complex surface Y together with a Hirzebruch-Jung string $Z\subset Y$ and a holomorphic map $f:Y\to X$ such that $f(Z)=F$ and, for every open neighborhood $U\subset Y$ of Z, we have that $f(U)$ is a neighborhood of F.*

Proof. Let $F=L_1\cup L_2\cup\cdots\cup L_n$ be the decomposition of F into irreducible components. After performig a finite number of blow-ups we may assume that F_j, $j=1,2,\ldots,n$, satisfy conditions a) - d) mentioned at the beginning of this section.

Let $-k_j=L_j\cdot L_j$. Then $k_j\geq 1$.

We choose $U_j\subset X$ an open and relatively compact neighborhood of L_j (open in the analytic topology) which is biholomorphic to a neighborhood

V_j of the zero section of $\mathcal{O}(-k_j)$ and we denote by $\chi_j : V_j \to U_j$ this biholomorphism which is the identity on L_j (the existence of V_j is guaranteed by Proposition 5). For each j we consider $\phi_j : \mathcal{O}(-2k_j) \to \mathcal{O}(-k_j)$ the morphism given by (*).

By shrinking U_j, if necessary, we may assume that:
- $\overline{U}_j \cap \overline{U}_k = \emptyset$ if $L_j \cap L_k = \emptyset$,
- $\overline{U}_j \cap \overline{U}_k \cap \overline{U}_l = \emptyset$ if $j,\ k,\ l$ are distinct.
- if $L_j \cap L_k \neq \emptyset$ and $j \neq k$ then $\chi_j^{-1}(U_j \cap U_k)$ contains no ramification point of ϕ_j (this is possible if ϕ_j are chosen generically enough),
- if $L_j \cap L_k \neq \emptyset$ and $j \neq k$ then $(\chi_j \circ \phi_j)^{-1}(U_j \cap U_k) = W^k_{j,1} \cup W^k_{j,2}$, where $\overline{W^k_{j,1}}$ and $\overline{W^k_{j,2}}$ are disjoint and, on each $W^k_{j,1}$ and $W^k_{j,2}$, ϕ_j is a biholomorphism onto $\chi_j^{-1}(U_j \cap U_k)$.

Let $L_{j_1} L_{j_2} \cdots L_{j_p}$ be the word given by Lemma 8. For each $s = 1, 2 \ldots, p$ we consider $\Omega_s = (\chi_{j_s} \circ \phi_{j_s})^{-1}(U_{j_s})$. Our surface is obtained by "gluing" together Ω_s. Namely we set

$$Y = \left(\bigsqcup_{s=1}^{p} \Omega_s \right) \Big/ \sim$$

where $\sim$ identifies $W^{j_{s+1}}_{j_s,1}$ with $W^{j_s}_{j_{s+1},2}$ via $(\chi_{j_s} \circ \phi_{j_s})^{-1} \circ (\chi_{j_{s+1}} \circ \phi_{j_{s+1}})$.

Notice that in each Ω_s we have the following "distinguished" open subsets: $W^{j_{s-1}}_{j_s,1}$, $W^{j_{s-1}}_{j_s,2}$, $W^{j_{s+1}}_{j_s,1}$, $W^{j_{s+1}}_{j_s,2}$ and the ones "affected" by the identification $\sim$ are $W^{j_{s-1}}_{j_s,2}$ and $W^{j_{s+1}}_{j_s,1}$. It may happen that $j_{s-1} = j_{s+1}$ and hence $W^{j_{s-1}}_{j_s,1} = W^{j_{s+1}}_{j_s,1}$ and $W^{j_{s-1}}_{j_s,2} = W^{j_{s+1}}_{j_s,2}$. We note that $\overline{W^{j_{s+1}}_{j_s,1}} \cap \overline{W^{j_{s-1}}_{j_s,2}} = \emptyset$. Indeed:
- if $j_{s-1} \neq j_{s+1}$ this holds because $\overline{U}_{j_{s-1}} \cap \overline{U}_{j_s} \cap \overline{U}_{j_{s+1}} = \emptyset$,
- if $j_{s-1} = j_{s+1}$ this holds because $\overline{W^k_{j,1}} \cap \overline{W^k_{j,2}} = \emptyset$ for $j \neq k$.

Let $f : Y \to X$ defined by $f_{|\Omega_s} = \chi_{j_s} \circ \phi_{j_s}$. Taking into account the definition of $\sim$ and using the fact that $W^{j_{s+1}}_{j_s,1} \cap W^{j_{s-1}}_{j_s,2} = \emptyset$ we deduce that f is a well-defined holomorphic map.

If Λ_s is the zero section in Ω_s then $\Lambda_s \cdot \Lambda_s \leq -2$. Given the properties of the word $L_{j_1} L_{j_2} \cdots L_{j_p}$, we deduce that $Z := \bigcup_{s=1}^{p} \Lambda_s$ is a Hirzebruch-Jung string in Y.

At the same time for every open neighborhood $U \subset Y$ of Z, we have that $f(U)$ is a neighborhood of F since each ϕ_j has a similar property. □

Remark. Outside the exceptional divisor, the holomorphic map $f : Y \to X$ constructed above has finite fibers. However f is not proper and hence it is not a finite map.

To finish the proof of the second statement of Theorem 7 we just have to combine Propositions 11 and 12. Indeed, we apply Proposition 12 to $\tilde{X}$ and the exceptional divisor F and we obtain a smooth complex surface $\tilde{Y}$, a Hirzebruch-Jung string $Z \subset \tilde{Y}$, and a holomorphic map $\tilde{Y} \to \tilde{X}$. We blow-down Z and we obtain a normal singularity (Y, y_0). The contraction map $(\tilde{Y}, Z) \to (Y, y_0)$ gives a biholomorphism $\tilde{Y} \setminus Z \to Y \setminus \{y_0\}$ and therefore we obtain a map $Y \setminus \{y_0\} \to X$. Because X is Stein and Y is normal, by Hartogs' extension theorem, this map extends to a map $Y \to X$. Proposition 12 implies that we obtain a surjective morphism of germs $(Y, y_0) \to (X, x_0)$. Proposition 11 gives a surjective morphism of germs $(\mathbb{C}^2, 0) \to (Y, y_0)$ and therefore we obtain a surjective morphism of germs $(\mathbb{C}^2, 0) \to (X, x_0)$.

Remark. One possible attempt to prove the above proposition is to apply the the standard plumbing construction as follows: start with the word $L_{j_1} L_{j_2} \cdots L_{j_p}$ and the corresponding string of $\mathbb{P}^1$, "glue" together V_{j_1}, V_{j_2}, $\ldots$,V_{j_p} in a straightforward manner obtaining in this way a surface Y, and map biholomorphically each V_{j_s} in Y over the corresponding U_{j_s} in X.

There are two issues with this attempt. Suppose, for example, that $F = L \cup L_1 \cup L_2 \cup L_3$ where L, L_j are biholomorphic to $\mathbb{P}^1$, L_1, L_2, L_2 are pairwise disjoint, $L \cap L_j = \{a_j\}$ and the word $L_1 L L_2 L L_3$. Then the curve in the Hirzebruch-Jung string that corresponds to L_2 in this word will intersect the adjacent curves (these adjacent curves correspond to the two copies of L in the word) in two distinct points that must be both mapped in a_2.

On the other hand, it might happen that an irreducible component of F has self intersection -1 and then the corresponding copies in Y would have self-intersection -1 as well. However in a Hirzebruch-Jung string all components have self-intersection ≤ -2.

Working with ramified coverings and twistings solves both these issues since the plumbing is over non-ramified points.

Acknowledgments: We would like to thank the referee for very useful remarks.

Both authors were supported by CNCS grant PN-II-ID-PCE-2011-3-0269.

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