The Levi problem in the blow-up *

M. Colţoiu and C. Joiţa

Abstract

We prove that a locally Stein open subset of the blow-up of $\mathbb{C}^n$ at a point is Stein if and only if it does not contain a subset of the form $U \setminus A$ where A is the exceptional divisor and U is an open neighborhood of A. We also study an analogous statement for locally Stein open subsets of line bundles over $\mathbb{P}^n$.

1 Introduction

Let $U \subset \mathbb{C}^n$ be an open set which is locally Stein. Then $-\log d$ is a plurisubharmonic function on U (d denotes here the distance to the boundary of U) and therefore by Oka's theorem (see [6]) U is Stein. A similar result holds (see Fujita [2]) if U is an open locally Stein subset of $\mathbb{P}^n$, $U \neq \mathbb{P}^n$.

If U is a locally Stein open subset of $\mathbb{P}^n \times \mathbb{C}$ (which can be identified with $\mathcal{O}(0)$ over $\mathbb{P}^n$) then it was proved in [1] that U is Stein if and only if U does not contain any compact fiber $\mathbb{P}^n \times \{x\}$ for some $x \in \mathbb{C}$.

On the other hand, S. Yu. Nemirovskii studied in [7] the Levi problem for open sets U in $\tilde{\mathbb{C}}^n$, the blow-up of $\mathbb{C}^n$ at a point, in the particular case when the intersection of U with the exceptional set is strongly pseudoconvex. In this particular case he uses the fact that there exists a smooth strongly plurisubharmonic function in a neighborhood of $\overline{U}$ and therefore one can apply a theorem of A. Takeuchi [8] to deduce that U is Stein. In this context let us remark that if $\pi : F \to C$ is a negative line bundle over a compact complex curve and $D \subset\subset F$ is a smoothly bounded domain then C (identified with the zero section of F) cannot be contained in ∂D, since otherwise F would be topologically trivial. Therefore if D, as above, is locally Stein it

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follows easily from this remark that it is Stein. It seems unknown what happens if D is locally Stein, ∂D is not smooth, $C \subset \partial D$ and the genus of C is greater or equal to 1.

In this paper we consider locally Stein open subsets of the blow-up $\tilde{\mathbb{C}}^n$ of $\mathbb{C}^n$ at a point. However one can identify $\tilde{\mathbb{C}}^n$ with $\mathcal{O}(-1)$, the holomorphic line bundle of degree -1 over $\mathbb{P}^{n-1}$ and one can consider the more general case $X = \mathcal{O}(r)$, for $r < 0$ or $r > 0$. We want to decide under what additional geometrical conditions a locally Stein open subset of X is Stein. In this direction we prove:

Theorem 1. *Let $X = \mathcal{O}(r)$ be the degree r line bundle on $\mathbb{P}^n$ and let $\Omega \subset X$ be a locally Stein open subset of X.*

1) If $r < 0$ then Ω is Stein if and only it does not contain an open subset of the form $U \setminus A$ where $A = \mathbb{P}^n$ is the zero section and U is an open neighborhood of A.

2) If $r > 0$ then Ω is Stein if and only it does not contain a neighborhood of the section at infinity.

2 Proof of Theorem 1

The proof of the theorem is based on the Lemmas 1 and 2 bellow.

Lemma 1. *Let $F : Z \to Y$ be a holomorphic line bundle over a complex manifold Y and let Z_0 be its zero section. If $Z \setminus Z_0$ is Stein then Y is also Stein.*

This is a particular case of a more general result (see Theorem 5, page 151 in [5]).

Definition 1. *Suppose that M is a complex manifold, A is an analytic subset of M, $\operatorname{codim}(A) > 0$, and D is an open subset of $M \setminus A$. A point $z \in \partial D \cap A$ is called removable along A if there exists an open neighborhood U of z such that $U \setminus A \subset D$.*

Lemma 2. *Let A be a closed analytic subset of a Stein manifold M, $\operatorname{codim}(A) > 0$, and $D \subset M \setminus A$ be an open subset. If D is locally Stein at every point $z \in \partial D \setminus A$ and if no boundary point $z \in \partial D \cap A$ is removable along A then D is Stein.*

This lemma is due to Grauert and Remmert [3] (see also Ueda [9]).

We begin now the proof of Theorem 1 and we start with a few notations. Let $\pi : \mathcal{O}(r) \to \mathbb{P}^n$ be the vector bundle projection, $z_0, z_1 \ldots, z_n$ be the coordinate functions in $\mathbb{C}^{n+1}$, and, for $k = 0, \ldots, n$, we let $U_k = \{[z] = [z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n : z_k \neq 0\}$. We denote by $\psi_k : \pi^{-1}(U_k) \to U_k \times \mathbb{C}$ the local trivializations. It follows that

$$(\psi_j \circ \psi_k^{-1})([z], \lambda) = \left([z], \frac{z_k^r}{z_j^r}\lambda\right) \quad \forall [z] = [z_0 : z_1 : \cdots : z_n] \in U_j \cap U_k. \quad (1)$$

We will define now a holomorphic map $F : (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} \to \mathcal{O}(r)$ as follows. We set $W_k = \{(z, \lambda) \in (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} : z_k \neq 0\}$ for $k = 0, \ldots, n$ and we define

$$F(z, \lambda) = \psi_k^{-1}\left([z], \frac{\lambda}{z_k^r}\right) \quad \forall\ (z, \lambda) \in W_k.$$

We have to check of course that F is well defined. However it follows from (1) that $(\psi_j \circ \psi_k^{-1})([z], \frac{\lambda}{z_k^r}) = ([z], \frac{\lambda}{z_j^r})$ and hence $\psi_k^{-1}([z], \frac{\lambda}{z_k^r}) = \psi_j^{-1}([z], \frac{\lambda}{z_j^r})$. As the map $(z, \lambda) \in W_k \longrightarrow ([z], \frac{\lambda}{z_k^r}) \in U_k \times \mathbb{C}$ is surjective, it follows that $F_{|W_k} : W_k \to \pi^{-1}(U_k)$ is surjective as well. We claim that F is a local trivial fibration with fiber $\mathbb{C}^*$ and the transition functions are linear on each fiber. In other words there exists a holomorphic line bundle $\tilde{F} : Z \to \mathcal{O}(r)$ such that if we denote by Z_0 its zero section, then $Z \setminus Z_0 = (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}$ and $F = \tilde{F}_{|(\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}}$.

We define $\Phi_k : W_k \to \pi^{-1}(U_k) \times \mathbb{C}^*$, $\Phi(z, \lambda) = (F(z, \lambda), z_k)$. We will show that Φ_k is invertible and we will compute $\Phi_j \circ \Phi_k^{-1}$.

Note that if we set

$$\tilde{\Phi}_k : W_k \to (U_k \times \mathbb{C}) \times \mathbb{C}^*, \quad \tilde{\Phi}_k(z, \lambda) = \left(([z], \frac{1}{z_k^r}\lambda), z_k\right)$$

$$\chi_k : (U_k \times \mathbb{C}) \times \mathbb{C}^* \to \pi^{-1}(U_k) \times \mathbb{C}^*, \quad \chi_k((z, \lambda), \mu) = (\psi_k^{-1}(z, \lambda), \mu)$$

then $\Phi_k = \chi_k \circ \tilde{\Phi}_k$.

It is easy to see that $\tilde{\Phi}_k$ is invertible and its inverse is

$$\tilde{\Phi}_k^{-1}\left(([z], \lambda), \mu\right) = (\frac{\mu}{z_k} z, \lambda \mu^r). \quad (2)$$

Obviously χ_k is invertible and its inverse is

$$\chi_k^{-1}(\zeta,\mu) = (\psi_k(\zeta),\mu). \tag{3}$$

Therefore Φ_k is invertible and from (1), (2) and (3) we deduce that

$$\forall (\zeta,\mu) \in \pi^{-1}(U_j \cap U_k), \text{ if } \pi(\zeta) = [z] = [z_0 : \cdots : z_n] \text{ then}$$

$$(\Phi_j \circ \Phi_k^{-1})(\zeta,\mu) = (\zeta, \frac{z_j}{z_k}\mu)$$

and our claim is proved.

Let Ω be an open subset of $\mathcal{O}(r)$ which is locally Stein but it is not Stein. It follows from Lemma 1 that, as an open subset of $\mathbb{C}^{n+1} \times \mathbb{C}$, $F^{-1}(\Omega)$ is locally Stein at every point of $(\partial F^{-1}(\Omega)) \setminus (\{0\} \times \mathbb{C})$ and is not Stein. From Lemma 2 we conclude that there exists $\lambda_0 \in \mathbb{C}$ such that $(0,\lambda_0) \in (\partial F^{-1}(\Omega)) \cap (\{0\} \times \mathbb{C}) \subset \mathbb{C}^{n+1} \times \mathbb{C}$ is removable along $\{0\} \times \mathbb{C}$ and therefore there exists $\epsilon > 0$ such that

$$F^{-1}(\Omega) \supset \{(z,\lambda) \in (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C} : |z_j| < \epsilon \ \forall\ j = \overline{0,n} \text{ and } |\lambda - \lambda_0| < \epsilon\}.$$

Part 1: $r > 0$.

We have to show that Ω contains a neighborhood of the section at infinity. This is the same thing as showing that for every $[z] \in \mathbb{P}^n$ and every $k \in \{0,1,\dots,n\}$ such that $[z] \in U_k$ there exists an open set, V, in $\mathbb{P}^n$ and $M \in (0,\infty)$ such that $[z] \in V$ and $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : |\lambda| > M\}$.

Let $[\tilde{z}]$ be a fixed point in U_k and let T be a real number such that $T > \max\left\{\frac{|\tilde{z}_j|}{|\tilde{z}_k|} : j = \overline{0,n}\right\}$ (in particular $T > 1$). We set $V := \left\{[z] \in U_k : \frac{|z_j|}{|z_k|} < T\right\}$ which is an open neighborhood of $[\tilde{z}]$. Let $\lambda_1 \in \mathbb{C}$ and $M \in \mathbb{R}$ be such that $\lambda_1 \neq 0$, $|\lambda_1 - \lambda_0| < \epsilon$ and $M > \frac{|\lambda_1| T^r}{\epsilon^r}$. We claim that $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : |\lambda| > M\}$. Indeed, let $([w],\nu) \in V \times \{\lambda \in \mathbb{C} : |\lambda| > M\}$ and let μ be a complex number such that $\mu^r = \frac{\lambda_1}{\nu}$. We set $z := \frac{\mu}{w_k} w \in \mathbb{C}^{n+1} \setminus \{0\}$ (z depends only on $[w]$ and not on a representative of this class). In particular $z_k = \mu \neq 0$. We note that, for every $j \in \{0,\dots,n\}$, $|z_j| = |\mu| \frac{|w_j|}{|w_k|} < |\mu| T$. However $|\mu| = \left(\frac{|\lambda_1|}{|\nu|}\right)^{\frac{1}{r}} < \left(\frac{|\lambda_1|}{M}\right)^{\frac{1}{r}} < \frac{\epsilon}{T}$ and therefore $|z_j| < \epsilon$. It follows that $(z,\lambda_1) \in F^{-1}(\Omega)$ and hence $F(z,\lambda_1) \in \Omega$. As $(z,\lambda_1) \in W_k$ we have that $F(z,\lambda_1) = \psi_k^{-1}([z], \frac{\lambda_1}{z_k^r})$ and hence $\psi_k(F(z,\lambda_1)) = ([z], \frac{\lambda_1}{z_k^r}) = ([w],\nu)$

Part 2: $r < 0$.

We have to show that Ω contains an open subset of the form $U \setminus A$ where A is the zero section and U is an open neighborhood of A. That is, we must show that for every $[z] \in \mathbb{P}^n$ and every $k \in \{0, 1, \ldots, n\}$ such that $[z] \in U_k$ there exists an open set, V, in $\mathbb{P}^n$ and $\delta \in (0, \infty)$ such that $[z] \in V$ and $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$.

Let $[\tilde{z}]$ be a fixed point in U_k and let T be a real number such that $T > \max\left\{\frac{|\tilde{z}_j|}{|\tilde{z}_k|} : j = \overline{0, n}\right\}$. Let $\lambda_1 \in \mathbb{C}$ and $\delta \in \mathbb{R}$ be such that $\lambda_1 \neq 0$, $|\lambda_1 - \lambda_0| < \epsilon$ and $\delta < \left(\frac{\epsilon}{T}\right)^{-r} |\lambda_1|$. We claim that $\psi_k(\Omega \cap \pi^{-1}(U_k)) \supset V \times \{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$. Indeed let $([w], \nu) \in V \times \{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\}$. and let μ be a complex number such that $\mu^{-r} = \frac{\nu}{\lambda_1}$ (hence $\mu \neq 0$). It follows that $|\mu|^{-r} < \frac{\delta}{|\lambda_1|} < \left(\frac{\epsilon}{T}\right)^{-r}$ and therefore $|\mu| < \frac{\epsilon}{T}$. Let $z = \frac{\mu}{w_k} w$. In particular $z_k = \mu$, hence $z_k \neq 0$ and therefore $(z, \lambda_1) \in W_k$. For every $j = 0, 1, \ldots, n$ $|z_j| = \frac{|\mu \cdot w_j|}{|w_k|} \leq |\mu| T < \epsilon$. We deduce that $(z, \lambda_1) \in F^{-1}(\Omega) \cap W_k$. Therefore $F(z, \lambda_1) \in \Omega \cap \pi^{-1}(U_k)$ and $F(z, \lambda_1) = \psi_k^{-1}([z], \frac{\lambda_1}{z_k^r})$, which implies that $\psi_k(F(z, \lambda_1)) = ([z], \frac{\lambda_1}{z_k^r}) = ([w], \frac{\lambda_1}{\mu^r}) = ([w], \nu)$.

Remarks: 1) We used in the proof the fact that the pull-back of $\mathcal{O}(r)$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is trivial, but we had to work with a fixed trivialization in order to apply the Grauert-Remmert removability theorem (Lemma 2).

2) The blow-up of a Stein manifold at a point is not infinitesimally homogeneous and therefore the results in [4] cannot be applied in our situation.

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Mihnea Colţoiu
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest 014700
ROMANIA
E-mail address: Mihnea.Coltoiu@imar.ro

Cezar Joiţa
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest 014700
ROMANIA
E-mail address: Cezar.Joita@imar.ro