Dedicated to the memory of Gabriela Kohr

Polynomial convexity properties of closure of domains biholomorphic to balls

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Abstract

We discuss the connections between the polynomial convexity properties of a domain biholomorphic to ball and its closure.

1 Introduction

A classical theorem of Runge states that for every simply connected open subset U of $\mathbb{C}$, the restriction morphism $\mathcal{O}(\mathbb{C}) \to \mathcal{O}(U)$ has dense image. As usual, the topology on the space of holomorphic functions is the topology of uniform convergence on compacts. We say then that U is Runge in $\mathbb{C}$. This is not longer true in $\mathbb{C}^n$ for $n \geq 2$. It was shown in [13], [14], [15] that there are open subsets of $\mathbb{C}^n$ that are biholomorphic to a polydisc and are not Runge in $\mathbb{C}^n$. E. F. Wold proved in [16] that there are Fatou-Bieberbach domains that are not Runge and hence any open subset of $\mathbb{C}^n$, $n \geq 2$, is biholomorphic to a non-Runge open subset of $\mathbb{C}^n$. In [5] it was given an example of a bounded open subset of $\mathbb{C}^3$ which is biholomorphic to a ball and it is not Runge in any strictly larger open subset of $\mathbb{C}^3$.

In this short paper, motivated by [9], which in turn is based on [7], we want to discuss the possible connections between the polynomial convexity properties of $f(B^n)$ and $\overline{f(B^n)}$ where $f : B^n \to \mathbb{C}^n$ is biholomorphic map onto its image. More precisely we will show that, in general, there is no such connection.

2 Results

We start be recalling a few basic notions.

Definition 1. Let M be a complex manifold. By $\mathcal{O}(M)$ we will denote the set of holomorphic functions defined on M. If $K \subset M$ is a compact subset we denote by $\widehat{K}^M$ the holomorphically convex hull of K,

$$\widehat{K}^M = \{z \in M : |f(z)| \leq \sup_{x \in K} |f(x)|, \ \forall f \in \mathcal{O}(M)\}.$$

K is called holomorphically convex in M if $\widehat{K}^M = K$.

If $M = \mathbb{C}^n$, then $\widehat{K}^{\mathbb{C}^n}$ is the same as the polynomially covex hull of K,

$$\{z \in M : |f(z)| \leq \sup_{x \in K} |f(x)|, \ \forall \text{ polynomial function } f\}.$$

Definition 2. If M is a Stein manifold and U is a Stein open subset then U is called Runge in M if the restriction morphism $\mathcal{O}(M) \to \mathcal{O}(U)$ has dense image

It is well-known, see e.g. [8], that, in the above setting, the following statements are equivalent:

1. U is Runge in M.
2. For every compact set $K \subset U$ we have $\widehat{K}^U = \widehat{K}^M$.
3. For every compact set $K \subset U$ we have $\widehat{K}^M \subset U$.

We recall that a Fatou-Bieberbach domain is a proper open subset of $\mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$. We will need the precise statement of the main theorem of [16] mentioned in the introduction. This is the following.

Theorem 3. *There exits a Fatou-Bieberbach domain $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ which is Runge in $\mathbb{C} \times \mathbb{C}^*$ but not in $\mathbb{C}^2$.*

We will move now to our discussion of the closure of domains in $\mathbb{C}^n$ that are biholomorphic to a ball. We denote by B^n the unit ball in $\mathbb{C}^n$ centered at the origin. We will begin with some remarks.

Remark 4.

- If U is a bounded Runge open subset of $\mathbb{C}$ then it is simply connected and hence biholomorphic to a disc. In general $\overline{U}$ might not be holomorphically convex. It is easy to give such an example. However, if U has smooth boundary, then $\overline{U}$ is holomorphically convex.
- If $n \geq 2$ on can construct a bounded Runge open subset of $\mathbb{C}^n$ biholomorphic to a ball and with smooth boundary such that $\overline{U}$ is not holomorphically convex. One possible construction is the following: start with $F : B^2 \to \mathbb{C}^2$ biholomorphic onto its image such that $F(B^2)$ is not Runge in $\mathbb{C}^2$. Let $B(0, r) \subset \mathbb{C}^2$ be the ball centered at the origin and of radius r. It is easy to see that if r is small enough then $F(B(0, r))$ is Runge. Let $r_0 = \sup\{r : F(B(0, r)) \text{ is Runge}\}$. Because an increasing union of Runge domains is Runge as well we have that $r_0 < 1$ and $F(B(0, r_0))$ is Runge. It was noticed in [10] that $\overline{F(B(0, r_0))}$ is not polynomially convex.
- The interior of a polynomially convex compact set is Runge. Hence if one is trying to find $F : B^2 \to \mathbb{C}^2$ which is a biholomorpism onto its image such that $F(B^2)$ is not Runge and $\overline{F(B^2)}$ is polynomially convex then one must have that the interior of $\overline{F(B^2)}$ is strictly larger then $F(B^2)$.

Proposition 5. *Suppose that M is a connected complex manifold, $\overline{\Gamma}$ and $\overline{\Delta}$ two closed sets, U and V two open sets such that $\overline{\Gamma} \subset U \subset \overline{\Delta} \subset V$. Moreover, we assume that there exist an open set $\tilde{U} \subset \mathbb{C}^n$ containing a closed ball $\overline{B}$, a biholomorphism $F : \tilde{U} \to U$ such that $F(\overline{B}) = \overline{\Gamma}$, an open set $\tilde{V} \subset \mathbb{C}^n$ containing a closed polydisc $\overline{P}$, and a biholomorphism $G : \tilde{V} \to V$ such that $G(\overline{P}) = \overline{\Gamma}$. Then there exists an open and dense subset of M which is biholomorphic to a ball and contains $\overline{\Gamma}$.*

Proof. This proposition is simply a consequence of some of the results and the proofs given in [3], [4] and [2]. For the reader's convenience, we we will recall the main steps needed to prove the proposition. Actually in [3] and [2] the authors prove more than density results: they obtain full-measure embeddings.

We recall that a complex manifold M is called taut if for every complex manifold N (in fact it suffices to work with the unit disc in $\mathbb{C}$, see [1]) the space of holomorphic maps from N to M is a normal family.

• It was noticed in [3] that in any complex manifold M there exists $M_1 \subset M$ a Stein, dense, open subset.

• Another remark from [3] is that for any Stein manifold, M_1, there exists $M_2 \subset M_1$ a taut dense open subset.

• It was proved in [3] that in a taut manifold an increasing union of open sets each one biholomorphic to a polydisc is biholomorphic to a polydisc. A similar statement holds for an increasing union of balls instead of polydiscs.

• A consequence of Theorem II.4 in [4] is the following: if $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset U of a complex manifold M, $\overline{\Delta} = F(\overline{P})$ and x is any point in M then there exists an open subset Δ_1 of M, biholomorphic to a polydisc, such that $\overline{\Delta} \cup \{x\} \subset \Delta_1$.

• This last statement implies easily that if $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset U of a complex manifold M and $\overline{\Delta} = F(\overline{P})$ then there exists an increasing sequence of open subsets biholomorphic to polydiscs in M, $\Delta_1 = \Delta \Subset \Delta_2 \Subset \cdots$ such that $\bigcup \Delta_j$ is dense in M. Indeed, it suffices to consider a dense sequence $\{x_k\}_{k \geq 1} \subset M$ and to construct inductively the polydiscs such that $\{x_1, \dots, x_k\} \subset \overline{\Delta}_k$.

It follows then from the previous statements that:

• If M is any complex manifold, $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset U of M and $\overline{\Delta} = F(\overline{P})$ then there exists a dense open subset of M biholomorphic to polydisc that contains $\overline{\Delta}$.

• Lemma 2.1 in [2] implies the following statement: suppose that P is a polydisc in $\mathbb{C}^n$, U is an open subset of P such that there exists $\tilde{U} \subset \mathbb{C}^n$ an open neighborhood of a closed ball $\overline{B}$ and a biholomorphism $F : \tilde{U} \to U$. If $\overline{\Gamma} = F(\overline{B})$ and x is any point in P then there exists an open subset Γ_1 of P, biholomorphic to a ball, such that $\overline{\Delta} \cup \{x\} \subset \Gamma_1$. As before we deduce that there exists an open and dense subset of P that contains $\overline{\Gamma}$.

The conclusion of the proposition is now straightforward. □

The following corollary answers Question 3.19 in [7].

Corollary 6. *There exists $F : B^2 \to \mathbb{C}^2$ wich is biholomorphic onto its image and such that $F(B^2)$ is not Runge in $\mathbb{C}^2$, and that $\overline{F(B^2)}$ is a holomorphically convex compact subset of $\mathbb{C}^2$.*

Proof. Let $\Omega \subset \mathbb{C}^2$ be a Fatou-Bieberbach domain which is not Runge in $\mathbb{C}^2$. Such a domain exists by Theorem 3. Let also $F : \mathbb{C}^2 \to \Omega$ be a biholomorphism.

As Ω is not Runge in $\mathbb{C}^2$, there exists a compact $K \subset \Omega$ such that $\widehat{K}^{\mathbb{C}^2} \not\subset \Omega$. Choose a point $a \in \widehat{K}^{\mathbb{C}^2} \setminus \Omega$. Choose also a ball B and a polydisc P in $\mathbb{C}^2$ such that

$$F^{-1}(K) \subset B \subset \overline{B} \subset P,$$

and an open ball $U \subset \mathbb{C}^2$ such that $\{a\} \cup F(\overline{P}) \subset U$.

We apply now Proposition 5 for $M = U \setminus \{a\}$ and we deduce that there exists a dense open subset Γ of $U \setminus \{a\}$ which is biholomorphic to a ball and contains $F(\overline{B})$. In particular it contains K while it does not contain a. This implies that Γ is not Runge in $\mathbb{C}^2$. The closure of Γ is, of course, $\overline{U}$ which is polynomially convex. □

Proposition 5 and Corollary 6 are geometric in nature in the sense that they are not concerned with the behaviour of the map $F : B^2 \to \mathbb{C}^2$ (except that it is biholomorphic onto its image). Our next theorem exhibits a somehow stranger behaviour of the map.

Theorem 7. *There exists $F : B^2 \to \mathbb{C}^2$ biholomorphic onto its image such that $F(B^2)$ is not Runge in $\mathbb{C}^2$ and for every open set $V \in \mathbb{C}^2$ with $V \cap \partial B^2 \neq \emptyset$ we have $\overline{F(B^2 \cap V)} \supset (\mathbb{C}^2 \setminus F(B))$.*

Before we prove the theorem, we need some preliminaries.

For the following definition, see [11].

Definition 8. A complex manifold M has the density property if every holomorphic vector field on M can be approximated locally uniformly by Lie combinations of complete vector fields.

Manifolds with the density property have been studied in [11] and [12]. In particular one has:

Proposition 9. $\mathbb{C} \times \mathbb{C}^*$ *has the density property.*

The following theorem is a particular case of Theorem 0.2 in [12]. If $M = \mathbb{C}^n$, it is Corollary 2.2 in [6].

Theorem 10. *Suppose that M is a conected Stein manifold that satisfies the density property. Let K be a holomorphically convex compact subset of M and g a metric on M. Suppose also given: ε a positive number, A a finite subset of K, and $\{x_1, \ldots, x_s\}$, $\{y_1, \ldots, y_s\}$ two finite subsets of $M \setminus K$ of same cardinality. Then there exists an automorphism $F : M \to M$ such that:*

1. $\sup_{x \in K} d_g(F(x), x) < \varepsilon$ *where d_g is the distance induced by g,*

2. $F(a) = a$ *and* $\mathrm{d}F(a) = Id$ *for every* $a \in A$,

3. $F(x_j) = y_j$ *for every* $j = 1, \ldots, s$.

We need also the following elementary lemma.

Lemma 11. *Suppose that* U, V, Ω *are connected open subsets of* $\mathbb{C}^n$ *with* $V \Subset U \Subset \Omega$. *Let* $r > 0$ *be such that there exists a ball* $B(x_0, r)$ *of radius* r *with* $B(x_0, r) \subset V$ *and let* δ *be the distance between* $\overline{V}$ *and* ∂U. *If* $F : \Omega \to F(\Omega) \subset \mathbb{C}^n$ *is a biholomorphism onto its image and* $\sup_{x \in \overline{U}} \|F(x) - x\| < \min\{\delta, r\}$ *then* $\overline{V} \subset F(U)$.

Proof. Because $\sup_{x\in\overline{U}} \|F(x) - x\| < \delta$, we get that $F(\partial U) \cap \overline{V} = \emptyset$. In particular $V \subset F(U) \cup (\mathbb{C}^n \setminus \overline{U})$. At the same time $\sup_{x\in\overline{U}} \|F(x) - x\| < r$ implies that $F(x_0) \in B(x_0, r)$ and hence $F(U) \cap V \neq \emptyset$. As V is connected, we deduce that $V \subset F(U)$. Finally, $F(\partial U) \cap \overline{V} = \emptyset$ implies that $\overline{V} \subset F(U)$. □

Proof of Theorem 7. We consider the Fatou-Beiberbach domain $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ given by Theorem 3 which is Runge in $\mathbb{C} \times \mathbb{C}^*$ but not in $\mathbb{C}^2$. Let K be a compact subset of Ω such that $\widehat{K}^{\mathbb{C}^2} \not\subset \mathbb{C} \times \mathbb{C}^*$. Let $F_0 : \mathbb{C}^2 \to \Omega$ be a Fatou-Beiberbach map. Of course we may assume that $F_0(B^2) \supset K$. We fix also a point $a \in K$.

We choose a strictly increasing sequence of open balls, $\{B_s\}_{s\geq -1}$, centered at the origin, such that $\bigcup_s B_s = B^2$ and such that $B_{-1} \supset F_0^{-1}(K)$.

We will construct inductively a sequence of automorphisms $\{H_s\}_{s\geq 0}$ of $\mathbb{C} \times \mathbb{C}^*$ such that, if we set $F_s = H_s \circ \cdots \circ H_0 \circ F_0 \in \mathcal{S}(B^2)$, then the map we are looking for will be $F = \lim_s F_s$. Note that $F(B^2)$ will be also a subset of $\mathbb{C} \times \mathbb{C}^*$ because $\mathbb{C} \times \mathbb{C}^*$ is Stein.

We have to make sure that the sequence converges to a nondegenerate map on B^2. At the same time we would like to have $F_0(B_{-1}) \subset F(B^2)$. If this is the case, we will have $K \subset F(B^2)$ and this will imply that $F(B^2)$ is not Runge in $\mathbb{C}^2$. In fact we will need more that that, namely we would like to have $F_s(\overline{B}_{s-1}) \subset F(B^2)$ for every s. To force this inclusion we will apply Lemma 11. Hence we will introduce a sequence of positive real numbers $\{\varepsilon_s\}_{s\geq 0}$ that will act as the bounds needed in that lemma.

For the remaining property, we will need to introduce an increasing sequence of of finite subsets of B^2, $\{A_s\}_s \in \mathbb{N}$, $A_s \subset A_{s+1}$ that will help "spreading" the image of F.

• We consider $\{x_n\}_{n\geq 1} \subset \partial B^2$ a dense sequence. For each $n \in \mathbb{N}$ we consider $\{x_n^p\}_{p\in\mathbb{N}} \subset B^2$ a sequence that converges to x_n. Moreover we assume that $x_n \neq x_m$ for $n \neq m$ and $x_n^p \neq x_m^q$ for $(n, p) \neq (m, q)$.

• We set H_0 to be the identity and $A_0 = \{a\}$, $\varepsilon_0 = 1$.

• We assume that we have constructed $H_0, \ldots, H_s$, $A_0, \ldots, A_s$, $\varepsilon_0, \ldots, \varepsilon_s$ and that $H_j(a) = a$ for $j \leq s$ and we will construct H_{s+1}, A_{s+1}, and ε_{s+1}.

We choose $T_1^{s+1}, \ldots, T_{s+1}^{s+1}$ pairwise disjoint, finite, subsets of $\mathbb{C} \times \mathbb{C}^*$, , such that for every $j = 1, \ldots, s+1$ we have

◇ $T_j^{s+1} \cap (F_s(\overline{B}_s) \cup F_s(A_s)) = \emptyset$ and

◇ $\bigcup_{z\in T_j^{s+1}} B(z,\frac{1}{s}) \supset \{z\in\mathbb{C}^2\setminus F_s(B_s) : d(z,F_s(\overline{B}_s))\le s\}$.

Here $d(z,F_s(\overline{B}_s))$ stands for the distance between z and the compact set $F_s(\overline{B}_s)$.

After we chose these finite sets T_j^{s+1}, we choose, for each $j=1,\dots,s+1$, a finite subset, A_j^{s+1}, of $\{x_j^p : p\in\mathbb{N}\}$ such that:

◇ $\#A_j^{s+1}=\#T_j^{s+1}$,

◇ $A_j^{s+1}\cap(\overline{B}_s\cup A_s)=\emptyset$,

◇ $\|x_j-x\|<\frac{1}{s}$ for every $x\in A_j^{s+1}$.

We set

$$A_{s+1}=A_s\cup\left(\bigcup_{j=1}^{s+1}A_j^{s+1}\right).$$

Let δ_s denote the distance between $F_s(\overline{B}_{s-1})$ and $\partial F_s(\overline{B}_s)$. $F_s(B_{s-1})$ is an open subset of $\mathbb{C}\times\mathbb{C}^*$. Let $r_s>0$ be such that there exists a ball of radius r_s included in $F_s(B_{s-1})$.

We define

$$\varepsilon_{s+1}:=\frac{1}{2^{s+1}}\min\{\delta_s,r_s,\varepsilon_0\dots,\varepsilon_s\}.$$

Because H_j, $j\le s$, are automorphisms of $\mathbb{C}\times\mathbb{C}^*$ we have that $F_s(B^2)$ is Runge in $\mathbb{C}\times\mathbb{C}^*$ and hence $F_s(\overline{B}_s)$ is holomorphically convex in $\mathbb{C}\times\mathbb{C}^*$. As A_s is a finite set, $F_s(\overline{B}_s\cup A_s)$ is holomorphically convex in $\mathbb{C}\times\mathbb{C}^*$.

We apply Theorem 10 and we deduce that there exists an automorphism H_{s+1} of $\mathbb{C}\times\mathbb{C}^*$ such that

1. $\|H_{s+1}(z)-z\|<\varepsilon_{s+1}$ for every $z\in F_s(\overline{B}_s)$,
2. $H_{s+1}(z)=z$ for every $z\in F_s(A_s)$ (in particular $H_{s+1}(a)=a$),
3. $\mathrm{d}H_{s+1}(a)=I_2$,
4. $H_{s+1}(F_s(A_j^{s+1}))=T_j^{s+1}$ for every $j=1,\dots s+1$.

Note now that property 1 implies that $F=\lim_s F_s$ (where $F_s=H_s\circ\cdots\circ H_0\circ F_0$) is holomorphic and property 3 that it is nondegenerate. Hence F is biholomorphic on B^2. Also property 2, together with Lemma 11, imply that $F_s(\overline{B}_{s-1})\subset F(B^2)$ (in fact it implies that $F_s(\overline{B}_{s-1})\subset F(B_s)$) for every s. In particular $K\subset F(B^2)$ and therefore $F(B^2)$ is not Runge in $\mathbb{C}^2$.

It remains to check that for every $V\in\mathbb{C}^2$ with $V\cap\partial B^2\ne\emptyset$ we have $\overline{F(B^2\cap V)}\supset(\mathbb{C}^2\setminus F(B))$. Fix then such an open set V and a point $p\in\mathbb{C}^2\setminus F(B^2)$. We recall that the sequence $\{x_n\}$ was chose to be dense in ∂B^2. Let $x_j\in V\cap\partial B^2$. Let $m\in\mathbb{N}$ be large enough such that $m>j$, $\|p-a\|<m$, and $B(x_j,\frac{1}{m})\subset V$.

We distinguish now two cases:

a) $p\notin F_m(\overline{B}_m)$. Note that $\|p-a\|<m$ implies, in particular that $d(p,F_m(\overline{B}_m))<m$. According to our choice of T_j^{m+1}, there exists a point $z\in T_j^{m+1}$ such that $\|p-z\|<\frac{1}{m}$. By property 4 in the construction of $\{H_s\}$, there exists $x \in A_j^{s+1}$ such that $H_{m+1}(F_m(x)) = z$. According to the choice of A_j^{s+1}, we have that $\|x_j - x\| < \frac{1}{m}$ and hence $x \in V$. Note also that property 2 in the construction of $\{H_s\}$ implies that $F(x) = z$.

b) $p \in F_m(\overline{B}_m)$. Since $F_{m+1}(\overline{B}_m) \subset F(B^2)$ and $p \notin F(B^2)$, we have that $p \notin F_{m+1}(\overline{B}_m)$. Let $q = H_{m+1}(p)$. It follows that $q \in F_{m+1}(\overline{B}_m)$. At the same time, property 1 in the construction of $\{H_s\}$ implies that $\|q - p\| < \frac{1}{2^{m+1}}$. It follows that $d(p, F_{m+1}(\overline{B}_m)) < \frac{1}{2^{m+1}}$ and therefore $d(p, \partial F_{m+1}(B_m)) < \frac{1}{2^{m+1}}$. Let $v \in \partial F_{m+1}(B_m))$ be such that $\|p - v\| < \frac{1}{2^{m+1}}$. However $\partial F_{m+1}(B_m)) = H_{m+1}(\partial F_m(B_m)$ and we let $u \in \partial F_m(B_m)$ such that $H_{m+1}(u) = v$. We have then $\|u - v\| < \frac{1}{2^{m+1}}$. We use again our choice of T_j^{m+1} and we find a point $z \in T_j^{m+1}$ such that $\|u - z\| < \frac{1}{m}$. Hence $\|p - z\| < \frac{1}{m} + \frac{1}{2^m}$. As above we obtain a point $x \in V$ such that $F(x) = z$.

In both cases we found $x \in V$ such that $\|p - F(x)\| < \frac{1}{m} + \frac{1}{2^m}$. As m can be chosen arbitrarily large, this finishes the proof. $\square$

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