q-completeness with corners of unbranched Riemann domains *

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Abstract

We prove that if we consider $p : Y \to X$ to be an unbranched Riemann domain between two complex spaces with isolated singularities, X is Stein and p is locally q-complete with corners, then Y is q-complete with corners.

1 Introduction

The "local Steinness problem" or the "Levi problem on singular spaces" is one of the most important and difficult problems in several complex variables. A survey concerning the Levi problem on Stein spaces is [5] or [22]. For complex spaces with isolated singularities this problem was solved by Andreotti and Narasimhan in [2]. Namely they proved that if X is a Stein space with isolated singularities and $Y \subset X$ is a locally Stein open subset of X, then Y is Stein.

This result was generalized by Colţoiu and Diederich in [6], as they showed that if $p : Y \to X$ is an unbranched Riemann domain between two complex spaces with isolated singularities such that X is Stein and p is a Stein morphism, then Y is also Stein.

The notions of q-convex and q-complete complex spaces were introduced by Andreotti and Grauert in [1]. They proved finiteness and vanishing theorems for the cohomology of q-convex and q-complete spaces with values in a coherent analytic sheaf.

* Mathematics Subject Classification (2010). Primary: 32F10, 32C55. Secondary: 32C15.

Key Words: q-complete with corners space, unbranched Riemann domain.

The author was supported by CNCS grant PN-II-ID-PCE-2011-3-0269.

The purpose of this paper is to prove the following result which generalizes [6] (p being locally 1-complete with corners corresponds to p being a Stein morphism):

Theorem 1. *Let X and Y be complex spaces with isolated singularities and $p : Y \to X$ an unbranched Riemann domain. Assume that X is Stein and that p is locally q-complete with corners, i.e., each point $x \in X$ has a neighbourhood $V = V(x)$ such that $p^{-1}(V)$ is q-complete with corners. Then Y is q-complete with corners.*

If p is the inclusion map, then the above theorem was proved by Vâjâitu in [23]. Also if X and Y are smooth, Vâjâitu showed that if X is r-complete with corners and p is locally q-complete with corners, then Y is $(q + r - 1)$-complete with corners (see [24]).

Aknowledgements: I am very grateful to Professor Mihnea Colţoiu for suggesting me this problem and for his helpful advice.

2 Preliminaries

In the following all complex spaces are assumed to be reduced and with countable topology. If X be a complex space, then by T_xX we denote the holomorphic tangent space of X at a point $x \in X$. As usual, set $TX := \bigcup_{x \in X} T_xX$. If we are referring to $\mathbb{C}^n$, then the tangent space of $\mathbb{C}^n$ at any point $x \in \mathbb{C}^n$ is just $\mathbb{C}^n$.

Let X be a complex space and U an open subset of X. A (local) chart of X is a holomorphic embedding $i : U \hookrightarrow U'$, where U' is an open subset of $\mathbb{C}^n$ (for some integer $n \geq 1$) such that $i(U)$ is an analytic subset of U' and U and $i(U)$ are biholomorphic (via i).

Now, following Andreotti and Grauert [1], we define the notion of a q-convex function on a complex space.

Let $f : D \to \mathbb{R}$ be a smooth real function defined on some open subset $D \subset \mathbb{C}^n$. If $z_0 \in D$, then we denote by $L(f, z_0)$ the Levi form of f at z_0, namely

$$L(f, z_0)(\xi, \eta) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i \bar{\eta}_j,$$

where $\xi, \eta \in \mathbb{C}^n$. If $\xi = \eta$, then we simply denote the Levi form of f at z_0 by $L(f, z_0)\xi$. A function f is called *q-convex* if its Levi form has at least $n - q + 1$ positive (> 0) eigenvalues at any point of D.

Using local charts the above notion can be generalized to arbitrary complex spaces.

Definition 1. *Let X be a complex space. A function $f : X \to \mathbb{R}$ is called q-convex at a point $x \in X$ if there exists a local chart $i : U \hookrightarrow U'$ of X, $x \in U$ and a smooth real q-convex function $f' : U' \to \mathbb{R}$ such that $f' \circ i = f|_U$.*

The function f is said to be *q-convex on a subset* $W \subset X$ if it is q-convex at every point of W.

An upper semi-continuous function $f : X \to \mathbb{R}$ is said to be an *exhaustion* function on X if the sublevel sets $\{x \in X : f(x) < c\}$ are relatively compact for any $c \in \mathbb{R}$.

Definition 2. *A complex space X is said to be q-convex, if there exists a compact subset K of X and a smooth exhaustion function $f : X \to \mathbb{R}$, which is q-convex on $X \backslash K$. If we can choose $K = \emptyset$, then X is said to be q-complete.*

By [9] and [19], the function $f \in C^0(X, \mathbb{R})$ is called *q-convex with corners* if for every point $x \in X$ there is an open neighbourhood U of x and finitely many q-convex functions $f_1, \dots, f_k$ on U such that

$$f|_U = \max(f_1, \dots, f_k).$$

We denote by $\mathrm{F}_q(X)$ the set of all q-convex functions with corners on X.

Definition 3. *A complex space X is said to be q-convex with corners, if there exists a compact subset K of X and an exhaustion function $f : X \to \mathbb{R}$, which is q-convex with corners on $X \backslash K$. If we can choose $K = \emptyset$, then X is said to be q-complete with corners.*

Definition 4. *Let $D \subset \mathbb{C}^n$ be an open subset and $f : D \to [-\infty, \infty)$ an upper semicontinuous function. We say that f is subpluriharmonic if for every relatively compact subset $G \subset\subset D$ and for every pluriharmonic function h defined on a neighbourhood of $\overline{G}$ (i.e., h is locally the real part of a holomorphic function) such that $f|_{\partial G} \le h|_{\partial G}$ we have also $f \le h$ on $\overline{G}$.*

Definition 5. *Let X be a complex space, $f : X \to [-\infty, \infty)$ an upper semicontinuous function and q a positive integer. We say that f is:*

(1) q-plurisubharmonic if for every open set $G \subset \mathbb{C}^q$ and every holomorphic map $g : G \to X$, the function $f \circ g$ is subpluriharmonic;

(2) strongly q-plurisubharmonic if for every $\theta \in C_0^\infty(X, \mathbb{R})$ there exists $\epsilon > 0$ such that the function $f + \epsilon\theta$ is q-plurisubharmonic on X.

For a complex space X, let $\mathrm{P}_q(X)$ be the set of all q-plurisubharmonic functions on X and $\mathrm{SP}_q(X)$ the set of strictly q-plurisubharmonic functions.

In the literature (see [10], [11], [14], [16], [17], [21]) the concepts of subpluriharmonicity and (strongly) q-plurisubharmonicity are defined in various ways. For example, in [14], a function defined on an open subset $D \subset \mathbb{C}^n$ and with values in $[-\infty, \infty)$ is called q-plurisubharmonic $(1 \leq q \leq n)$ in D if it is upper semicontinuous and if it is subpluriharmonic on the intersection of every q-dimensional complex plane with D. Using local embeddings we can gene-ralize this definition to arbitrary complex spaces. Fujita proved in [11] that for X smooth the above notions coincide. This happens also in the singular case for $q = 1$ (see [12]). For $q > 1$, but only for continuous functions, this result was announced by Popa-Fischer in [20] and proved in [21].

The next approximation result was proved by Bungart [3] for open subsets of some $\mathbb{C}^n$, but Matsumoto [17] remarked that it is also true for complex manifolds.

Theorem 2. *Let X be a complex manifold and $f : X \to \mathbb{R}$ a continuous strongly q-plurisubharmonic function. Then for an arbitrary continuous function $\delta : X \to (0, \infty)$ there exists a function $\widetilde{f} \in F_q(X)$ such that $|\widetilde{f} - f| < \delta$.*

In [7] Colţoiu and Mihalache proved the following result.

Theorem 3. *Let X be a 1-convex complex space. Then X carries a strongly plurisubharmonic exhaustion function $\Phi : X \to [-\infty, \infty)$. Moreover, Φ can be chosen $-\infty$ exactly on the exceptional set S of X and real analytic outside S.*

We give now a criterion for q-completeness with corners of Vâjâitu [24].

Proposition 1. *Let X be a complex space and $\Phi \in F_q(X)$ such that for every $c \in \mathbb{R}$ the set $X_c := \{\Phi < c\}$ is q-complete with corners. Then X is q-complete with corners.*

From [4] we quote:

Lemma 1. *Consider X to be a complex space, $A \subset X$ an analytic subset and $f \in F_q(A)$. Then for every $\eta \in C^0(A, \mathbb{R})$, $\eta > 0$ there is an open neighbourhood U of A in X and $\widetilde{f} \in F_q(U)$ such that $\left|\widetilde{f}|_A - f\right| \leq \eta$.*

The following result was proved by M. Peternell in [18].

Lemma 2. *Let X be a complex space and $A \subset X$ an analytic subset. Then there exists $h \in C^\infty(X, \mathbb{R})$, $h \geq 0$ such that:*

(a) $\{h = 0\} = A$;

(b) for every $x \in X$ *there exists an open neighbourhood* U *of* x *and a smooth function* $\sigma : U \to \mathbb{R}$ *such that* $\log(h|_{U\setminus A}) + \sigma|_{U\setminus A}$ *is plurisubharmonic.*

The function $\log h$ is locally equal to the sum of a plurisubharmonic function and a smooth function. Such a function is called *almost plurisubharmonic* or *quasi-plurisubharmonic* (see [8]).

3 The Proof

First we assume that $\mathrm{Sing}(X)$ is a finite set and that $p(Y)$ is relatively compact in X. Let us even assume that $\mathrm{Sing}(X) = \{x_0\}$. Otherwise the proof is almost the same.

There are two possibilities:

Case 1: $x_0 \notin p(Y)$. Consider $\pi : \widetilde{X} \to X$ to be a resolution of the singularity x_0. Thus $\widetilde{X}$ is a 1-convex manifold. From Theorem 3 we get a strongly plurisubharmonic exhaustion function $\varphi : \widetilde{X} \to [-\infty, \infty)$ which can be chosen $-\infty$ exactly on the exceptional set of $\widetilde{X}$.

Since $x_0 \notin p(Y)$, we consider $p_1 : Y \to \widetilde{X}$ to be the Riemann domain such that $p = \pi \circ p_1$. Because p is locally q-complete with corners, we have that p_1 is also locally q-complete with corners. We denote by $U \subset\subset \widetilde{X}$ a strongly pseudoconvex neighbourhood of B (see [13]). This neighbourhood can be chosen such that $p_1^{-1}(U)$ is q-complete with corners. Also on $\widetilde{X}$ we have a smooth plurisubharmonic function $\alpha : \widetilde{X} \to \mathbb{R}_+$ such that α is 0 on U and α is ≥ 0 and strongly plurisubharmonic on $\widetilde{X}\setminus\overline{U}$. We choose strongly pseudocovex neighbourhoods V and V' of B with $B \subset U \subset\subset V \subset\subset V'$ and such that $p_1^{-1}(V')$ is q-complete with corners. Also we may assume that $\varphi \geq 0$ outside U. Since $p_1^{-1}(V')$ is q-complete with corners, we denote by $h_1 : p_1^{-1}(V') \to \mathbb{R}_+$ a smooth q-convex with corners exhaustion function.

In what follows, some arguments are similar to those in [6]. However, for reader's convenience, we repeat them here.

Since $K := \overline{p_1(Y)} \subset \widetilde{X}$ there exists a finite number of open balls $\{U_i\}$, such that $Y_i := p_1^{-1}(U_i)$ is q-complete with corners. Thus $-\log \delta_i$ is a q-plurisubharmonic function (see Proposition 7, page 513 in [24]), where δ_i represents the boundary distance measured in the euclidian metric for the Riemann domain $Y_i \to U_i \subset\subset \mathbb{C}^n$. Consider now concentric balls $V_i \subset\subset U_i$ such that K is still covered by V_i. By Lemma 3 in Matsumoto [16] (or M. Peternell [19]) the quotients δ_i/δ_j are bounded on $p_1^{-1}(V_i \cap V_j)$. Therefore, the differences $\log \delta_j - \log \delta_i$ are also bounded.

For each i we can suitably choose a function $\theta_i \in C_0^\infty(V_i)$, $\theta_i \geq 0$ such that the function

$$l(y) := \max_{p_1(y)\in V_i} (-\log \delta_i(y) + \theta_i(p_1(y)))$$

is continuous on Y. Moreover, for a sufficiently large constant $A > 0$, the function

$$q := A \cdot \varphi \circ p_1 + l$$

is strongly q-plurisubharmonic on Y and has the following property:

$$p_1(\{q < c\}) \subset\subset \widetilde{X}, \text{ for every } c \in \mathbb{R}.$$

For the details one should consult Matsumoto [16] (pages 107-108).

The idea is to use Lemma 1 and 2 for $A = \mathrm{Sing}(Y)$. Since Y is with isolated singularities we have that $\dim A = 0$; hence A is q-complete with corners. Lemma 1 gives us an open neighbourhood U of A and a function $g \in \mathrm{F}_q(U)$.

Using Lemma 2, there is a smooth function $h : Y \to [0, \infty)$ such that $\{h = 0\} = A$ and $\log h$ is almost plurisubharmonic. Now select $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ strictly increasing and convex such that $q' := \chi \circ q + \log h \in \mathrm{SP}_q(Y\backslash A)$ and $q' > 1 + g$ on ∂U. By Theorem 2 there is $f \in \mathrm{F}_q(Y\backslash A)$ such that $|f - q'| < 1$ on $Y\backslash A$. Finally we can define a function $\Phi : Y \to \mathbb{R}$ such that $\Phi \in \mathrm{F}_q(Y)$:

$$\Phi := \begin{cases} f \text{ on } Y\backslash U \\ \max(f, g) \text{ on } U\backslash A \\ g \text{ near } A. \end{cases}$$

In order to prove that Y is q-complete with corners, we will apply Proposition 1. We have to show that for every $c \in \mathbb{R}$ the set $\{\Phi < c\}$ is q-complete with corners, i.e., to define for every $c \in \mathbb{R}$ an exhaustion function $\eta_c : \{\Phi < c\} \to \mathbb{R}$ that is q-convex with corners.

Let's denote by g a fixed Riemannian metric on $\widetilde{X}$ and by g^* its pull-back to Y. We define for each $\epsilon > 0$ the set

$$Y_\epsilon := \{y \in Y : \delta(y) > \epsilon\},$$

where δ is the induced boundary distance on the Riemann domain $p_1 : Y \to \widetilde{X}$.

Using the regularization method in Hörmander [15] (see pages 141-142), Colţoiu and Diederich [6] constructed a $\mathcal{C}^2$-function $\phi = \phi_\epsilon : Y_\epsilon \to \mathbb{R}_+$ such that ϕ is a Lipschitz vertical exhaustion function on Y_ϵ and the Levi form of ϕ is bounded from below.

We define the function $\mu := \phi \cdot \widetilde{\alpha} : \{\Phi < c\} \to \mathbb{R}$, where $\widetilde{\alpha} = \alpha \circ p_1$. In order for μ to be well-defined we choose a small enough $\epsilon = \epsilon_c > 0$ such that $\{\Phi < c\} \backslash p_1^{-1}(U) \subset \{\delta > \epsilon\}$.

From the formula

$$L(\mu) = \widetilde{\alpha} L(\phi) + \phi L(\widetilde{\alpha}) + 2\mathrm{Re}(\partial\phi)(\bar{\partial}\widetilde{\alpha})$$

we get that $L(\mu)$ is bounded from below on $\{\Phi < c\}$, μ is a vertical exhaustion function outside $p_1^{-1}(\overline{U})$ and it is identically 0 on $p_1^{-1}(\overline{U})$ Now choose a strongly pseudoconvex neighbourhood $U' \subset\subset U$ of B such that there exists a smooth 1-convex function $\psi : \widetilde{X} \to \mathbb{R}_+$ which enjoys the same properties as the function α. So, for a sufficiently large constant C, the function

$$\mu + C \cdot \psi \circ p_1 : \{\Phi < c\} \to \mathbb{R}$$

is 1-convex on all $\{\Phi < c\}$, relatively exhausting outside $p_1^{-1}(\overline{U})$ and identically 0 on $p_1^{-1}(\overline{U}')$. Now we select a smooth rapidly increasing strictly convex function $\chi : [0, \infty) \to [0, \infty)$, $\chi(0) = 0$ such that

$$\chi \circ (\mu + C \cdot \psi \circ p_1) > h_1 \text{ on } \{\Phi < c\} \cap p_1^{-1}(\partial V).$$

Remember that the goal is to construct a continuous q-convex with corners exhaustion function on $\{\Phi < c\}$. Now consider the maximum between h_1 and $\chi \circ (\mu + C \cdot \psi \circ p_1)$ over $\{\Phi < c\} \cap p_1^{-1}(V)$. Next we extentend this function by $\chi \circ (\mu + C \cdot \psi \circ p_1)$ and we obtain a q-convex with corners function on $\{\Phi < c\}$; let's denote it by λ_c. Therefore, the function $\eta_c := \lambda_c + \dfrac{1}{c - \Phi}$ is a continuous q-convex with corners exhaustion function on $\{\Phi < c\}$. Using Proposition 1 we get that Y is q-complete with corners.

Case 2: $x_0 \in p(Y)$. Let $\pi : \widetilde{X} \to X$ be the local desingularization at x_0. Now consider the fiber product of $p : Y \to X$ and π, that is the set $\widetilde{Y} = \{(y, \tilde{x}) \in Y \times \widetilde{X} : p(y) = \pi(\tilde{x})\}$. We have obtained two projection maps: one on Y which will be denoted by π_1 and another one on $\widetilde{X}$ which will be denoted by p_1. As before let $B := \pi^{-1}(\{x_0\})$ be the exceptional set of $\widetilde{X}$. We have that π_1 is a proper modification of Y at the discrete set $p^{-1}(\{x_0\}) = \{a_n\}_n$ and $p_1 : \widetilde{Y} \to \widetilde{X}$ is a Riemann domain over $\widetilde{X}$. Now the proof is almost the same as in case 1: p_1 is locally q-complete with corners, thus there exists a strongly pseudoconvex neighbourhood $U \subset\subset \widetilde{X}$ of B such that $p_1^{-1}(U)$ is a proper modification at a discrete subset of a q-complete with corners space. We get a strongly plurisubharmonic exhaustion function $\widetilde{\varphi} : \widetilde{Y} \to \mathbb{R}$ which can be chosen $-\infty$ exactly on the exceptional set $\widetilde{B}$ of $\widetilde{Y}$. Finally, for each $c \in \mathbb{R}$, we have on the open set $\{\widetilde{\Phi} < c\} \subset \widetilde{Y}$

a continuous real valued q-convex with corners exhaustion function. Using the fact that π_1 is biholomorphic outside $\widetilde{B}$, we can define $\Phi = \widetilde{\Phi} \circ \pi_1^{-1}$ on $Y \backslash (p^{-1}(\{x_0\})$ and $\Phi = -\infty$ on $p^{-1}(\{x_0\})$. From Proposition 1 we get that Y is q-complete with corners.

Now we only have to prove that it is enough to assume $\mathrm{Sing}(X)$ to be a finite set, i.e., knowing that $\mathrm{Sing}(X)$ is finite we should infer that Y is q-complete with corners. Since X is Stein, let's denote by ψ a continuous strongly plurisubharmonic exhaustion function on X. We write $X = \bigcup_c X_c$, where $X_c = \{\psi < c\}$. Since X has isolated singularities we have that $\mathrm{Sing}(X_c)$ is a finite set. We put $Y_c = p^{-1}(X_c)$ and we get that each Y_c is q-complete with corners. We have that $\Phi := \psi \circ p$ is 1-convex on Y. Since $Y_c = \{\psi \circ p < c\}$ and using Proposition 1 we have that Y is q-complete with corners.

Using the same argument as before, we can assume that $p(Y) \subset\subset X$.

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