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# Hausdorff Dimension of the Limit Set of Countable Conformal Iterated Function Systems with Overlaps

Eugen Mihailescu and Mariusz Urbański

Abstract. We provide lower and upper estimates for the Hausdorff dimension of the limit sets of conformal iterated function systems with overlaps. What is most important is that the alphabet of those system, though countable, is allowed to be infinite. As in [**4**], where the case of finite alphabet was explored, these estimates are expressed in terms of the topological pressure and the function $d(\cdot)$ counting overlaps. However, the infinite case introduces new difficulties. In the case when the function $d(\cdot)$ is constant, we get an exact formula for the Hausdorff dimension. We also prove that in certain cases this formula holds if and only if the function $d(\cdot)$ is constant. In the end, we also give examples of countable IFS with overlaps.

## 1. Introduction

In the paper [**4**] we have dealt with conformal iterated function systems with overlaps. We always assumed there that the alphabet of the system is finite. We have provided lower and upper estimates for the Hausdorff dimension of the limit sets of such systems expressing them in terms of the topological pressure and the function $d$ counting overlaps.

In the present paper we consider conformal iterated function systems with overlaps built over a countable alphabet which is allowed to be infinite. As in [**4**] we work on one fixed system rather than, as it has been common in the theory of iterated function sytems with overlaps, a family of systems. In particular we do not utilize the celebrated transversality condition.

The difficulties of having an infinite alphabet are manifold, particular ones arise from the set $\mathcal{S}(\infty)$ being non-empty. In this paper we deal with them and pay bigger attention to the boundaries of images of generators of the system. This permits us not only to handle the case of infinite alphabet but also to improve the estimates, both lower and upper, of the Hausdorff dimension of the limit set, even in the case of finite alphabet. This is particularly transparent in the last section of our paper when we address the question of when our estimates are optimal.

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The general strategy of our proofs in the present paper develops the one from [**4**], and as we have already said, it goes beyond.

## 2. IFS Preliminaries

Fix an integer $q \geq 1$ and a real number $s \in (0,1)$. Let $X$ be a compact subset of $\mathbb{R}^q$ such that $X = \overline{\mathrm{Int} X}$. Suppose that $V$ is a bounded connected open subset of $\mathbb{R}^q$ such that $X \subset V$. For every closed set $F \subset X$ and every map $g : F \to \mathbb{R}$ put

$$||g|| := \sup\{|g(x)| : x \in F\} \in [0, +\infty].$$

Also fix an arbitrary countable, either finite or infinte, set $E$ called in the sequel an alphabet. A system

$$\mathcal{S} = \{\phi_e : V \to V\}_{e \in E}$$

of $C^{1+\varepsilon}$ conformal injective maps from $V$ to $V$ is called a *conformal iterated function system* if the following conditions are satisfied.

(a)
$$\phi_e(X) \subset X$$
for all $e \in E$.

(b) There exists $s \in (0,1)$ such that
$$||\phi_e'|| = \sup\{|\phi_e'(x)| : x \in X\} \leq s < 1$$
for all $e \in E$. Here, $\phi_e'(x) : \mathbb{R}^q \to \mathbb{R}^q$ is the derivative of the map $\phi_e : V \to V$ evaluated at the point $x$, it is a similarity map, and $|\phi_e'(x)|$ is its operator norm, or equivalently, its scaling factor.

(c) (Refined Distortion Property) There are two constants $L \geq 1$ and $\alpha > 0$ such that
$$\big||\phi_e'(y)| - |\phi_e'(x)|\big| \leq L \|(\phi_e')^{-1}\|^{-1} \|y - x\|^{\alpha}$$
for all $x, y \in V$ and all $e \in E$.

(d) If the alphabet $E$ is infinite, then
$$\lim_{e \to \infty} \mathrm{diam}(\phi_e(X)) = 0.$$

We will usually assume that the system $\mathcal{S}$ is *irreducible* meaning that

$$J_{\mathcal{S}} \not\subset \partial X \quad \text{or equivalently} \quad J_{\mathcal{S}} \cap \mathrm{Int}(X) \neq \emptyset,$$

where $J_S$ is defined next in this Section.

If, in addition, the system $\mathcal{S}$ satisfies the Open Set Condition (OSC), meaning that the interiors of the sets $\phi_e(X)$, $e \in E$, are mutually disjoint (perhaps with different set $X$), then there is a fully systematic and fairly complete account of the fractal properties of its limit set; see [**1**] and [**2**] for example. If the alphabet $E$ is finite, then this condition is equivalent (see [**5**]) to the the Strong Open Set Condition (SOSC) which apart from (OSC) requires that $J_{\mathcal{S}} \cap \mathrm{Int}(X) \neq \emptyset$. If the alphabet $E$ is infinite, then (OSC) does not in general imply (SOSC), see [**6**]. None of these conditions should be mixed with the qualitatively stronger, Strong Separation Condition (SSC) which requires all the sets $\phi_e(X)$, $e \in E$, to be mutually disjoint. Let us however emphasize that we do not assume any sort of such conditions, we assume no open set condition at all, i.e. we do allow any overlaps of the sets $\phi_a(X)$ and $\phi_b(X)$, where $a, b \in E$ with $a \neq b$. This is what makes the theory of such systems qualitatively different from the one with(OSC). Let

$$E^* = \bigcup_{n=0}^{\infty} E^n \text{ and } E^\infty = \{(\omega_n)_{n=1}^{\infty} : \forall (n \geq 1)\, \omega_n \in E\}.$$

If $\tau \in E^\infty$ and $n \geq 0$, we put $\tau|_n = \tau_1 \dots \tau_n$. Now fix $\omega \in E^\infty$ and notice that $\big(\phi_{\omega|_n}(X)\big)_{n=1}^{\infty}$ is a descending sequence of compact sets such that

$$\operatorname{diam}\big(\phi_{\omega|_n}(X)\big) \leq \tilde{D} s^n \operatorname{diam}(X),$$

where the number $\tilde{D} \geq 1$ is due to the fact that we do not assume the set $X$ to be convex. Therefore, the intersection $\bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$ is a singleton, and we denote its only element by $\pi(\omega)$. So, we have defined a map

$$\pi : E^\infty \to X$$

which is Lipschitz continuous if $E^\infty$ is endowed with the metric

$$d_s(\omega, \tau) = s^{|\omega \wedge \tau|},$$

where $\omega \wedge \tau$ is the longest common initial block of $\omega$ and $\tau$ and $|\omega \wedge \tau|$ is its length. We also set $s^\infty = 0$. The *limit set* (or the *attractor*) $J = J_{\mathcal{S}}$ of the system $\mathcal{S}$ is defined to be equal to $\pi(E^\infty)$. Clearly $J_{\mathcal{S}}$ satisfies the following self-conformality condition

$$J_{\mathcal{S}} = \bigcup_{e \in E} \phi_e(J_{\mathcal{S}}),$$

and, by induction,

$$J_{\mathcal{S}} = \bigcup_{|\omega|=n} \phi_\omega(J_{\mathcal{S}})$$

for all $n \geq 1$. Let $\sigma : E^\infty \to E^\infty$ be the (one sided) shift map, i.e.

$$\sigma\big((\omega_n)_{n=1}^{\infty}\big) = \big((\omega_{n+1})_{n=1}^{\infty}\big).$$

By the definition of $J_{\mathcal{S}}$ we have that

$$J_{\mathcal{S}} = \bigcup_{\omega \in E^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X).$$

However the order of the union and the intersection **cannot** be exchanged always, i.e. in general it is not true that $J_{\mathcal{S}} = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in E^n} \phi_\omega(X)$. The former is contained in the latter, and equality holds if, for example the families $\{\phi_\omega(X) : \omega \in E^n\}$ are pointwise bounded for all $n \geq 1$. This is in particular the case if the system $\mathcal{S}$ satisfies the Open Set Condition.

Let now $\psi : E^\infty \to \mathbb{R}$ be the function defined by the following formula,

$$\psi(\omega) = \log |\phi'_{\omega_1}(\pi(\sigma(\omega)))|,\ \omega \in E^\infty.$$

It is well known and easy to prove that the following two lemmas hold.

LEMMA 2.1. The function $\psi : E^\infty \to \mathbb{R}$ is Hölder continuous.

For every $\omega \in E^*$, say $\omega \in E^n$, let us define the (initial) *cylinder* initiated by $\omega$:

$$[\omega] = \{\tau \in E^\infty : \tau|_n = \omega\}.$$

Let also $\mathcal{F}in(\mathcal{S})$ be the set of all $t \in \mathbb{R}$ such that

$$\sum_{e\in E} \|\phi'_\omega\|_\infty^t < +\infty.$$

Obviously this series converges if and only if the following series converges.

$$\sum_{e\in E} \exp(\sup(t\psi|_{[e]})).$$

We say then that the potential $t\psi$ is *summable*. Following [**1**] and [**2**], we denote

$$\theta_{\mathcal{S}} := \inf(\mathcal{F}in(\mathcal{S})).$$

LEMMA 2.2. If $g : E^\infty \to \mathbb{R}$ is Hölder continuous, then there exists a constant $C_g > 0$ such that

$$\left|\sum_{j=0}^{n-1} g(\sigma^j(\omega)) - \sum_{j=0}^{n-1} g(\sigma^j(\tau))\right| \le C_g$$

for all $n \ge 1$ and all $\omega, \tau \in E^\infty$ such that $\omega|_n = \tau|_n$.

Recalling the notation $J = J_S$, let us define now a function $d : J \to \mathbb{N}$ by the following formula,

$$d(x) = \#\{e \in E : x \in \phi_e(J)\}.$$

Immediately from this definition we get the following trivial, but very useful, formula

$$\sum_{e\in E:\, x\in\phi_e(J)} \frac{1}{d(x)} = 1 \tag{2.1}$$

for all $x \in J$. Now let $\kappa : E^\infty \to [1, +\infty)$ be a (not necessarily bounded) Hölder continuous function and, for an arbitrary parameter $t \in \mathbb{R}$, consider the potentials $\psi_{\kappa,t} : E^\infty \to \mathbb{R}$ defined as follows:

$$\psi_{\kappa,t}(\omega) = t\psi(\omega) - \log\kappa(\omega) = t\log|\phi'_{\omega_1}(\pi(\sigma(\omega)))| - \log\kappa(\omega),$$

for all $\omega \in E^\infty$. One can check easily that $\psi_{\kappa,t}$ is Hölder continuous, by using Lemma 2.1 and the Hölder continuity of $\kappa$. Since the function $\log\kappa$ is non-negative, the set

$$\mathcal{F}in^\kappa(\mathcal{S}) = \{t \in \mathbb{R} : \sum_{e\in E} \exp(\sup(\psi_{\kappa,t}|_{[e]})) < +\infty\},$$

that is the set of those parameters $t \in \mathbb{R}$ for which the potential $\psi_{\kappa,t}$ is summable, contains $\mathcal{F}in(\mathcal{S})$. For any $t \ge 0$, let $\mathrm{P}(\psi_{\kappa,t})$ be the *topological pressure*, as defined in [**2**], of the potential $\psi_{\kappa,t}$ with respect to the dynamical system $\sigma : E^\infty \to E^\infty$. Precisely,

$$\mathrm{P}(\psi_{\kappa,t}) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in E^n} \exp\left(\sup\left(\sum_{j=0}^{n-1} \psi_{\kappa,t}\circ\sigma^j|_{[\omega]}\right)\right).$$

This limit exists since the sequence involved is subadditive. Since $\log|\phi'_{\omega_1}(\pi(\sigma(\omega)))| \le \log s < 0$, it is straightforward to check that the function $\mathcal{F}in^\kappa(\mathcal{S}) \ni t \mapsto \mathrm{P}(\psi_{\kappa,t}) \in \mathbb{R}$ is convex, continuous, strictly decreasing, and $\lim_{t\to+\infty} \mathrm{P}(\psi_{\kappa,t}) = -\infty$.

From now on we will frequently denote $\mathrm{P}(\psi_{\kappa,t})$ simply by $\mathrm{P}(t)$. If it will be needed to be more specific, we will write also $\mathrm{P}_{\mathcal{S}}(t)$ or $\mathrm{P}_E(t)$ for $\mathrm{P}(t)$. Define now

$$h_\kappa := \inf\{t \geq 0 : \mathrm{P}(\psi_{\kappa,t}) \leq 0\}.$$

Like with the pressure, we write $h_\kappa(\mathcal{S})$ or $h_\kappa(E)$ if we want to be more specific. If there exists $t \geq 0$ such that $\mathrm{P}(\psi_{\kappa,t}) = 0$, then such a $t$ is unique and is equal to $h_\kappa$. If $t \in \mathcal{F}in^\kappa(\mathcal{S})$, then (see [**2**]) there exists a unique shift-invariant Gibbs (equilibrium) state $\tilde{\mu}_t$ of the Hölder continuous potential $\psi_{\kappa,t} : E^\infty \to \mathbb{R}$. Being shift-invariant it is uniquely characterized by the (Gibbs) property that

$$\tilde{\mu}_t([\omega|_n]) \asymp e^{-\mathrm{P}(t)n} ||\phi'_{\omega|_n}||^t \prod_{j=0}^{n-1} \frac{1}{\kappa(\pi(\sigma^j(\omega)))} \tag{2.2}$$

for every $\omega \in E^\infty$ and every $n \geq 1$. Let

$$\mu_t = \tilde{\mu}_t \circ \pi^{-1}. \tag{2.3}$$

Clearly, $\mu_t(J) = 1$. If $A$ is an arbitrary Borel subset of $J$ and $\mathcal{F} \subset E^*$ is a family of mutually incomparable words, meaning that none is extension of another, such that $\pi^{-1}(A) \subset \bigcup_{\omega \in \mathcal{F}} [\omega]$, then

$$\mu_t(A) \leq \sum_{\omega \in \mathcal{F}} \tilde{\mu}_t([\omega]). \tag{2.4}$$

We say that a set $F \subset X$ is $\mathcal{S}$-invariant if

$$\bigcup_{e \in E} \phi_e(F) \subset F.$$

We say that a Borel probability measure $\mu$ on $X$ is $\mathcal{S}$-invariant if there exists a Borel probability shift-invariant measure $\tilde{\mu}$ on $E^\infty$ such that

$$\mu = \tilde{\mu} \circ \pi^{-1}.$$

Then obviously $\mu(J_{\mathcal{S}}) = 1$. We further say that such a measure $\mu$ is *ergodic* if the measure $\tilde{\mu}$ is ergodic, that is $\mu$ is ergodic if and only if for an $\mathcal{S}$-invariant Borel subset $F$ of $X$ either $\mu(F) = 0$ or $\mu(F) = 1$. Let

$$\partial_\infty X := \bigcup_{\omega \in E^*} \phi_\omega(\partial X).$$

Of course $\partial_\infty X$ is an $\mathcal{S}$–invariant subset of $X$. We shall prove the following.

Lemma 2.3. If $\mathcal{S}$ is a conformal IFS and $\mu$ is a Borel probability $\mathcal{S}$–invariant measure such that $\mu(\partial_\infty X) < 1$, then $\mu(\partial_\infty X) = 0$.

Proof. Since $\mu(\partial X) < 1$, we have that $\tilde{\mu}(\pi^{-1}(\partial X)) < 1$. Since $\sigma(\pi^{-1}(\partial X)) \subset \pi^{-1}(\partial X)$, ergodicity of $\tilde{\mu}$ implies that $\mu(\partial X) = \tilde{\mu}(\pi^{-1}(\partial X)) = 0$. Since $\pi^{-1}(\partial_\infty X) \subset \bigcup_{n=0}^\infty \sigma^{-n}(\pi^{-1}(\partial X))$ and since the measure $\tilde{\mu}$ is $\sigma$-invariant, we thus conclude that $\mu(\partial_\infty X) = 0$. $\square$

Recall from [**2**] that $\mathcal{S}(\infty)$, the *boundary at infinity* of the system $\mathcal{S}$, is defined to consist of all cluster points of all sequences $(x_n)_{n=1}^\infty$, where $x_n \in \phi_{e_n}(X)$ with some

$e_n \in E$, and all elements $e_n$, $n \geq 1$, are mutually distinct. Obviously $\mathcal{S}(\infty)$ is a closed subset of $X$. We put

$$\mathcal{S}^+(\infty) := \bigcup_{\omega \in E^*} \phi_\omega(\mathcal{S}(\infty)).$$

So, $\mathcal{S}^+(\infty)$ is a Borel (in fact $F_\sigma$) $\mathcal{S}$-invariant subset of $X$. We say that the system $\mathcal{S}$ is *small at infinity* if

$$\mu(S(\infty)) = 0$$

for every Borel $\mathcal{S}$-invariant probability measure $\mu$ on $J_{\mathcal{S}}$ such that $\mu(\partial X) = 0$. Let us make the following straightforward observation.

Observation 2.4. Assume that $\mathcal{S}$ is a conformal irreducible IFS. If any of the following conditions holds, then $\mathcal{S}$ is small at infinity.

(a) $\mathcal{S}(\infty) \subset \partial_\infty X$.
(b) $\mathcal{S}(\infty)$ is countable.
(c) $\mathcal{S}(\infty) \cap J_{\mathcal{S}} = \emptyset$
(d) $\mathcal{S}(\infty) = \emptyset$ meaning that the alphabet $E$ is finite.

Proof. To see the sufficiency of (a) just notice that the argument used in the last part of the proof of Lemma 2.3 yields $\mu(\partial_\infty X) = 0$. The sufficiency of either (b), (c), or (d) is obvious. $\square$

Employing the same argument as in the proof of Lemma 2.3, we get the folowing.

Lemma 2.5. If $t \in \mathcal{F}in^\kappa(\mathcal{S})$ and $\mu_t(S(\infty)) = 0$, then $\mu_t(\mathcal{S}^+(\infty)) = 0$. In particular, if $\mathcal{S}$ is small at infinity, then $\mu_t(\mathcal{S}^+(\infty)) = 0$ for a dense set of parameters $t \in (\theta_{\mathcal{S}}, q)$.

## 3. Upper Bound

Following closely the proof of the corresponding theorem in [**4**], which was formulated and proved there assuming that the alphabet $E$ is finite, we shall prove in this section Theorem 3.1. It gives an upper bound for the Hausdorff dimension of the limit set $J_{\mathcal{S}}$ from which an invariant subset is taken away. In [**4**] this set was empty.

Theorem 3.1. Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a conformal iterated function system. Let $H$ be an $\mathcal{S}$-invariant subset of $J_{\mathcal{S}}$. If $k \geq 1$ is an integer satisfying $d(x) \geq k$ for all $x \in J_{\mathcal{S}} \setminus H$, then $\mathrm{HD}(J_{\mathcal{S}} \setminus H) \leq h_k$.

*Proof.* Fix $t > h_k$. Then $\mathrm{P}(t) < 0$ and therefore

$$\sum_{|\omega|=n} ||\phi'_\omega||^t k^{-n} \leq e^{\frac{1}{2}\mathrm{P}(t)n} \tag{3.1}$$

for all $n \geq 1$ large enough, say $n \geq n_0$. For every $\omega \in E^n$ consider the smallest closed ball $B_\omega$ containing $\phi_\omega(X)$. Then

$$\mathrm{diam}(B_\omega) \leq 2\mathrm{diam}(\phi_\omega(X)) \leq 2\tilde{D}\mathrm{diam}(X)||\phi'_\omega||. \tag{3.2}$$

Since $\{B_\omega\}_{\omega\in E^n}$ is a cover of the set $J_{\mathcal S}\setminus H$ by closed balls, in virtue of the $4r$-Covering Theorem (see [**3**]), there exists a set $I_1\subset E^n$ with the following properties.

(a) $B_\omega\cap B_\tau=\emptyset$ for all $\omega,\tau\in I_1$ with $\omega\neq\tau$.
(b) $\bigcup_{\omega\in I_1}4B_\omega\supset J_{\mathcal S}\setminus H$.

Suppose now by induction that the sets $I_1,I_2,\dots,I_l$, $1\le l<k^n$ have been defined with the following properties:

(c) $I_i\cap I_j=\emptyset$ for all $1\le i<j\le l$.
(d) $\forall(1\le j\le l)\ \forall(\omega,\tau\in I_j)\ \omega\neq\tau\Rightarrow B_\omega\cap B_\tau=\emptyset$.
(e) $\forall(1\le j\le l)\ \bigcup_{\omega\in I_j}4B_\omega\supset J_{\mathcal S}\setminus H$.

Because of (c) and (d), each point of $J_{\mathcal S}\setminus H$ belongs to at most $l$ elements of the family $\{B_\omega:\omega\in I_1\cup\ldots\cup I_l\}$. But, as $d\ge k$, and the set $H$ is $\mathcal S$-invariant, each element of $J_{\mathcal S}\setminus H$ belongs to at least $k^n>l$ elements of the family $\{\phi_\omega(J):|\omega|=n\}$, and thus, to at least $k^n>l$ elements of the family $\{\phi_\omega(X):|\omega|=n\}$, and eventually to at least $k^n>l$ elements of the family $\{B_\omega:|\omega|=n\}$. Thus, the family $\{B_\omega:\omega\in E^n\setminus(I_1\cup\ldots\cup I_l)\}$ covers $J_{\mathcal S}\setminus H$, and it therefore follows from the $4r$-Covering Theorem (see [**3**]) that one can find a set $I_{l+1}\subset E^n\setminus(I_1\cup\ldots\cup I_l)$ such that

(f) If $\omega,\tau\in I_{l+1}$ and $\omega\neq\tau$, then $B_\omega\cap B_\tau=\emptyset$.
(g) $\bigcup_{\omega\in I_{l+1}}4B_\omega\supset J_{\mathcal S}\setminus H$.

So, we have constructed by induction a family of sets $I_1,I_2,\dots,I_{k^n}\subset E^n$ such that the conditions (c), (d), and (e) hold with $l=k^n$.

Choose now $1\le j\le k^n$ so that the sum $\sum_{\omega\in I_j}\mathrm{diam}^t(B_\omega)$ is the smallest. Then by (3.2), (3.1) and (c), (d), (e), we get that

$$\begin{aligned}\sum_{\omega\in I_j}\mathrm{diam}^t(4B_\omega)&=4^t\sum_{\omega\in I_j}\mathrm{diam}^t(B_\omega)\le\frac{4^t}{k^n}\sum_{i=1}^{k^n}\sum_{\omega\in I_i}\mathrm{diam}^t(B_\omega)\\&\le 4^tk^{-n}\sum_{|\omega|=n}\mathrm{diam}^t(B_\omega)\le(8\tilde D\mathrm{diam}(X))^t\sum_{|\omega|=n}||\phi_\omega'||^tk^{-n}\\&\le(8\tilde D\mathrm{diam}(X))^te^{\frac12\mathrm P(t)n}.\end{aligned}$$

Denoting by $\mathrm H_t$ the $t$-dimensional Hausdorff measure, because of (e) and since $\mathrm P(t)<0$, we thus conclude that $\mathrm H_t(J_{\mathcal S}\setminus H)=0$; so $\mathrm{HD}(J_{\mathcal S}\setminus H)\le t$. By the arbitrariness of $t>h_k$, this yields $\mathrm{HD}(J_{\mathcal S}\setminus H)\le h_k$. We are done. □

Substituting $H=\emptyset$ in this theorem, we get the following.

Corollary 3.2. Let $\mathcal S=\{\phi_e\}_{e\in E}$ be a conformal iterated function system. If $k\ge1$ is an integer satisfying $d(x)\ge k$ for all $x\in J_{\mathcal S}$, then $\mathrm{HD}(J_{\mathcal S})\le h_k$.

## 4. Lower Bound

In this section we prove the lower bound for the Hausdorff dimension. Although we also follow rather closely the proof of the corresponding result in [**4**], the difference in the formulation and in the proof is even larger than the one pertaining to the upper bound.

THEOREM 4.1. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be an irreducible conformal iterated function system which is small at infinity. If $\hat{\kappa} : J_{\mathcal{S}} \to [1, +\infty)$ is a Hölder continuous function such that $d(x) \le \hat{\kappa}(x)$ for all $x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))$, then $\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \ge h_\kappa$, where $\kappa = \hat{\kappa} \circ \pi : E^\infty \to \mathbb{R}$.

*Proof.* Since the system $\mathcal{S}$ is irreducible, there exists a non-empty finite set $H_0 \subset E$ such that $J_{H_0} \not\subset \partial X$. Fix $H$, an arbitrary finite subset of $E$ containing $H_0$. We first shall prove that

$$\mathrm{HD}(J_H \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \ge h_\kappa(H), \tag{4.1}$$

where, we emphasize, the set $\partial_\infty X$ is understood with respect to the full original system $\mathcal{S}$. Given $\mathcal{F}$, a finite family of mutually incomparable elements of $H^*$, set

$$\mathcal{F}(\emptyset) = \mathcal{F},$$

where $\emptyset$ stands here for the empty word, and define

$$\mathcal{F}_1 := \{\omega_1 \in H : \omega \in \mathcal{F}\},$$

and then, for all $e \in \mathcal{F}_1$,

$$\mathcal{F}(e) := \{\omega \in H^* : e\omega \in \mathcal{F}\}.$$

Observe that

$$\mathcal{F}(e) \ne \emptyset \quad \Leftrightarrow \quad e \in \mathcal{F}_1.$$

Notice also that for each $e \in \mathcal{F}_1$, the family $\mathcal{F}(e)$ consists of mutually incomparable words. Define further by induction,

$$\mathcal{F}(\omega) := \mathcal{F}\big(\omega|_{|\omega|-1}\big)(\omega_{|\omega|}).$$

Of course,

$$\mathcal{F}(\omega) \ne \emptyset \quad \Rightarrow \quad \mathcal{F}(\omega|_n) \ne \emptyset \ \ \forall\, n \le |\omega|.$$

Since $\mathrm{HD}(J_H \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \ge 0$, we may also assume without loss of generality that $h_\kappa(H) > 0$. Then, fix an arbitrary $t \in (0, h_\kappa(H))$. So, $\mathrm{P}_H(t) > 0$, where $\mathrm{P}_H(t)$ is the pressure function generates by the iterated function system $\mathcal{S}_H = \{\phi_e\}_{e\in H}$. Let $\mu_{H,t}$ be the corresponding measure produced in (2.3) for the system $\mathcal{S}_H$. Since the topological support of the measure $\mu_{H,t}$ is equal to $J_H$ and since $J_H \not\subset \partial X$, we conclude that $\mu_{H,t}(\partial X) < 1$. It therefore follows from Lemma 2.3 that $\mu_{H,t}(\partial_\infty X) = 0$. Since the systems $\mathcal{S}$ is small at infinity, we thus conclude that

$$\mu_{H,t}(\partial_\infty X \cup \mathcal{S}^+(\infty)) = 0. \tag{4.2}$$

Consider the restricted function

$$\hat{\kappa}_H =: \hat{\kappa}|_{J_H} : J_H \to [1, +\infty).$$

It is obviously Hölder continuous. In particular, the function $1/\hat{\kappa}_H : J_H \to (0, 1]$ is uniformly continuous, whence there exists $\eta > 0$ so small that

$$\frac{1}{\hat{\kappa}(y)} \le e^{\mathrm{P}_H(t)} \frac{1}{\hat{\kappa}(x)}$$

for all $x, y \in J_H$ with $||y - x|| < \eta$. Since the set $H$ is finite, for every $z \in J_H$ there exists $R(z) \in (0, \eta)$ such that if $B(z, R(z)) \cap \phi_e(J_H) \ne \emptyset$, then $z \in \phi_e(J_H)$. Since the set $J_H$ is compact and $\{B(z,R(z)/2) : z \in J_H\}$ is an open cover of $J_H$, there exists a finite set $F \subset J_H$ such that

$$J_H \subset \bigcup_{z\in F} B(z,R(z)/2). \tag{4.3}$$

Let

$$R_* := \frac{1}{5}\min\{\mathrm{diam}(J), \min\{R(z) : z \in F\}\}.$$

Now fix an arbitrary $x \in J_H$ and $0 < r < R_*$. By (4.3) there exists $z_x \in F$ such that $x \in B(z_x, R(z_x)/2)$. Given a set $B \subset B(x,r)$, we say that a family $\mathcal{F} \subset E^*$ consisting of mutually incomparable words is *properly placed with respect to the triple* $(x,B,r)$, if for all $\omega \in \mathcal{F}$ we have that

$$B \cap \phi_\omega(J_H) \neq \emptyset. \tag{4.4}$$

It immediately follows from this definition, the definition of $R_*$ and the restriction on $r > 0$, that

$$z_x \in \phi_{\omega_1}(J_H) \tag{4.5}$$

for all $\omega \in \mathcal{F}$. In other words

$$\mathcal{F}_1 \subset \{e \in H : z_x \in \phi_e(J_H)\}. \tag{4.6}$$

Now fix an arbitrary $\tau \in H^\infty$, and a family $\mathcal{F} \subset E^*$ which is properly placed with respect to $(x,B,r)$ for some $B \subset B(x,r)$. We then have

$$\begin{aligned}
\Sigma(\mathcal{F}) :&= \sum_{\omega\in\mathcal{F}} e^{-\mathrm{P}_H(t)|\omega|}\frac{1}{\kappa(\omega\tau)}\cdot\frac{1}{\kappa(\sigma(\omega\tau))}\cdot\ldots\cdot\frac{1}{\kappa(\sigma^{|\omega|-1}(\omega\tau))}\\
&\le \sum_{\omega\in\mathcal{F}} e^{-\mathrm{P}_H(t)|\omega|}e^{\mathrm{P}_H(t)}\frac{1}{\hat\kappa(z_x)}\cdot\frac{1}{\kappa(\sigma(\omega\tau))}\cdot\ldots\cdot\frac{1}{\kappa(\sigma^{|\omega|-1}(\omega\tau))}\\
&\le \sum_{\omega\in\mathcal{F}} e^{-\mathrm{P}_H(t)(|\omega|-1)}\frac{1}{d(z_x)}\cdot\frac{1}{\kappa(\sigma(\omega\tau))}\cdot\ldots\cdot\frac{1}{\kappa(\sigma^{|\omega|-1}(\omega\tau))}\\
&= \sum_{e\in\mathcal{F}_1}\frac{1}{d(z_x)}\cdot\sum_{\gamma\in\mathcal{F}(e)} e^{-\mathrm{P}_H(t)|\gamma|}\frac{1}{\kappa(\gamma\tau)}\cdot\frac{1}{\kappa(\sigma(\gamma\tau))}\cdot\ldots\cdot\frac{1}{\kappa(\sigma^{|\gamma|-1}(\gamma\tau))}\\
&= \sum_{e\in\mathcal{F}_1}\frac{1}{d(z_x)}\cdot\Sigma(\mathcal{F}(e)).
\end{aligned} \tag{4.7}$$

If $\omega \in \mathcal{F}(e)$, then we have

$$\emptyset \neq \phi_e^{-1}(\phi_{e\omega}(J_H)\cap B) = \phi_\omega(J_H)\cap\phi_e^{-1}(B)$$

and

$$\phi_e^{-1}(B) \subset B\big(\phi_e^{-1}(x_{B,e}), \mathrm{diam}\big(\phi_e^{-1}(B)\big)\big),$$

where $x_{B,e}$ is an arbitrary point in $\phi_e(J_H)\cap B$, independent of $\omega$. We say that the letter $e$ is *$B$-proper* if

$$\mathrm{diam}\big(\phi_e^{-1}(B)\big) < R_*.$$

We say further by induction that a word $\omega \in H^*$ with $|\omega| \ge 2$ is $B$-proper if $\omega|_{|\omega|-1}$ is $B$-proper, $\omega_{|\omega|} \in \mathcal{F}\big(\omega|_{|\omega|-1}\big)_1$, and $\mathcal{F}(\omega)$ is properly placed with respect to the triple

$$\left(\phi_{\omega_{|\omega|}}^{-1}\big(x_{\phi_{\omega_{|\omega|}}^{-1}\big(\phi_{\omega|_{|\omega|-1}}^{-1}(B),\omega_{|\omega|}\big)}\big), \phi_{\omega_{|\omega|}}^{-1}\big(\phi_{\omega|_{|\omega|-1}}^{-1}(B)\big), \mathrm{diam}\big(\phi_{\omega_{|\omega|}}^{-1}\big(\phi_{\omega|_{|\omega|-1}}^{-1}(B)\big)\big)\right).$$

Let $\mathcal{F}_B \subset H^*$ be the family of all finite words $\omega \in H^*$ such that $\omega$ is not $B$-proper but $\omega|_{|\omega|-1}$ is and $\omega_{|\omega|} \in \mathcal{F}\big(\omega|_{|\omega|-1}\big)_1$. Clearly, $\mathcal{F}_B \subset H^*$ is a maximal antichain, meaning that all it elements are mutually incomparable and their union is equal to $H^\infty$. Expanding (4.5) we get that

$$\Sigma(\mathcal{F}) = \sum_{\omega\in\mathcal{F}_B} \frac{1}{d\big(z_{\phi_{\omega_1}^{-1}(x_{\omega_1})}\big)} \cdot \frac{1}{d\left(z_{\phi_{\omega_2}^{-1}\left(x_{\omega_1\omega_2}\right)}\right)} \cdot \ldots \cdot \frac{1}{d\left(z_{\phi^{-1}_{\omega_{|\omega|-1}}\left(x_{\omega|_{|\omega|-1}}\right)}\right)} \Sigma(\mathcal{F}(\omega)),$$

where we abbreviated

$$x_{\omega|_k} := x_{\phi^{-1}_{\omega|_{k-1}}(B),\omega_k}.$$

Now, we shall prove the following.

**Claim:** If $\Sigma(\mathcal{F}(\omega)) \le M$ for some $M \ge 0$ and all $\omega \in \mathcal{F}_B$, then $\Sigma(\mathcal{F}) \le M$.

Proof. Let $n \ge 1$ be the longest word in $\mathcal{F}_B$. For every $0 \le k \le n-1$ define

$$F_B^k := \{\omega|_k : |\omega| \ge k\} \cup \{\omega \in \mathcal{F}_B : |\omega| < k\}.$$

In particular, $\mathcal{F}_B^0 = \emptyset$. Then for every $0 \le k \le n-1$ put

$$S(k) := \sum_{\omega\in\mathcal{F}_B} \frac{1}{d(z_x)} \prod_{j=1}^{|\omega|} \frac{1}{d\big(z_{\phi_{\omega_j}^{-1}(x_{\omega|_j})}\big)}.$$

If $0 \le k \le n-2$, then by (2.1) and (4.6) we get that

$$\begin{aligned} S(k+1) &= \sum_{\substack{\omega\in\mathcal{F}_B\\|\omega|\le k}} \frac{1}{d(z_x)} \prod_{j=1}^{|\omega|} \frac{1}{d\big(z_{\phi_{\omega_j}^{-1}(x_{\omega|_j})}\big)} + \sum_{\substack{\omega\in\mathcal{F}_B^{k+1}\\|\omega|=k+1}} \frac{1}{d(z_x)} \prod_{j=1}^{k+1} \frac{1}{d\big(z_{\phi_{\omega_j}^{-1}(x_{\omega|_j})}\big)} \\ &= \sum_{\substack{\omega\in\mathcal{F}_B\\|\omega|\le k}} \frac{1}{d(z_x)} \prod_{j=1}^{|\omega|} \frac{1}{d\big(z_{\phi_{\omega_j}^{-1}(x_{\omega|_j})}\big)} + \\ &\qquad + \sum_{\gamma\in\mathcal{F}_B^k} \frac{1}{d(z_x)} \prod_{j=1}^{k} \frac{1}{d\big(z_{\phi_{\gamma_j}^{-1}(x_{\gamma|_j})}\big)} \sum_{e\in\mathcal{F}(\gamma)_1} \frac{1}{d\big(z_{\phi_e^{-1}(x_{\gamma e})}\big)} \\ &\le \sum_{\substack{\omega\in\mathcal{F}_B\\|\omega|\le k}} \frac{1}{d(z_x)} \prod_{j=1}^{|\omega|} \frac{1}{d\big(z_{\phi_{\omega_j}^{-1}(x_{\omega|_j})}\big)} = S_k. \end{aligned}$$

Thus,

$$S_{n-1} \le S_0 = \sum_{\omega\in\mathcal{F}_M^0} \frac{1}{d(z_x)} \le 1.$$

Therefore, $\Sigma(\mathcal{F}) \le MS_{n-1} \le M$. The claim is proved. □

Now we define a special family, which is properly placed with respect to the triple $(x, B(x,r), r)$, with $r \in (0, R_*)$, namely:

$$\mathcal{F}_*(x,r) := \big\{\omega \in H^* : B(x,r) \cap \phi_\omega(J_H) \ne \emptyset,\ \phi_\omega(J_H) \subset B(x,2r), \text{ and } \phi_{\omega|_{|\omega|-1}}(J_H) \not\subset B(x,2r)\big\}.$$

Aiming to apply the claim, we want to estimate from above the number of elements of $\mathcal{F}_*(x,r)(\omega)$ for every $\omega \in \mathcal{F}_*(x,r)$. So, fix such $\omega$ and consider an arbitrary word $\gamma \in \mathcal{F}_*(x,r)(\omega)$. Then $\omega\gamma \in \mathcal{F}_*(x,r)$, and, by the definition of $\mathcal{F}_*(x,r)$, this yields

$$||\phi'_{\omega\gamma}|| \le C_1 r \quad \text{and} \quad ||\phi'_{\omega\gamma|_{|\omega\gamma|-1}}|| \ge c_2^{-1} r \tag{4.8}$$

with some $C_1, C_2 \in (0,+\infty)$ independent of $x$ and $r$. On the other hand since $\omega|_{n-1}$ is $B(x,r)$-proper (with $n = |\omega|$) but $\omega$ is not, and since $\omega_n \in \mathcal{F}_*(x,r)\big(\omega|_{n-1}\big)$, we must have that $\operatorname{diam}(\phi_\omega^{-1}(B(x,r)) \ge R_*$. This implies that $2Kr||\phi'_\omega||^{-1} \ge R_*$. Along with (4.8), this gives,

$$||\phi'_\omega|| \le 2KR_*^{-1} r \le 2C_2KR_*^{-1}||\phi'_{\omega\gamma|_{|\omega\gamma|-1}}|| \le 2C_2KR_*^{-1}||\phi'_\omega|| \cdot ||\phi_\gamma|'_{|\gamma|-1}||.$$

Hence,

$$||\phi'_{\gamma|_{|\gamma|-1}}|| \ge (2C_2K)^{-1}R_*.$$

Consequently, $\#\{\gamma|_{|\gamma|-1} : \gamma \in \mathcal{F}_*(x,r)(\omega)\}$ is bounded above by a constant $M_1$ depending only on the system $\mathcal{S}_H$ and the number $(2C_2K)^{-1}R_*$. Since the set $H$ is finite, we thus conclude that $\#\mathcal{F}_*(x,r)(\omega) \le M_2$, a constant which also depends only on $\mathcal{S}_H$ and $(2C_2K)^{-1}R_*$. In conclusion, there exists a constant $M > 0$ such that

$$\Sigma\mathcal{F}_*(x,r)(\omega) \le M$$

for all $x \in J_H$, all $r \in (0,R_*)$ and all $\omega \in \mathcal{F}_{*,B(x,r)}(x,r)$. Therefore, applying the above Claim, we get that

$$\Sigma\mathcal{F}_*(x,r) \le M. \tag{4.9}$$

Since $\mathcal{F}_*(x,r)$ consists of mutually incomparable words and

$$\pi^{-1}(B(x,r)) \subset \bigcup_{\omega \in \mathcal{F}_*(x,r)} [\omega],$$

we get from (2.4), (2.2), the very first line of (4.7), the first formula of (4.8) (with $\omega\gamma$ being now $\omega$), and (4.9) that

$$\begin{aligned}
\mu_t(B(x,r)) &\preceq \sum_{\omega \in \mathcal{F}_*(x,r)} e^{-\mathrm{P}_H(t)|\omega|}||\phi'_\omega||^t \prod_{j=0}^{|\omega|-1} \frac{1}{\kappa(\pi(\sigma^j(\omega\tau)))} \\
&\le C_1^t r^t \sum_{\omega \in \mathcal{F}_*(x,r)} e^{-\mathrm{P}_H(t)|\omega|} \prod_{j=0}^{|\omega|-1} \frac{1}{\kappa(\pi(\sigma^j(\omega\tau)))} \\
&= C_1^t r^t \Sigma(\mathcal{F}_*(x,r)) \\
&\le MC_1^t r^t.
\end{aligned}$$

Therefore, invoking also (4.2), it follows from the Converse Frostman Lemma (see for example [**3**]) that $\mathrm{H}_t(J_H \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) > 0$; consequently $\mathrm{HD}(J_H \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \ge t$. Since $t > 0$ was an arbitrary number smaller than $h_\kappa$, we thus conclude that formula (4.1) holds.

For the general case fix $0 < t < h_\kappa$ arbitrary. Then $\mathrm{P}(t) > 0$. It then follows from Theorem 2.1.5 in [**2**] that

$$0 < \mathrm{P}(t) = \sup\{\mathrm{P}_H(t) : H_0 \subset H, \text{ finite}\}$$

Therefore, there exists a finite set $H_0 \subset H \subset E$ such that $\mathrm{P}_H(t) > 0$. But this means that $t < h_\kappa(H)$, and further, by (4.1),

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \geq \mathrm{HD}(J_H \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \geq h_\kappa(H) > t.$$

Because of arbitrariness of $t < h_\kappa$, we thus get that

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \geq h_\kappa.$$

The proof is complete. □

If the alphabet $E$ is finite, then $\mathcal{S}(\infty) = \emptyset$ and therefore a simplified version of the above proof gives this.

Theorem 4.2. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a conformal iterated function system with a finite alphabet $E$. If $\hat{\kappa} : J_{\mathcal{S}} \to [1, +\infty)$ is a Hölder continuous function such that $d(x) \leq \hat{\kappa}(x)$ for all $x \in J_{\mathcal{S}}$, then $\mathrm{HD}(J_{\mathcal{S}}) \geq h_\kappa$, where $\kappa = \hat{\kappa} \circ \pi : E^\infty \to \mathbb{R}$.

## 5. Exact Dimensions

The first main theorem in this section is the following.

Theorem 5.1. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be an irreducible conformal iterated function system which is small at infinity. Assume that

$$D := \sup\{d(x) : x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))\}$$

is finite; in particular the supremum becomes a maximum. Then we obtain:

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) = h_D \quad \Leftrightarrow \quad d(x) = D, \ \forall\, x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty)).$$

*Proof.* Assume first that $d(x) = D$ forall $x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))$. Since $\partial_\infty X \cup \mathcal{S}^+(\infty)$ is a closed set, it follows from Theorem 3.1 that

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \leq h_D. \tag{5.1}$$

On the other hand, it directly follows from Theorem 4.1 that $\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \geq h_D$. Along with (5.1) this gives that $\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) = h_D$ completeing this part of the implication.

Now assume that

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) = h_D.$$

Seeking contradiction assume that $d(z) \leq D-1$ for some $z \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))$. Since $z \notin \mathcal{S}^+(\infty)$, there thus exists an open neighborhood $V \subset \mathbb{R}^q$ of $z$ such that $d(x) \leq D-1$ for all $x \in V$. Now, by a refined version of Urysohn's Lemma, there exists a Lipschitz continuous function $\hat{\kappa} : X \to [D-1, D]$ such that

$$\hat{\kappa}(z) = D - 1 \tag{5.2}$$

and $\hat{\kappa}(x) = D$ for all $x \in X \setminus V$. Note that by the definition of $\hat{\kappa}$, we have

$$d(x) \leq \hat{\kappa}(x) \leq D \tag{5.3}$$

for all $x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))$ and

$$\kappa \leq D \tag{5.4}$$

on $E^\infty$. Thus, by Theorem 4.1, we get that

$$h_D = \mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) \geq h_\kappa \geq h_D.$$

Hence,

$$h_\kappa = h_D. \tag{5.5}$$

Let $\tilde{\mu}_D$ be the unique equilibrium (Gibbs) state on $E^\infty$ of the potential $h_d\psi - \log D$. Since $\mathrm{P}(h_D\psi - \log D) = 0$, we have

$$\int_{E^\infty} (h_D\psi - \log D)\, d\tilde{\mu}_D + \mathrm{h}_{\tilde{\mu}_D}(\sigma) = 0, \tag{5.6}$$

where $\mathrm{h}_{\tilde{\mu}_D}(\sigma)$ is the Kolmogorov-Sinai metric entropy of the dynamical system $\sigma : E^\infty \to E^\infty$ with respect to the $\sigma$-invariant measure $\tilde{\mu}_D$. In virtue of the Variational Principle, we also have,

$$\int_{E^\infty} (h_D\psi - \log \kappa)\, d\tilde{\mu}_D + \mathrm{h}_{\tilde{\mu}_D}(\sigma) = \int_{E^\infty} (h_\kappa\psi - \log D)\, d\tilde{\mu}_D + \mathrm{h}_{\tilde{\mu}_D}(\sigma) \leq \mathrm{P}(h_\kappa\psi - \log \kappa) = 0.$$

This combined with (5.6), imply that

$$\int_{E^\infty} (\log D - \log \kappa)\, d\tilde{\mu}_D \leq 0. \tag{5.7}$$

Since the function $\log D - \log \kappa$ is continuous and since the equilibrium state $\tilde{\mu}_D$ (as a Gibbs state of a Hölder continuous function) is positive on non-empty open subsets of $E^\infty$, it follows from (5.7) and (5.4) that $\log \kappa = \log D$ on $E^\infty$. So, $\hat{\kappa} = D$ on $J$ and this contradiction finishes the proof.

□

As an immediate corollary of this theorem we get the following.

Corollary 5.2. Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be an irreducible conformal iterated function system which is small at infinity. Assume that

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) < \mathrm{HD}(J_{\mathcal{S}}).$$

Then:

$$\mathrm{HD}(J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty))) = h_D \iff d(x) = D \;\; \forall\, x \in J_{\mathcal{S}} \setminus (\partial_\infty X \cup \mathcal{S}^+(\infty)).$$

A slightly more involved proof is required to get this.

Theorem 5.3. Assume that $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a conformal iterated function system such that the function $d : J_{\mathcal{S}} \to [1, +\infty]$ is upper semi-continuous at all points of $J_{\mathcal{S}} \cap \mathcal{S}(\infty)$. Let

$$D := \max\{d(x) : x \in J_{\mathcal{S}}\}.$$

Then, with $D$ as above:

$$\mathrm{HD}(J_{\mathcal{S}}) = h_D \iff d(x) = D \;\; \forall\, x \in J_{\mathcal{S}}.$$

*Proof.* If $d(x) = D$ forall $x \in J_{\mathcal{S}}$, then it directly follows from Theorem 3.1 that

$$\mathrm{HD}(J_{\mathcal{S}}) \le h_D.$$

On the other hand, we get from Theorem 4.1 that $\mathrm{HD}(J_{\mathcal{S}}) \ge h_D$. So, $\mathrm{HD}(J_{\mathcal{S}}) = h_D$, and the proof of this part of the implication is complete.

So, assume that

$$\mathrm{HD}(J_{\mathcal{S}}) = h_D.$$

Seeking to obtain a contradiction, assume that $d(z) \le D-1$ for some $z \in J_{\mathcal{S}}$. Since the function $d : X \to [0,+\infty]$ is upper semi-continuous at all points of $J_{\mathcal{S}} \cap \mathcal{S}(\infty)$, there thus exists an open neighborhood $V \subset \mathbb{R}^q$ of $z$ such that $d(x) \le D-1$ for all $x \in V$. Now, by a refined version of Urysohn's Lemma, there exists a Lipschitz continuous function $\hat{\kappa} : X \to [D-1, D]$ such that

$$\hat{\kappa}(z) = D - 1 \tag{5.8}$$

and $\hat{\kappa}(x) = D$ for all $x \in X \setminus V$. Note that by the definition of $\hat{\kappa}$, we have

$$d(x) \le \hat{\kappa}(x) \le D \tag{5.9}$$

for all $x \in J_{\mathcal{S}}$ and

$$\kappa \le D \tag{5.10}$$

on $E^\infty$. Thus, by Theorem 4.1, we get that

$$h_D = \mathrm{HD}(J_{\mathcal{S}}) \ge h_\kappa \ge h_D.$$

Hence,

$$h_\kappa = h_D. \tag{5.11}$$

Now, the rest of the proof is exactly the same as the corresponding part of the proof of Theorem 5.1. □

Although Corollary 5.2 looks clumsier and more technical than elegantly formulated Theorem 5.3, it is Corollary 5.2 which frequently brings more information. Indeed, let us consider the following two examples supporting this claim.

*Example 5.5.* Let $X = [0,1]$ and let $\mathcal{S} = \{\phi_1, \phi_2\}$, where $\phi_1(x) = x/2$ and $\phi_1(x) = (x+1)/2$. Then $\mathcal{S}$ is obviously irreducible small at infinity as it satisfies condition (d) of Observation 2.4. Moreover, $J_{\mathcal{S}} = [0,1]$, $d(x) = 1$ for all $x \in J_{\mathcal{S}} \setminus \{1/2\}$ and $d(1/2) = 2$. So, Theorem 5.3 tells us only that $\mathrm{HD}(J_{\mathcal{S}}) \ne h_2$ whereas Corollary 5.2 tells us that $\mathrm{HD}(J_{\mathcal{S}}) = h_1$.

Although we immediately see anyway that $\mathrm{HD}(J_{\mathcal{S}}) = 1$, this example has its value. Indeed, compare with the following.

*Example 5.6.* Let $X = [-1,1]$. For every $n \in \mathbb{Z} \setminus \{0\}$ let $\phi_n : [-1,1] \to [-1,1]$ be given by the following formula.

$$\phi_n(x) = \frac{n}{|n|}\left(\frac{x}{4n^2} + 1 - \frac{1}{|n|}\right).$$

Then $\phi_n([-1,1]) \subset [-1,1]$, the system $\mathcal{S} := \{\phi_n\}_{n \in \mathbb{Z}\setminus\{0\}}$ is irreducible, $\mathcal{S}(\infty) = \{-1,1\}$, so $\mathcal{S}$ is small at infinity as it satisfies condition (c) of Observation 2.4, and

$d(x) = 1$ for all $x \in J_{\mathcal{S}} \setminus \{0\}$ while $d(0) = 2$. Therefore, Theorem 5.3 tells us only that $\mathrm{HD}(J_{\mathcal{S}}) \neq h_2$ whereas Corollary 5.2 tells us that $\mathrm{HD}(J_{\mathcal{S}}) = h_1$.

The next examples exhibits some strange unexpected phenomena, which may occur when overlaps are allowed, and indicates how our theorem can be used to estimate Hausdorff dimensions of the corresponding limit sets.

*Example 5.7.* Let $X = I = [0,1]$. Let $\phi_1 : X \to X$ be a strictly increasing differentiable (ex. linear) contraction such that

$$\phi_1(0) = 0 \quad \text{and} \quad \phi_1(1) < 1/2.$$

Then define recursively $\phi_{2(n+1)} : X \to X$ to be a strictly increasing differentiable (ex. linear) contraction such that

$$\phi_{2(n+1)}(0) = \phi_{2n+1}(1) \quad \text{and} \quad \phi_{2(n+1)}(1) < 1/2.$$

and

$$\phi_{2n+1}(0) > \phi_{2n}(1) \quad \text{and} \quad \phi_{2n+1}(1) < 1/2.$$

Similarly, let $\phi_{-1} : X \to X$ be a strictly increasing differentiable (ex. linear) contraction such that

$$\phi_0(1) = 1 \quad \text{and} \quad \phi_0(0) > 1/2.$$

Then define recursively $\phi_{2(n+1)} : X \to X$ to be a strictly increasing differentiable (ex. linear) contraction such that

$$1/2 < \phi_{-(n+1)}(1) < \phi_{-n}(0).$$

Consider the system

$$\mathcal{S} = \{\phi_n : n \in \mathbb{Z}\}.$$

Notice that both points $0 = \pi_{\mathcal{S}}(1^\infty)$ and $1 = \pi_{\mathcal{S}}(0^\infty)$ belong to $J_{\mathcal{S}}$. Thus all the points of type $\phi_j(0), \phi_j(1)$, $j \in \mathbb{Z}$, belong to $J_{\mathcal{S}}$. But for the contact points of type $\phi_{2j+1}(1)$, $j > 0$ we see that the preimage counting function $d$ is equal to 2, whereas for all other points in $J_S$ it is equal to 1. Also notice that $S(\infty)$ is countable, thus the system is small at infinity. Moreover, clearly $S$ is irreducible since $J_{\mathcal{S}} \not\subset \partial X$. Thus since these contact points are in $\partial_\infty(X)$, we can apply Corollary 5.2 and obtain that $\mathrm{HD}(J_S) = h_1$.

*Example 5.8.* Let $X = \overline{B}(0,1)$ be the closed unit disk in the complex plane. For every $n \in \mathbb{Z}$ let $\phi_n : \overline{B}(0,1) \to B(0,1)$ be a contracting similarity of the form $z \mapsto a_n z + b_n$, where both $a_n$ and $b_n$ are real and $0 < a_n < 1$. Then

$$\phi_n([-1,1]) \subset (-1,1)$$

and therefore

$$J_{\mathcal{S}} \subset [-1,1],$$

where $\mathcal{S} = \{\phi_n : n \in \mathbb{Z}\}$. We may select the numbers $a_n$ and $b_n$, $n \in \mathbb{Z}$, so that $\phi_n([-1,1]) \cap \phi_k([-1,1]) \neq \emptyset$ if and only if $|n-k| = 1$, and when this does hold then in addition $\phi_n((-1,1)) \cap \phi_k((-1,1)) \neq \emptyset$. We further require that $|a_n| \underset{n\to\infty}{\to} 0$; moreover assume that the sequence $(\phi_n(0))_{n\in\mathbb{Z}}$ is increasing and

$$\lim_{n\to+\infty} \phi_n(0) = 1$$

while

$$\lim_{n \to -\infty} \phi_n(0) = -1$$

In this example $S(\infty)$ being the doubleton $\{-1, 1\}$, it is of course countable, hence the system is small at infinity. From the conditions above we obtain also that $(-1, 1) \subset J_S$. However the function $d(\cdot)$ is jumping in $(-1, 1)$ from the value 1 to 2, and therefore it is not continuous on $J_S$.

By fitting now a well chosen Hölder continuous function $\tilde{\kappa}$ such that $d(x) \leq \tilde{\kappa}(x)$, we obtain from Theorem 4.1 that $\mathrm{HD}(J_{\mathcal{S}}) = 1 \leq h_\kappa$.

*Example 5.9.* Take $X = [0, 1]$. $\phi_{-2} : X \to X$ be an increasing contraction with $\phi_{-2}(0) > 1/2$ and $\phi_{-2}(1) = 1$. Let $\phi_{-1} : X \to X$ be an increasing contraction with $\phi_{-1}(0) = 0$ and $\phi_{-1}(1) < 1/2$. Then for every $n \geq 0$ let $\phi_{2n} : X \to X$ and $\phi_{2n+1} : X \to X$ be two increasing contractions defined inductively such that $\phi_{-1}(1) < \phi_0(0)$ and

$$\phi_{2n-1}(1) < \phi_{2n}(0) < \phi_{2n+1}(0) < \phi_{2n}(1) = \phi_{2n+1}(1) < 1/2.$$

We can arrange this construction so that $\lim_{n\to\infty} b_n = 1/2$, where $b_n := \phi_{2n}(1) = \phi_{2n+1}(1)$. Let

$$\mathcal{S} = \{\phi_k : k \geq -2\}.$$

Then $0, 1 \in J_{\mathcal{S}}$, $\frac{1}{2} = \phi_{-2}(0) \in J_{\mathcal{S}}$, and therefore $b_n \in J_{\mathcal{S}}$. The function $d : J_{\mathcal{S}} \to [1, \infty]$ takes then on the following form

$$d(x) = \begin{cases} 1 & \text{if } \ x \in J_{\mathcal{S}} \cap [0, \phi_{-1}(1)] \\ 1 & \text{if } \ x \in J_{\mathcal{S}} \cap [\phi_{2n}(0), \phi_{2n+1}(0)) \\ 2 & \text{if } \ x \in J_{\mathcal{S}} \cap [\phi_{2n+1}(0), \phi_{2n+1}(1)] \\ 1 & \text{if } \ x = 1/2 \\ 1 & \text{if } \ x \in J_{\mathcal{S}} \cap [\phi_{-2}(0), 1]. \end{cases}$$

In particular $d(b_n) = 2$ for all $n \geq 0$ and $d(1/2) = 1$. Since $\lim_{n\to\infty} b_n = 1/2$ this implies that the function $d : J_{\mathcal{S}} \to [1, \infty)$ is not upper semi-continuous. Taking a Hölder continuous function $\tilde{\kappa} : J_{\mathcal{S}} \to [1, \infty)$ such that $d(x) \leq \tilde{\kappa}$ for all $x \in J_{\mathcal{S}}$, we obtain $\mathrm{HD}(J_S) \geq h_\kappa$ in virtue of Theorem 4.1.

*Example 5.10.* In this example we construct an IFS $S$ where the function $d(\cdot)$ increases indefinitely on $J_{\mathcal{S}}$, but the set $J_{\mathcal{S}}$ is not compact, as frequently is the case for infinite alphabets. Just modify first Example 5.8 by requiring that the sets $\{\phi_n([-1, 1])\}_{n\in\mathbb{Z}}$ are mutually disjoint rather then having some intersections; everything else stays the same. Now form the system $\mathcal{S}$ by repeating each copy of $\phi_n$ $n$ times, for $n > 0$.

As in Example 5.8, $\mathcal{S}(\infty) = \{-1, 1\}$, so in this case the system is small at infinity. It is irreducible since $J_S \not\subset \partial X$. However in this example the function $d(\cdot)$ increases indefinitely on $J_S$, thus in Theorem 4.1 we have to take a function $\tilde{\kappa}$ which increases indefinitely on $J_S$.

Note that this example serves simultaneously as one on the unit disk $\overline{B}(0, 1)$, as well as just on $[0, 1]$. Generalizations to higher phase spaces are obvious.

*Example 5.11.* As above let $X = \overline{B}(0, 1)$ be the closed unit disk and for every integer $n \geq 1$ let $C_n$ be the circle centered at the origin $(0, 0)$ with some radius $r_n \in (0, 1)$.

We chose these radii so that they form an increasing sequence converging to 1. Cover then for each $n \geq 1$ the circle $C_n$ with closed disks $D_n(i)_{i\in K_n}$, of the same small radius $r'_n$, where $K_n$ is a finite set such that each disk $D_n(i)$ intersects only two other disks of the form $D_n(j)$. Assume in addition that for any $m \neq n$ the families $\{D_m(i)\}_{i\in K_m}$ and $\{D_n(i)\}_{i\in K_n}$ consist of mutually disjoint disks.

Our iterated function system $\mathcal{S}$ is obtained by taking contracting similarities $\{\phi_{n,i} : X \to X : n \geq 0, i \in K_n\}$ whose respective images of $X$ are the disks $D_n(i), i \in K_n, n \geq 0$.

In this case the limit set $J_\mathcal{S}$ is non-compact and $\mathcal{S}(\infty) \subset \partial(X)$. Thus the infinite IFS $\mathcal{S}$ is small at infinity. Also clearly $\mathcal{S}$ is irreducible.

We may have points $x$ in $J_\mathcal{S}$ with $d(x) = 2$ and points $y$ with $d(y) = 1$, and this will strongly influence the choice of the upper bounding function $\tilde{\kappa}$.

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Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO 014700, Bucharest, Romania

*E-mail address*: Eugen.Mihailescu@imar.ro

*URL*: www.imar.ro/∼mihailes

Department of Mathematics, University of North Texas, P.O. Box 311430, Denton, Texas 76203-1430

*E-mail address*: urbanski@unt.edu

*URL*: www.math.unt.edu/∼urbanski