OBSTRUCTIONS TO THE EXISTENCE OF KÄHLER STRUCTURES ON COMPACT COMPLEX MANIFOLDS

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Abstract. We prove that a manifold in the Fujiki class $\mathcal{C}$ which supports a $i\partial\bar{\partial}$-closed metric is Kähler. This result implies that on a compact complex manifold in the Fujiki class $\mathcal{C}$ which is not Kähler there exists a nonzero $i\partial\bar{\partial}$-exact, positive current of bidimension $(1,1)$.

Introduction

In [HaLa], Harvey and Lawson proved that the obstruction to the existence of a Kähler metric on a given compact complex manifold X of dimension n is a positive, non-zero current of bidegree $(n-1,n-1)$ which is the $(n-1,n-1)$ component of a d-exact current on X. In general, such currents are not d-closed, therefore the theory of closed positive currents cannot be used to study them (although some results extending this theory to the case of $i\partial\bar{\partial}$-closed, positive currents do exist). The main result of this paper is that, in case when X is a manifold in the Fujiki class $\mathcal{C}$, the obstruction current can be chosen to be d-closed:

Theorem 0.1. *Let X be a compact complex manifold of dimension n in the Fujiki class $\mathcal{C}$ and which is not Kähler. Then there exists a positive, nonzero current T of bidegree $(n-1,n-1)$, which is $i\partial\bar{\partial}$-exact.*

Theorem 0.1 follows immediately from

Theorem 0.2. *Let X be a compact complex manifold of dimension n in the Fujiki class $\mathcal{C}$ and suppose there exists ω a strictly positive $(1,1)$ form on X such that $i\partial\bar{\partial}\omega = 0$. Then X is a Kähler manifold.*

The two theorems are generalizations to the analytic case of the algebraic case which was proved by Peternell [Pe]. Theorem 0.2 is

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similar to Moishezon's theorem which states that a Moishezon manifold which is Kähler is in fact projective.

A $(1,1)$-form ω as in Theorem 0.2 (i.e., positive defined, and $i\partial\bar{\partial}$-closed) is called a *strong Kähler with torsion* (SKT) metric. See for instance [FiTo] for an introduction to SKT metrics. Therefore, Theorem 0.2 states that a manifold in Fujiki class $\mathcal{C}$ which supports a SKT metric is in fact Kähler.

On surfaces the two Theorems 0.1 and 0.2 are vacuous since any surface in the Fujiki class $\mathcal{C}$ is Kähler. But on 3-folds, Theorem 0.1 implies that any closed obstruction contains a nonzero curve:

Theorem 0.3. *Let X be a compact complex 3-fold in class $\mathcal{C}$ which is not Kähler and let T be a positive, nonzero $(2,2)$ current which is $i\partial\bar{\partial}$-exact. Then there exists C an irreducible curve in X, $\lambda > 0$ and R a closed positive $(2,2)$ current on X such that $T = \lambda[C] + R$.*

Theorem 0.1, combined with a result of Lamari [La], Théorème 3.2 (see Theorem 1.5 below) implies the following general existence theorem, which is a refined version of Harvey and Lawson theorem:

Theorem 0.4. *Let X be a compact complex manifold of dimension n such that*

- (i) *there is no non-trivial nef pluriharmonic current on X of bidegree $(n-1, n-1)$, which is the $(n-1, n-1)$ component of a boundary*
- (ii) *there is no non-trivial positive, $\partial\bar{\partial}$-exact current of bidegree $(n-1, n-1)$ on X*

Then X is Kähler.

Therefore, on a non-Kähler manifold, the obstruction is either nef (when the manifold is not in $\mathcal{C}$) or closed (when the manifold is in the Fujiki class $\mathcal{C}$).

For the proof of Theorem 0.2, we first show that the cohomology class of the $i\partial\bar{\partial}$-closed form satisfies the numerical conditions of a Kähler class. The main result of [DePă] then implies that the cohomology class contains a Kähler current. We then proceed by induction on the dimension of the manifold to show that the cohomology class contains a Kähler form.

Theorem 0.1 follows at once from Theorem 0.2 by using a result of Harvey and Lawson [HaLa].

1. Preliminaries

In this section we gather some results needed for the proof of the above results.

1.1. **Intrinsic characterization of Kähler manifolds.** The main result of [HaLa] is the following

Theorem 1.1. *Let X be a compact manifold of dimension n. Then X is non-Kähler iff there exists a nonzero positive current which is the $(n-1, n-1)$ component of a boundary.*

In the same paper, the authors prove that the obstruction to the closedness of the obstruction current involves a strictly positive, $i\partial\bar{\partial}$-closed $(1,1)$ form:

Theorem 1.2. *Suppose X is a compact manifold of dimension n. Then X admits a closed real $(1,1)$-form $\eta = \bar{\partial}\alpha + \omega + \partial\bar{\alpha}$ where ω is a strictly positive $(1,1)$ form and α is a $(1,0)$ form on X iff X does not support a nonzero, d-closed, positive current which is the $(n-1, n-1)$ component of a boundary*

1.2. **Positive classes on compact manifolds.** The main result of [DePă] is the following

Theorem 1.3. *Let (X, λ) be a compact Kähler manifold and let $\{\eta\}$ be a $(1,1)$ cohomology class on X. Then $\{\eta\}$ is a Kähler cohomology class iff for every irreducible analytic set $Z \subset X$, $\dim Z = p$, and every $k = \overline{1, p}$,*

$$\int_Z \eta^k \wedge \lambda^{p-k} > 0 \tag{1.1}$$

In order to construct a Kähler metric, we will need the following result from [DePă]:

Theorem 1.4. *Let X be a compact complex space and let $\{\eta\}$ be a cohomology class of type $(1,1)$ on X. Assume that $\{\eta\}$ contains a Kähler current T and that the restriction $\{\eta\}|Y$ to every irreducible component Y in the Lelong sublevel sets $E_c(T)$ is a Kähler cohomology class. Then $\{\eta\}$ is a Kähler cohomology class on X.*

1.3. **Manifolds in the Fujiki class $\mathcal{C}$.** The manifolds in class $\mathcal{C}$ were first introduced by Fujiki as manifolds which are meromorphic images of Kähler manifolds [Fu]:

Definition 1.1. A compact complex manifold X is in class $\mathcal{C}$ if there exists a complex Kähler space Y and a surjective meromorphic map $h\colon Y \to X$.

There are several other ways of characterizing the manifolds in the Fujiki class. A current T of bidegree $(n-1, n-1)$ on a compact complex manifold X of dimension n is *nef pluriharmonic* if it is weak limit of

Gauduchon metrics. A closed current of bidegree $(1,1)$ is a *Kähler current* if it dominates some strictly positive, smooth $(1,1)$-form. Then we have:

Theorem 1.5. *Let X be a compact complex manifold of dimension n. Then the following are equivalent:*

i) *X is in the Fujiki class $\mathcal{C}$*
ii) *there exists Y a Kähler manifold and $h\colon Y \to X$ a proper transform of X ([Va])*
iii) *there exists T a Kähler current on X ([DePă])*
iv) *if R is a $(n-1,n-1)$ nef pluriharmonic current on X which is the $(n-1,n-1)$ component of a boundary, then $R=0$ ([La])*

The Fujiki class $\mathcal{C}$ is stable under most natural operations, except under small deformations. The Hodge decomposition is valid on manifolds in the Fujiki class $\mathcal{C}$, in particular the $\partial\bar{\partial}$ lemma is valid on such manifolds.

2. Non-Kähler manifolds in the Fujiki class $\mathcal{C}$

In this section we prove the two Theorems 0.1 and 0.2. We first need the following lemma which will be used later to show that a certain cohomology class satisfies the numerical inequalities of a Kähler class:

Lemma 2.1. *Let X be a compact complex manifold of dimension n, $\eta = \partial\bar{\alpha} + \omega + \bar{\partial}\alpha$ be a closed $(1,1)$ form where α is a $(1,0)$ form on X and ω is a strictly positive $(1,1)$ form on X, and λ be a closed real $(n-k,n-k)$ form on X. Then*

$$\int_X \eta^k \wedge \lambda = \sum_{2i+j=k} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} 2i \\ i \end{pmatrix} \int_X \omega^j \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \wedge \lambda \qquad (2.1)$$

Proof. Since η is closed, it follows that $\partial\omega = \bar{\partial}\partial\alpha$ and $\bar{\partial}\omega = \partial\bar{\partial}\bar{\alpha}$. We prove the statement by induction on k. For $k=1$, the above equation becomes

$$\int_X \eta \wedge \lambda = \int_X \omega \wedge \lambda$$

and it follows from Stokes' theorem since λ is closed. Suppose the formula is true for k. Then for $k+1$ we have

$$\int_X \eta^{k+1} \wedge \lambda = \int_X \eta^k \wedge \eta \wedge \lambda =$$

$$\sum_{2i+j=k} \binom{k}{j} \binom{2i}{i} \int_X \omega^j \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \wedge (\bar{\partial}\alpha + \omega + \partial\bar{\alpha}) \wedge \lambda =$$

$$\sum_{2i+j=k} \binom{k}{j} \binom{2i}{i} \int_X \omega^{j+1} \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \wedge \lambda +$$

$$\sum_{2i+j=k} \binom{k}{j} \binom{2i}{i} \int_X \bar{\partial}\alpha \wedge \omega^j \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \wedge \lambda +$$

$$\sum_{2i+j=k} \binom{k}{j} \binom{2i}{i} \int_X \partial\bar{\alpha} \wedge \omega^j \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \wedge \lambda$$

The third term in the above sum is the conjugate of the second term, so we focus only on the second term. Stokes' theorem implies that it is equal to

$$\sum_{2i+j=k} j \binom{k}{j} \binom{2i}{i} \int_X \alpha \wedge \partial\bar{\partial}\bar{\alpha} \wedge \omega^{j-1} \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \wedge \lambda +$$

$$\sum_{2i+j=k} i \binom{k}{j} \binom{2i}{i} \int_X \alpha \wedge \partial\omega \wedge \omega^j \wedge \bar{\partial}\bar{\alpha} \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^{i-1} \wedge \lambda$$

where we have used $\bar{\partial}\omega = \partial\bar{\partial}\alpha$ and $\bar{\partial}\partial\alpha = \partial\omega$. Now

$$\partial \left[\sum_{2i+j=k-1} \frac{j+1}{i+1} \binom{k}{j+1} \binom{2i}{i} \bar{\partial}\bar{\alpha} \wedge \omega^j \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i \right] =$$

$$\sum_{2i+j=k} j \binom{k}{j} \binom{2i}{i} \partial\bar{\partial}\bar{\alpha} \wedge \omega^{j-1} \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^i +$$

$$\sum_{2i+j=k} i \binom{k}{j} \binom{2i}{i} \partial\omega \wedge \omega^j \wedge \bar{\partial}\bar{\alpha} \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^{i-1}$$

and again Stokes' theorem implies that the second term is equal to

$$\sum_{2i+j=k-1} \frac{j+1}{i+1} \binom{k}{j+1} \binom{2i}{i} \int_X \omega^j \wedge (\partial\alpha \wedge \bar{\partial}\bar{\alpha})^{i+1} \wedge \lambda$$

Now the formula for $k+1$ follows from rearranging the terms of the sums, along with some trivial combinatorial identities. □

Now we can prove

Theorem 2.2. *Let X be a compact complex manifold of dimension n in the Fujiki class $\mathcal{C}$ and suppose there exists ω a strictly positive $(1,1)$ form on X such that $i\partial\bar{\partial}\omega = 0$. Then X is a Kähler manifold.*

Proof. Note that since X is in the Fujiki class $\mathcal{C}$, the $\partial\bar{\partial}$ lemma is valid on X, and therefore the condition $i\partial\bar{\partial}\omega = 0$ implies the existence of α and η as in Lemma 2.1 (apply the $i\partial\bar{\partial}$-lemma to the d-closed and ∂-exact form $\partial\omega$). We prove by induction on n that $\{\eta\}$ is a Kähler cohomology class.

First assume that X can be made Kähler by a single blow-up along a smooth submanifold $Y \subset X$, $\dim Y \leq n-2$. Denote by $\pi : \widetilde{X} \to X$ the blow up of X along Y and denote by $\widetilde{Y} \subset \widetilde{X}$ the exceptional divisor and suppose that $\widetilde{\lambda}$ is a Kähler form on $\widetilde{X}$. It is well-known that there exists a smooth d-closed $(1,1)$ form $\widetilde{u}$ on $\widetilde{X}$, in the cohomology class of $[\widetilde{Y}]$, such that $\widetilde{\omega} = \pi^*\omega - \varepsilon\widetilde{u}$ is strictly positive on $\widetilde{X}$ for some small $\varepsilon > 0$. Moreover, $\widetilde{u} = [\widetilde{Y}] + i\partial\bar{\partial}\widetilde{\psi}$, where $\widetilde{\psi}$ is smooth on $\widetilde{X} \setminus \widetilde{Y}$. Set $\widetilde{\alpha} = \pi^*\alpha$ and $\widetilde{\eta} = \partial\bar{\widetilde{\alpha}} + \widetilde{\omega} + \bar{\partial}\widetilde{\alpha}$. Then we can use Lemma 2.1 for $\widetilde{\eta}$ on $\widetilde{X}$.

Note that

$$(\partial\widetilde{\alpha} \wedge \bar{\partial}\bar{\widetilde{\alpha}})^i = (\partial\widetilde{\alpha})^i \wedge (\bar{\partial}\bar{\widetilde{\alpha}})^i$$

is a weakly positive $(2i,2i)$-form on $\widetilde{X}$. Hence, when we multiply it with the strongly positive form $\widetilde{\omega}^j \wedge \widetilde{\lambda}^k$, we obtain a positive multiple of the volume form. Therefore we obtain that

$$\int_{\widetilde{Z}} \widetilde{\eta}^k \wedge \widetilde{\lambda}^{p-k} > 0 \tag{2.2}$$

for every irreducible analytic subset $\widetilde{Z} \subset \widetilde{X}$, $\dim \widetilde{\mathrm{Z}} = \mathrm{p}$ and every $k = \overline{1,p}$. Indeed, when $\widetilde{Z}$ is smooth, this follows from Lemma 2.1.

In case when $\widetilde{Z}$ is not smooth, we use Hironaka's resolution of singularities, and we obtain a manifold $\widetilde{X}'$ which is a sequence of blow-ups with smooth centers of $\widetilde{X}$, and a smooth submanifold $\widetilde{Z}'$ which resolves the singularities of $\widetilde{Z}$. It is clear the integral in (2.2) is equal to the corresponding integral over $\widetilde{Z}'$, and hence is non-negative.

Theorem 1.3 implies that $\{\widetilde{\eta}\}$ is a Kähler cohomology class on $\widetilde{X}$, i.e., there exists $\widetilde{\varphi} \in \mathcal{C}^\infty(\widetilde{X}, \mathbb{R})$ such that $\widetilde{\eta} + i\partial\bar{\partial}\widetilde{\varphi} > 0$.

Now we push everything forward to X. At the cohomology level, we have on $\widetilde{X}$ that

$$\{\widetilde{\eta}\} = \pi^*\{\eta\} - \varepsilon\{[\widetilde{Y}]\}$$

The push-forward of $[\widetilde{Y}]$ is 0, so it follows that $\pi_*\{\widetilde{\eta}\} = \{\eta\}$ contains a Kähler current which is smooth on $X \setminus Y$.

Note that a smooth submanifold of a manifold in Fujiki class $\mathcal{C}$ is also in Fujiki class $\mathcal{C}$. Hence, by induction, we obtain that the restriction of $\{\eta\}$ to Y is a Kähler cohomology class, and by Theorem 1.4, it follows that $\{\eta\}$ is a Kähler cohomolgy class.

In general, suppose that X can be made Kähler by a sequence of blow-ups with smooth centers $X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = X$ and choose r to be minimal and suppose $r \geq 1$. We can easily construct a strictly positive $(1,1)$-form ω_{r-1} on X_{r-1} such that $i\partial\bar{\partial}\omega_{r-1} = 0$. Then X_{r-1} is Kähler, contradicting the minimality of r. □

On 3-folds we can prove a stronger result:

Theorem 2.3. *Let X be a 3-fold in the Fujiki class $\mathcal{C}$ and ω a strictly positive $(1,1)$-form on X such that either $i\partial\bar{\partial}\omega \geq 0$ or $i\partial\bar{\partial}\omega \leq 0$. Then X is Kähler.*

Proof. Since X is in the Fujiki class $\mathcal{C}$, there exists $T \geq \gamma$ a Kähler current on X, where γ is a strictly positive $(1,1)$-form on X. If $i\partial\bar{\partial}\omega \geq 0$, then

$$\langle i\partial\bar{\partial}\omega, \gamma\rangle \leq \langle i\partial\bar{\partial}\omega, T\rangle = \langle \omega, i\partial\bar{\partial}T\rangle = 0 \tag{2.3}$$

hence $i\partial\bar{\partial}\omega = 0$ and similarly when $i\partial\bar{\partial}\omega \leq 0$. Therefore $i\partial\bar{\partial}\omega = 0$ and the conclusion follows from Theorem 2.2. □

Now we can easily prove

Theorem 2.4. *Let X be a compact complex manifold of dimension n in the Fujiki class $\mathcal{C}$ and which is not Kähler. Then there exists a positive, nonzero current T of bidegree $(n-1, n-1)$, which is $i\partial\bar{\partial}$-exact.*

Proof. By Theorem 1.2, it is enough to prove that if X supports a strictly positive, $i\partial\bar{\partial}$-closed, $(1,1)$-form, then X is Kähler. But this is just the statement of Theorem 2.2. The $i\partial\bar{\partial}$-exactness of T follows immediately from the $\partial\bar{\partial}$ lemma. □

3. Non Kähler 3-folds

In this section we show that on any 3-fold in class $\mathcal{C}$ which is not Kähler there exists a curve which is part of the obstruction.

Theorem 3.1. *Let X be a 3-fold in class $\mathcal{C}$ which is not Kähler and let T be a closed, positive, nonzero $(2,2)$ current which is $i\partial\bar{\partial}$-exact. Then there exists C an irreducible curve in X, $\lambda > 0$ and R a closed positive $(2,2)$ current on X such that $T = \lambda[C] + R$.*

Remark 3.1. The above theorem is no longer true in higher dimensions. For instance, let Y be the 3-fold constructed by Hironaka [Hi] which is a proper modification of the projective space $\mathbb{P}^3$ and which contains a positive linear combination of curves which is homologuous to 0. Denote this obstruction by C. Let S be an arbitrary Riemann surface and ω_S a positive $(1,1)$-form on S. Let $X = Y \times S$ and let p_1 and p_2 the two projections. Set $T = p_1^*C \wedge p_2^*\omega_S$. Then T is a closed positive $(3,3)$ current, which is $i\partial\bar{\partial}$-exact, and it is a residual current.

Remark 3.2. Theorem 3.1 states that on 3-folds, **any** closed obstruction contains a curve. The above example shows that in higher dimensions there are obstructions which do not contain any curves. It's not clear whether on any manifold in class $\mathcal{C}$ there are obstructions which contain curves.

Before we prove Theorem 3.1, we need the following

Proposition 3.2. *Let X be a compact complex manifold of dimension n in the Fujiki class $\mathcal{C}$ and let $\pi : \widetilde{X} \to X$ be the blow-up of X along a smooth submanifold Y of dimension $\leq n-2$ and let $\widetilde{Y}$ the exceptional divisor. Let T be a closed positive $(n-1,n-1)$ current on X such that $\chi_Y T = 0$. Then there exists $\widetilde{T}$ a closed positive $(n-1,n-1)$ current on $\widetilde{X}$ such that $\chi_{\widetilde{Y}}\widetilde{T} = 0$ and $\pi_*\widetilde{T} = T$. Moreover, $\{\widetilde{T}\} = \pi^*\{T\} - \lambda\{[F]\}$, where F is a curve in the fibre of $\pi|\widetilde{Y} : \widetilde{Y} \to Y$ and $\lambda \geq 0$.*

Proof. The existence of $\widetilde{T}$ (the strict transform of T) is proved in [AlBa2]. Let F be a curve in a fibre of $\pi|\widetilde{Y} \to Y$. We prove that there exists $\lambda \in \mathbb{R}$ such that $\{\widetilde{T}\} = \pi^*\{T\} - \lambda\{[F]\}$ by duality. Let $\{\widetilde{\alpha}\} \in H^{1,1}(\widetilde{X})$; it is well-known that $\{\widetilde{\alpha}\} = \pi^*\{\alpha\} + \gamma\{[\widetilde{Y}]\}$ where $\{\alpha\} \in H^{1,1}(X)$. Then

$$\langle\{\widetilde{T}\} - \pi^*\{T\} - \lambda\{[F]\}, \{\widetilde{\alpha}\}\rangle = \gamma\left(\langle\{\widetilde{T}\}, \{[\widetilde{Y}]\}\rangle + \lambda\langle\{[F]\}, \{[\widetilde{Y}]\}\rangle\right)$$

It follows that for

$$\lambda = -\frac{\langle\{\widetilde{T}\}, \{[\widetilde{Y}]\}\rangle}{\langle\{[F]\}, \{[\widetilde{Y}]\}\rangle}$$

we have $\{\widetilde{T}\} = \pi^*\{T\} + \lambda\{[F]\}$. Since $\chi_{\widetilde{Y}}\widetilde{T} = 0$, it follows (see [AlBa1]) that $\langle\{\widetilde{T}\}, \{[\widetilde{Y}]\}\rangle \leq 0$. Now $\langle\{[F]\}, \{[\widetilde{Y}]\}\rangle < 0$ and therefore $\lambda \geq 0$. □

Proof. of Theorem 3.1 By Siu's decomposition theorem, T can be written $T = \sum_j \lambda_j[C_j] + R$, where C_j are curves in X, $\lambda_j > 0$ and R is a residual current, i.e., the Lelong sublevel sets $E_c(R)$ are 0 dimensional for every $c > 0$. So it is enough to prove that T cannot be residual.

Suppose T is residual. Then, if Y is a submanifold of X of dimension ≤ 1, then $\chi_Y T = 0$. Now suppose that X can be made Kähler by a sequence of blow-ups with smooth centers $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_r$ where X_r is Kähler. Denote by Y_s the center of the blow-up $X_{s+1} \to X_s$. The centers of the blow-ups are either smooth curves or points. Start off with the current $T = T_0$ on $X = X_0$. Clearly $\chi_{Y_0} T = 0$. We construct by induction $i\partial\bar{\partial}$-exact, positive currents T_s on X_s such that $\chi_{Y_s} T_s = 0$ and $(\pi_s)_* T_s = T_{s-1}$. Suppose T_s has been constructed. From Proposition 3.2 we obtain a closed positive current T'_{s+1} on X_{s+1} and $\lambda_{s+1} \geq 0$ such that $T'_{s+1} + \lambda_{s+1}[F_{s+1}]$ is $i\partial\bar{\partial}$-exact and its push-forward is T_s. If $Y_{s+1} \neq F_{s+1}$, we set $T_{s+1} = T'_{s+1} + \lambda_{s+1}[F_{s+1}]$. If $Y_{s+1} = F_{s+1}$, we set $T_{s+1} = T'_{s+1} + \lambda_{s+1}[F'_{s+1}]$ where $F'_{s+1} \neq F_{s+1}$ is another curve in the cohomology class $\{[F_{s+1}]\}$. It is clear that $\chi_{Y_{s+1}} T_{s+1} = 0$ and that $(\pi_{s+1})_* T_{s+1} = T_s$. On X_r we obtain a closed positive current T_r which is $i\partial\bar{\partial}$-exact. Since X_r is Kähler, it follows that $T_r = 0$ and therefore $T_0 = T = 0$. Contradiction. $\square$

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