HABILITATION THESIS

Specialization: Mathematics

Andaluzia Cristina Matei

Variational approaches in the study of nonlinear problems arising in Contact Mechanics

Contents

Abstract

The present work is a collection of results in the study of nonlinear problems by means of three variational approaches: a variational approach via Lagrange multipliers, a variational approach via bipotentials and a variational approach via history-dependent quasivariational inequalities on unbounded time intervals.

The study we present in this thesis has an interdisciplinary character and a strong applicability feature, combining mathematical areas as PDEs, Nonlinear Analysis, Convex Analysis and Calculus of Variations with Mechanics of Continua, Mechanics of Materials and Contact Mechanics.

All the problems we discuss in this thesis are related to models in Contact Mechanics for several types of deformable solid materials. The scientific results mentioned in the present thesis represent a part of the scientific results of the candidate, published after obtaining the Ph.D. degree in Mathematics. The results we focus on are presented without proofs; details can be found in the papers mentioned in the Publications of the Thesis (just after the Abstract in Romanian).

The thesis comprises SCIENTIFIC RESULTS, CAREER EVOLUTION AND DEVELOPMENT PLANS and BIBLIOGRAPHY.

The presentation of the SCIENTIFIC RESULTS is organized into three parts.

Part I, devoted to the study of a class of contact models by *a variational approach with Lagrange multipliers*, is a collection of new mixed variational problems. The variational formulations via Lagrange multipliers in non-smooth mechanics are suitable formulations to efficiently approximate the weak solutions; this motivates the research in this direction. Structured in seven chapters, Part I discusses slip-independent frictional contact problems/slip-dependent frictional contact problems, contact problems for several types of nonlinearly elastic materials, frictional contact viscoelastic problems, frictionless contact problems involving electro-elastic or viscoplastic materials, contact problems involving multi-contact zones, unilateral frictional contact problems, focusing on their weak solvability. Presenting new abstract results as nonlinear analysis tools is also under attention. The abstract problems we consider herein are new abstract variational systems. In order to solve them, the main techniques we use rely on saddle point or fixed point techniques.

Part II adopts *a variational approach via bipotentials* in the weak solvability of a class of nonlinearly elastic contact problems. This second part comprises two chapters, Chapters 8-9. In Chapter 8 it is analyzed a unilateral frictionless contact model while in Chapter 9 it is analyzed a frictional contact model, both models leading to new variational systems. In order to solve these systems, the main technique we use is a minimization technique. Using a separated bipotential we get existence and uniqueness results suitable to discuss a simultaneous computation of the displacement field and the Cauchy stress tensor. The results presented in Part II represent first

steps in a new research direction, more complex models related to non-separated bipotentials being also envisaged.

Part III discusses the *variational analysis via history-dependent quasivariational inequalities* for a class of viscoplastic or electro-elasto-viscoplastic contact problems on *unbounded* time interval. This third part comprises three chapters, Chapters 10-12. Some preliminaries are presented in Chapter 10: a fixed point result and an existence and uniqueness result for an auxiliary problem consisting of an abstract history-dependent quasivariational inequality formulated on unbounded time interval. Using these preliminaries, we analyze in Chapter 11 two viscoplastic problems and in Chapter 12 an electro-elasto-viscoplastic problem. The weak formulations we deliver are new variational problems. Working on the interval $[0, \infty)$, a continuation of the research going to the Asymptotic Analysis in Contact Mechanics is envisaged.

The main contributions:

▶ the statement and the solution of *three new classes of abstract problems*

- *stationary mixed variational problems governed by nonlinear maps*
- *evolutionary mixed variational problems (with short-memory term)*
- *time-dependent mixed variational problems (with long-memory term)*

▶ the weakly solvability of contact models *by new variational techniques*

- for *nonlinearly elastic, viscoelastic, viscoplastic or electro-elastic materials* via a *variational approach with Lagrange multipliers*
- for *nonlinearly elastic materials governed by possibly set valued elastic operators* by means of *a variational approach via bipotentials theory*
- on *unbounded time interval.*

We end this thesis by presenting some CAREER EVOLUTION AND DEVELOPMENT PLANS. The presentation is structured in two chapters, Chapters 13-14. Chapter 13 presents further research directions such us: qualitative and numerical analysis in the study of mixed variational problems or in the study of variational systems via bipotentials; variational formulations in contact mechanics/ weak solutions via weighted Sobolev spaces or via Lebesgue spaces with variable exponent; optimal control problems in contact mechanics; mathematical techniques in the study of dissipative dynamic contact problems; asymptotic analysis in contact mechanics, regularity results; convergence results; viscoelastic problems via fractional differential operators/fractional calculus of variations. Chapter 14 presents further plans on the scientific, professional and academic career such as: to do a research activity allowing to continue to publish in international journals of hight level, to participate to international meetings in order to disseminate the best results, to organize scientific meetings, to extend the editorial activities for scientific journals, to continue the collaborations started in the past and to establish new contacts, to apply for national/international/interdisciplinary research projects as manager or member, to publish Lecture Notes and new monographs addressed to the students or researchers, to extend the advising activity to Ph.D. theses.

Rezumat

Prezenta lucrare este o colecţie de rezultate în studiul unor probleme neliniare, studiu realizat prin intermediul a trei abordări variaţionale: o abordare variaţională cu multiplicatori Lagrange, o abordare variaţională via bipotenţiali şi a abordare variaţională bazată pe teoria inegalităţilor cvasivariaţionale cu termen istoric-dependent formulate pe interval de timp nemărginit.

Studiul prezentat prin intermediul acestei teze are atât caracter interdisciplinar cât şi o puternică trăsătură aplicativă, îmbinând domenii de matematică aplicată cum ar fi Ecuaţii cu derivate parţiale, Analiză neliniară, Analiză convexă şi Calcul variaţional cu Mecanica mediilor continue, Mecanica materialelor şi Mecanica contactului.

Toate problemele discutate în această teză sunt in legătură cu modele în mecanica contactului pentru mai multe tipuri de materiale solide deformabile. Rezultatele ştiinţifice menţionate în prezenta teză reprezintă o parte dintre rezultatele ştiinţifice ale candidatei, publicate după obţinerea titlului de Doctor în Matematică. Rezultatele focalizate sunt prezentate fără demonstraţii; detalii pot fi găsite în lista de lucrări intitulată "Publicaţiile tezei", listă ce apare în prezentul manuscris imediat dupa rezumatul tezei in limba româna.

Teza cuprinde REZULTATE ŞTIINŢIFICE, PLANURI DE DEZVOLTARE SI EVOLUŢIE A CARIEREI şi BIBLIOGRAFIE.

Prezentarea REZULTATELOR ŞTIINŢIFICE este organizată în trei părţi.

Partea I, dedicată studiului unei clase de modele în mecanica contactului prin intermediul unei abordări variaţionale via multiplicatori Lagrange, este o colecţie de noi probleme variaţionale mixte. Formulările variaţionale via multiplicatori Lagrange în mecanica nenetedă sunt formulări care permit o eficientă aproximare a soluţiilor slabe; aceasta motivează cercetarea în această direcţie. Structurată in şapte capitole, Partea I analizează variaţional probleme de contact cu frecare independentă sau dependentă de alunecare, probleme de contact pentru diferite tipuri de materiale neliniar elastice, probleme vâscoelastice de contact cu frecare, probleme de contact cu frecare neglijabilă pentru materiale electro-elastice sau vâscoplastice, probleme ce implică mai multe zone de contact, probleme de contact unilateral cu frecare. Se are în vedere de asemenea prezentarea unor noi rezultate abstracte care pot fi considerate unelte utile de analiză neliniară. Problemele abstracte discutate în această parte a lucrării sunt noi sisteme variaţionale abstracte. Principalele tehnici utilizate în rezolvarea lor sunt tehnici de punct şa şi tehnici de punct fix.

Partea a-II-a adoptă o abordare variaţională via bipotenţiali în vederea rezolvării în sens slab a unei clase de probleme de contact pentru materiale neliniar elastice. Această a doua parte a lucrării are două capitole, Capitolele 8-9. În Capitolul 8 se analizează un model de contact unilateral fără frecare în timp ce în Capitolul 9 se analizează un model de contact cu frecare, ambele modele conducând la noi sisteme variaţionale. Principala tehnică utilizată în vederea rezolvării acestor sisteme este o tehnică de minimizare. Utilizându-se un bipotenţial separat se obţin rezultate de existenţă şi unicitate care permit un calcul simultan al câmpului deplasare şi al tensorului tensiune Cauchy. Rezultatele prezentate în această parte a tezei reprezintă primi

paşi într-o nouă direcţie de cercetare, fiind vizate de asemenea modele mai complexe care implică bipotenţiali neseparaţi.

Partea a-III-a prezintă rezultate în analiza variaţională, via inegalităţi cvasivariaţionale cu termen istoric-dependent, pentru o clasă de probleme de contact vâscoplastice sau electro-elasto-vâscoplastice, formulate pe interval de timp nemărginit. Această a treia parte are trei capitole, Capitolele 10-12. În Capitolul 10 sunt prezentate câteva preliminarii: un rezultat de punct fix şi un rezultat de existenţă şi unicitate pentru o problemă auxiliară ce constă dintr-o inegalitate cvasivariaţională abstractă cu termen istoric-dependent, formulată pe interval de timp nemărginit. Utilizând aceste preliminarii, în Capitolul 11 analizăm două probleme vâscoplastice şi în Capitolul 12 o problemă electro-elasto-vâscoplastică. Formulările variaţionale obţinute sunt noi probleme variaţionale. Lucrând pe intervalul $[0, \infty)$, se are în vedere o continuare a cercetarii in direcţia Analizei Asimptotice.

Principalele contribuţii:

▶ formularea şi rezolvarea a *trei noi tipuri de probleme abstracte:*

- *probleme variaţionale mixte staţionare guvernate de aplicaţii neliniare*
- *probleme variaţionale mixte de evoluţie (cu termen memorie scurtă)*
- *probleme variaţionale mixte dependente de timp (cu termen memorie lungă)*

▶ studiul soluţiilor slabe, prin intermediul unor noi tehnici de calcul variaţional, al unor modele în mecanica contactului

- pentru *materiale neliniar elastice, vâscoelastice, vâscoplastice sau electro-elastice* prin intermediul unei *abordări variaţionale cu multiplicatori Lagrange*
- pentru *materiale neliniar elastice guvernate de operatori elastici posibil multivoci*, prin intermediul unei *abordări variaţionale via bipotenţiali*
- pe *interval de timp nemărginit.*

Prezenta teză se încheie cu prezentarea unor PLANURI DE DEZVOLTARE ŞI EVOLUŢIE A CARIEREI. Această prezentare este structurată în două capitole, Capitolele 13-14. În Capitolul 13 sunt indicate direcţii de cercetare pe care candidata le are în vedere pentru perioada următoare, direcţii precum: analiză calitativă şi numerică în studiul unor probleme variaţionale mixte sau în studiul unor sisteme variaţionale via bipotenţiali; formulări variaţionale în mecanica contactului /soluţii slabe prin intermediul spaţiilor Sobolev cu pondere sau prin intermediul spaţiilor Lebesgue cu exponent variabil; probleme de control optimal în mecanica contactului; un studiu matematic pentru probleme de contact dinamice disipative, analiză asimptotică în mecanica contactului, rezultate de regularitate, rezultate de convergenţă, probleme vâscoelastice via operatori diferenţiali fracţionari/calcul variational fracţionar. În Capitolul 14 sunt prezentate planuri viitoare în carieră, atât din punct de vedere ştiinţific şi profesional cât şi din punct de vedere academic, precum: desfăşurarea unei activităţi de cercetare de calitate care să conducă la publicaţii în jurnale internaţionale de înalt nivel, participarea la evenimente ştiinţifice în cadrul cărora să fie diseminate principalele rezultate obţinute, organizarea de evenimente ştiinţifice, extinderea activităţii editoriale pentru jurnale ştiinţifice, continuarea colaborărilor începute în trecut şi stabilirea de noi contacte, aplicarea pentru proiecte naţionale/internaţionale/interdisciplinare ca director sau membru de echipă, publicarea de note de curs sau monografii adresate studenţilor sau cercetătorilor, extinderea activităţii de coordonare ştiinţifică, de la lucrări de licenţă sau disertaţii, la teze de doctorat.

Publications of the Thesis

The Publications of the Thesis, specifying them in the order of their appearance in the present manuscript, are the following.

[1] S. Hüeber, **A. Matei** and B. Wohlmuth, Efficient algorithms for problems with friction, *SIAM Journal on Scientific Computing*, DOI: 10.1137/050634141, **29**(1) (2007), 70-92.

[2] S. Hüeber, **A. Matei** and B. Wohlmuth, A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity, *Bull. Math. Soc. Math. Roumanie*, **48**(96)(2) (2005), 209-232.

[3] **A. Matei**, On the solvability of mixed variational problems with solution-dependent sets of Lagrange multipliers, *Proceedings of The Royal Society of Edinburgh, Section: A Mathematics*, **143**(05) (2013), 1047-1059, http://dx.doi.org/10.1017/S0308210512000637.

[4] **A. Matei**, A mixed variational formulation for a slip-dependent frictional contact model, Lecture Notes in Engineering and Computer Science: *Proceedings of The World Congress on Engineering 2014*, 2-4 July, 2014, London, U.K., pp 750-754 (ISBN: 978-988-19253-5-0, ISSN: 2078-0958).

[5] **A. Matei**, Weak Solutions via Lagrange Multipliers for a Slip-dependent Frictional Contact Model, *IAENG International Journal of Applied Mathematics*, **44**(3) (2014), 151-156.

[6] **A. Matei** and R. Ciurcea, Contact problems for nonlinearly elastic materials: weak solvability involving dual Lagrange multipliers, *Australian and New Zealand Industrial and Applied Mathematics Journal (ANZIAM JOURNAL)*, DOI:10.1017/S1446181111000629, **52** (2010), 160-178.

[7] **A. Matei** and R. Ciurcea, Weak solvability for a class of contact problems, *Annals of the Academy of Romanian Scientists Series on Mathematics and its Applications*, **2**(1) (2010), 25-44.

[8] **A. Matei**, Weak solvability via Lagrange multipliers for two frictional contact models, Proceedings of "11-ème Colloque Franco-Roumain de Mathématiques Appliquées", 2012, Bucharest, *Annals of the University of Bucharest (mathematical series)*, **4**(LXII) (2013), 179-191.

[9] **A. Matei**, An existence result for a mixed variational problem arising from Contact Mechanics, *Nonlinear Analysis Series B: Real World Application*, **20** (2014), 74-81,

DOI: 10.1016/j.nonrwa.2014.01.010.

[10] **A. Matei** and R. Ciurcea, Weak solutions for contact problems involving viscoelastic materials with long memory, *Mathematics and Mechanics of Solids*, DOI: 10.1177/1081286511400515, **16**(4) (2011), 393-405.

[11] **A. Matei**, An evolutionary mixed variational problem arising from frictional contact mechanics, *Mathematics and Mechanics of Solids*, DOI: 10.1177/1081286512462168, **19**(3) (2014), 225-241.

[12] **A. Matei**, A variational approach for an electro-elastic unilateral contact problem, *Mathematical Modelling and Analysis*, **14**(3) (2009), 323-334.

[13] S. Hüeber, **A. Matei**, B. Wohlmuth, A contact problem for electro-elastic materials, *ZAMM-Journal of Applied Mathematics and Mechanics*, Special Issue: Mathematical Modeling: Contact Mechanics, Phase Transition, Multiscale Problems, DOI:10.1002/zamm.201200235, **93**(10-11) (2013), 789-800.

[14] M. Barboteu, **A. Matei** and M. Sofonea, On the behavior of the solution of a viscoplastic contact problem, *Quarterly of Applied Mathematics*, DOI: http://dx.doi.org/10.1090/S0033-569X-2014-01345-4, **72**(4) (2014), 625-647.

[15] **A. Matei**, Weak solvability via Lagrange multipliers for contact problems involving multi-contact zones, *Mathematics and Mechanics of Solids*, DOI: 10.1177/1081286514541577, published online July 7, 2014.

[16] **A. Matei**, Two abstract mixed variational problems and applications in Contact Mechanics, *Nonlinear Analysis Series B: Real World Application*, published online October 13, 2014, http://dx.doi.org/10.1016/j.nonrwa.2014.09.014.

[17] **A. Matei**, A variational approach via bipotentials for unilateral contact problems, *Journal of Mathematical Analysis and Applications*, http://dx.doi.org/10.1016/j.jmaa.2012.07.065, **397**(1) (2013), 371-380.

[18] **A. Matei**, A variational approach via bipotentials for a class of frictional contact problems, *Acta Applicandae Mathematicae*, DOI: 10.1007/s10440-014-9868-1, (2014), **134**(1) (2014), 45-59.

[19] M. Barboteu, **A. Matei** and M. Sofonea, Analysis of Quasistatic Viscoplastic Contact Problems with Normal Compliance, *The Quarterly Journal of Mechanics and Applied Mathematics*, **65**(4) (2012), 555-579, DOI: 10.1093/qjmam/hbs016.

[20] M. Boureanu, **A. Matei** and M. Sofonea, Analysis of a Contact Problem for Electro-elastic-visco-plastic Materials, *Communications on Pure and Applied Analysis*, **11**(3) (2012), 1185-1203, DOI: 10.3934/cpaa.2012.11.1185.

SCIENTIFIC RESULTS

Preface

"Each progress in mathematics is based on
the discovery of stronger tools and easier methods,
which at the same time makes it easier to
understand earlier methods. By making these stronger
tools and easier methods his own, it is possible for
the individual researcher to oriented
himself in the different branches of mathematics."
David Hilbert, 1900

The present thesis is a collection of results in the weak solvability of a class of nonlinear problems by means of three variational approaches: a variational approach via Lagrange multipliers, a variational approach via bipotentials and a variational approach via history-dependent quasivariational inequalities on unbounded time intervals.

The research from this thesis has an interdisciplinary character. New trends in Advanced Applied Mathematics are required, combining mathematical areas as PDEs, Nonlinear Analysis, Convex Analysis and Calculus of Variations with Mechanics of Continua, Mechanics of Materials and Contact Mechanics; see e.g. [1, 24, 50, 57, 83, 92, 151, 160] for important mathematical tools, [131, 162] for numerical approximation techniques, [48, 49, 59, 81, 124, 128, 130, 138] for applied mathematics in contact mechanics, [79, 88, 161, 166] for an engineering approach in contact mechanics, [45, 73] for viscoplasticity and [76, 158] for piezoelectricity, to give just a few examples of foundational books. It is worth to underline also the strong applicability feature of the research from this thesis: all the problems we discuss are related to models in Contact Mechanics for various kind of deformable solid materials. Solving contact problems for nonlinearly materials is a challenging topic of non smooth mechanics. The contact models are very complex. Most of them are analyzed by variational methods because of the difficulty of finding strong solutions. After establishing the well-possedness of a contact model, the next target is the approximation of the weak solution. Currently, obtaining variational formulations which are suitable to an efficient approximation of the weak solutions is an issue of great interest.

The scientific results mentioned in the present thesis represent a part of the scientific results of the candidate after obtaining the Ph.D. degree. The results we focus on are presented without proofs; details can be found in the papers mentioned in the Publications of the Thesis, a list placed just after the Abstract in Romanian.

The thesis comprises SCIENTIFIC RESULTS, CAREER EVOLUTION AND DEVELOPMENT PLANS and BIBLIOGRAPHY.

The presentation of the SCIENTIFIC RESULTS is organized into three parts.
PART I - A variational approach via Lagrange multipliers
PART II - A variational approach via bipotentials

PART III - A variational approach via history-dependent quasivariational inequalities on unbounded time interval

Part I, devoted to the study of a class of contact models by a variational approach with Lagrange multipliers, is based on the papers [69, 68, 105, 109, 112, 99, 104, 100, 107, 101, 111, 98, 70, 11, 110, 113], specifying them in the order of their appearance in the present manuscript. In this part of the thesis we discuss slip-independent or slip-dependent frictional contact problems, contact problems for several types of nonlinearly elastic materials, frictional contact viscoelastic problems, frictionless contact problems involving electro-elastic or viscoplastic materials, contact problems involving multi-contact zones, unilateral frictional contact problems, focusing on their weak solvability. Presenting new abstract results as nonlinear analysis tools is also under attention. The abstract problems we consider herein are new abstract variational systems. After presenting their solution, we show how these abstract results were used to solve contact problems for different types of materials or different types of contact conditions, frictionless or frictional. The main techniques we use rely on a saddle point technique and fixed point techniques. The saddle point theory, who originates from Babušca-Brezzi works, was successfully developed and applied in a large number of publications, see e.g. the books [22, 23, 61, 129] and the papers [3, 63, 66, 67, 69, 132, 164] to give only a few examples. The first part of the thesis is a collection of new mixed variational problems. The mixed variational formulations in non-smooth mechanics are suitable formulations to efficiently approximate the weak solutions; this motivates the research in this direction.

Part I is structured in seven chapters.

Chapter 1, which is concerned with the analysis of a class of slip-independent frictional contact problems, comprises two sections: Section 1.1 based on the paper [69] and Section 1.2 based on the paper [68].

Section 1.1 focuses on an antiplane frictional contact model which is related to *a saddle point problem* while Section 1.2 discusses an elasto-piezoelectric frictional contact problem whose variational formulation is related to *a generalized saddle point problem with non-symmetric bilinear form* $a(\cdot,\cdot)$*.* From the variational point of view both problems have the following form:

$$\begin{aligned} a(u,v)+b(v,\lambda) &= (f,\,v)_X && \text{for all } v\in X,\\ b(u,\mu-\lambda) &\le 0 && \text{for all } \mu\in\Lambda. \end{aligned}$$

Chapter 2, devoted to the analysis of a class of slip-dependent frictional contact problems, comprises three sections: Section 2.1 and Section 2.2 are based on the paper [105] while Section 2.3 is based on the papers [109, 112].

Section 2.1 presents an abstract existence result in the study of the following mixed variational problem with solution dependent-set of Lagrange multipliers, $\Lambda=\Lambda(u)$.

Given $f\in X$*,* $f\neq 0_X$*, find* $(u,\lambda)\in X\times Y$ *such that* $\lambda\in\Lambda(u)\subset Y$ *and*

$$\begin{aligned} a(u,v)+b(v,\lambda) &= (f,\,v)_X && \text{for all } v\in X,\\ b(u,\mu-\lambda) &\le 0 && \text{for all } \mu\in\Lambda(u). \end{aligned}$$

The applicability of the abstract result we present is illustrated in Section 2.2 in the study of an antiplane problem and in Section 2.3 in the study of a 3D slip-dependent frictional contact problem.

Chapter 3, structured in three sections, is related to the analysis of a class of contact problems for nonlinearly elastic materials leading to weak formulations governed, in Section 3.1, by a strongly monoton and Lipschitz continuous operator, in Section 3.2 by a proper convex l.s.c functional and in Section 3.3 by a nonlinear hemicontinuous generalized monotone operator. Section 3.1 is based on the papers [99, 104], Section 3.2 is based on the papers [100, 104] and Section 3.3 is based on the paper [107].

Section 3.1 analyzes the case of single-valued elastic operators; herein the mixed variational formulation via Lagrange multipliers leads to a mathematical problem of the form below.

Given $f,\ h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} (Au, v)_X + b(v, \lambda) &= (f,\, v)_X && \text{for all } v \in X,\\ b(u, \mu - \lambda) &\leq b(h, \mu - \lambda) && \text{for all } \mu \in \Lambda, \end{aligned}$$

where A is *a strongly monoton and Lipschitz continuous operator.*

In Section 3.2 the constitutive law we use is expressed in a form of a subdifferential inclusion governed by a proper convex lower semicontinuous functional. Thus, we focus on the case of possibly multi-valued elastic operators. The mixed variational formulation via Lagrange multipliers leads us to a mathematical problem having the following form.

Given $f \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} J(v) - J(u) + b(v - u, \lambda) \geq (f, v - u)_X &\quad \text{for all } v \in X\\ b(u, \mu - \lambda) \leq 0 &\quad \text{for all } \mu \in \Lambda, \end{aligned}$$

where J is *a proper convex lower semicontinuous functional.*

In Section 3.3 we study the weak solvability via Lagrange multipliers of a class of nonlinearly elastic contact models leading to a mixed variational problem governed by a nonlinear, hemicontinuous, generalized monotone operator. Using a fixed point theorem for set valued mapping, we analyze here the existence of the solution of the following abstract mixed variational problem.

Given $f \in X'$, find $(u, \lambda) \in X \times \Lambda$ such that

$$\begin{aligned} (Au, v)_{X',X} + b(v, \lambda) &= (f,\, v)_{X',X} && \text{for all } v \in X,\\ b(u, \mu - \lambda) &\leq 0 && \text{for all } \mu \in \Lambda, \end{aligned}$$

where A is *a nonlinear, hemicontinuous, generalized monotone operator.* Then, we apply the abstract result to the analysis of an antiplane contact problem involving a class of nonlinearly elastic materials.

Chapter 4, devoted to a class of viscoelastic frictional contact problems has two sections, treating the case of the viscoelasticity with long memory as well as the case of the viscoelasticity with short memory. Section 4.1 is based on the paper [101] and Section 4.2 is based on the paper [111].

In Section 4.1 we can see haw the viscoelastic model with long memory leads to a *time-dependent mixed variational problem involving an integral operator*, which, in an abstract framework, has the following form.

Given $f : [0,T] \to X$, *find* $u : [0,T] \to X$ *and* $\lambda : [0,T] \to Y$ *such that, for all* $t \in [0,T]$, *we have* $\lambda(t) \in \Lambda$ *and*

$$\begin{aligned} (Au(t), v)_X + (\int_0^t B(t-s)u(s)ds, v)_X + b(v, \lambda(t)) &= (f(t), v)_X \quad \text{for all } v \in X \\ b(u(t), \mu - \lambda(t)) &\le 0 \quad \text{for all } \mu \in \Lambda. \end{aligned}$$

In Section 4.2 we study a viscoelastic model with short memory leading to *an evolutionary mixed variational problem having the form below.*

Given $f : [0,T] \to X$, $g \in W$ *and* $u_0 \in X$, *find* $u : [0,T] \to X$ *and* $\lambda : [0,T] \to \Lambda(g) \subset Y$ *such that for all* $t \in (0,T)$, *we have*

$$\begin{aligned} a(\dot{u}(t), v) + e(u(t), v) + b(v, \lambda(t)) &= (f(t), v)_X \quad \text{for all } v \in X, \\ b(\dot{u}(t), \mu - \lambda(t)) &\le 0 \quad \text{for all } \mu \in \Lambda(g), \\ u(0) &= u_0. \end{aligned}$$

Chapter 5, who studies a class of frictionless contact problems, comprises two sections. Section 5.1 is based on the papers [98, 70] and Section 5.2 is based on the paper [11].

Section 5.1 analyzes the case of electro-elastic materials, treating the case of nonconductive foundation as well as the case of conductive foundation. Both weak formulations we deliver are *generalized saddle point problem.* The variational formulation in the nonconductive case consists of the following nonhomogeneous and nonsymmetric mixed variational problem. *Given* f, $g \in X$, $g \neq 0_X$, *find* $u \in X$ *and* $\lambda \in \Lambda$ such that

$$\begin{aligned} a(u, v)_X + b(v, \lambda) &= (f, v)_X \quad \text{for all } v \in X, \\ b(u, \mu - \lambda) &\le b(g, \mu - \lambda) \quad \text{for all } \mu \in \Lambda. \end{aligned}$$

In the conductive case, the weak formulation consists of the following *coupled variational system.*

Given $\boldsymbol{f} \in X$ *and* $q \in Y$, *find* $(\boldsymbol{u}, \varphi, \lambda) \in X \times Y \times \Lambda$ *such that*

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + e(\boldsymbol{v}, \varphi) + b(\boldsymbol{v}, \lambda) &= (\boldsymbol{f}, \boldsymbol{v})_X \quad \text{for all } \boldsymbol{v} \in X, \\ c(\varphi, \psi) - e(\boldsymbol{u}, \psi) + j(\lambda, \varphi, \psi) &= (q, \psi)_Y \quad \text{for all } \psi \in Y, \\ b(\boldsymbol{u}, \mu - \lambda) &\le 0 \quad \text{for all } \mu \in \Lambda. \end{aligned}$$

In Section 5.2, contact models involving viscoplastic materials are studied. The models lead to a weak formulation via Lagrange multipliers which consists of a variational system coupled with an integral equation; see the problem below.

Find a displacement field $\boldsymbol{u} : [0,T] \to V$*, a viscoplastic stress field* $\boldsymbol{\beta} : [0,T] \to Q$ *and a Lagrange multiplier* $\boldsymbol{\lambda} : [0,T] \to \Lambda$ *such that, for all* $t \in [0,T]$*,*

$$\begin{aligned}
&(L\boldsymbol{u}(t), \boldsymbol{v})_V + (\boldsymbol{\beta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,\\
&b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \text{for all } \boldsymbol{\mu} \in \Lambda,\\
&\boldsymbol{\beta}(t) = \int_0^t \mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \boldsymbol{\beta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0).
\end{aligned}$$

Chapter 6, divided in two sections, focuses on the weak solvability of a class of contact problems involving two contact zones, for elastic materials. Section 6.1 is based on the paper [110] and Section 6.2 is based on Sections 3 and 4 of the paper [113].

Section 6.1 focuses on the case of linearly elastic materials. To start, we present existence, uniqueness and boundedness results for a class of abstract generalized saddle point problems, as well as abstract convergence results for a class of regularized problems. The abstract problem we analyze has the following form:

$$\begin{aligned}
a(u, v-u) + b(v-u, \lambda) + j(v) - j(u) &\geq (f, v-u)_X \qquad \text{for all } v \in X,\\
b(u, \mu - \lambda) &\leq 0 \qquad \text{for all } \mu \in \Lambda.
\end{aligned}$$

Based on the abstract results we get two models are investigated. Every model is mathematically described by a boundary value problem which consists of a system of partial differential equations associated with a displacement condition, a traction condition, a frictional contact condition and a frictionless unilateral contact condition. In both models the unilateral contact is described by Signorini's condition with non zero gap. The difference between the models is given by the frictional condition we use. Thus, in the first model we use a frictional condition with prescribed normal stress, while in the second one we use a frictional bilateral contact condition.

Section 6.2 focuses on the case of nonlinearly elastic materials. Firstly, we present abstract results in the study of a *generalized saddle point problem* having the following form:

$$\begin{aligned}
J(v) - J(u) + b(v-u, \lambda) + \varphi(v) - \varphi(u) &\geq (f, v-u)_X \qquad \text{for all } v \in X,\\
b(u, \mu - \lambda) &\leq 0 \qquad \text{for all } \mu \in \Lambda.
\end{aligned}$$

Nextly, we apply the abstract results to the weak solvability of two contact models.

The study made in Chapter 7 goes to the weak solvability of an unilateral frictional contact model. In Section 7.1 we study the existence and the uniqueness of the solution of an abstract mixed variational problem governed by a functional J and a bifunctional j as follows:

$$\begin{aligned} J(v) - J(u) + b(v-u,\lambda) + j(\lambda, v) - j(\lambda, u) &\geq (f,\, v-u)_X \qquad \text{for all } \; v \in X, \\ b(u, \mu - \lambda) &\leq 0 \qquad\qquad\qquad \text{for all } \; \mu \in \Lambda. \end{aligned}$$

Next, in Section 7.2, we apply the abstract results to the weak solvability of an unilateral frictional contact model.

Part II, devoted to a variational approach via bipotentials for a class of contact problems for nonlinearly elastic materials, relies on the papers [106, 108]. The presence of the bipotentials in mechanics of solid was noticed quite recently, but the literature covering this subject is growing. According to Buliga-Saxcé-Vallée, starting from an extension of Fenchel inequality, the bipotentials were introduced as non-smooth mechanics tools used to model various multivalued laws. Several bipotential functions are related to the Coulomb's friction law [26], Cam-Clay models in soil mechanics [134, 159], cyclic plasticity [16, 133], viscoplasticity of metals with non-linear kinematical hardening rule [64], Lemaitre's damage law [15], the coaxial laws [136, 155], the elastic-plastic bipotential of soils [13]. For other important results related to the bipotential theory we refer for instance to [25, 27, 28, 135]; see also the recent work [156].

Herein, two contact problems are focused: a unilateral frictionless contact problem and a frictional contact problem with prescribed normal stress. In order to solve them, the main technique we use is a minimization technique. Using a separated bipotential we investigate the existence and the uniqueness of the solutions. The unknown is the pair consisting of the displacement vector and the Cauchy stress tensor. The main advantage of this approach is that it allows to compute simultaneously the displacement field and the Cauchy stress tensor. Also we discuss the relevance of the approach reported to previous variational approaches: the primal variational formulation and the dual variational formulation. We recall that the *primal variational formulation* is the weak formulation in displacements, and the *dual variational formulation* is the weak formulation in terms of stress.

The problems we treat in this second part of the thesis lead to new variational systems governed by bipotentials. The investigation on this direction can be extended to more complex models governed by non-separated bipotentials attached to the constitutive map and its Fenchel conjugate.

Part II comprises two chapters, Chapter 8 and Chapter 9.

In Chapter 8 we present the results in the study of a class of unilateral frictionless contact problem obtained in the paper [106]. The variational approach we use leads us to a variational problem having the following form.

Find $\boldsymbol{u} \in U_0 \subset V$ *and* $\boldsymbol{\sigma} \in \Lambda \subset L^2_s(\Omega)^{3\times 3}$ *such that*

$$\begin{aligned} &b(\boldsymbol{v}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \qquad \text{for all } \boldsymbol{v} \in U_0 \\ &b(\boldsymbol{u}, \boldsymbol{\mu}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0 \qquad\qquad\quad\;\; \text{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

In Chapter 9 we present the results in the study of a class of frictional contact problem obtained in the paper [108]. In this case, the variational approach via bipotentials leads to a variational problem governed by a functional j as follows.

Find $\boldsymbol{u} \in V$ *and* $\boldsymbol{\sigma} \in \Lambda \subset L^2_s(\Omega)^{3\times 3}$ *such that*

$$\begin{aligned} b(\boldsymbol{v}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad & \text{for all } \boldsymbol{v} \in V \\ b(\boldsymbol{u}, \boldsymbol{\mu}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0 \quad & \text{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

Part III discusses the variational analysis via history-dependent quasivariational inequalities of a class of viscoplastic or electro-elasto-viscoplastic contact problem on unbounded time interval. Part III focuses on results obtained in the papers [10, 20]. The first study of a contact problem on the unbounded interval $[0, \infty)$ was made in [144]. The next important contribution was the paper [146], followed by [10, 20], and more recently by [148].

In this third part of the thesis we focus on new contact models related to quasivariational inequalities defined on unbounded time interval and governed by two nondifferentiable convex functional in which one depends on the history of the solution,

$$\begin{aligned} u(t) \in K, \quad & (Au(t), v - u(t))_X + \varphi(\mathcal{S}u(t), v) - \varphi(\mathcal{S}u(t), u(t)) \\ & + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t))_X \qquad \text{for all } v \in K. \end{aligned}$$

These inequalities have a special structure, involving a history-dependent term. In addition, working on the time interval $[0, \infty)$, a continuation of the research going to the Asymptotic Analysis in Contact Mechanics is envisaged.

Part III comprises three chapters, Chapter 10-Chapter 12. Some auxiliary abstract results are presented in Chapter 10: a fixed point result and an existence and uniqueness result.

In Chapter 11 we present results obtained in [10] in the study of two viscoplastic problems and in Chapter 12 we present results obtained in [20] in the study of an electro-elasto-viscoplastic problem.

The names of the three parts in the presentation of the SCIENTIFIC RESULTS indicate the main research directions in which the candidate had original contributions. Let us nominee here *the main contributions*:

▶ the statement and the solution of *three new classes of abstract problems*:

- *abstract stationary mixed variational problems governed by nonlinear maps*
- *abstract evolutionary mixed variational problems (with short-memory term)*
- *abstract time-dependent mixed variational problems (with long-memory term)*

▶ the weakly solvability of contact models (*by new variational techniques*)

- for *nonlinearly elastic, viscoelastic, viscoplastic or electro-elastic materials* via a *variational approach with Lagrange multipliers*
- for *nonlinearly elastic materials governed by possibly set valued elastic operators* by means of *a variational approach via bipotentials theory*
- on *unbounded time interval.*

We end this thesis by presenting some CAREER EVOLUTION AND DEVELOPMENT PLANS. The presentation is structured in two chapters, Chapters 13-14. Chapter 13 presents further research directions such us: qualitative and numerical analysis in the study of mixed variational problems or in the study of variational systems via bipotentials; variational formulations in contact mechanics/ weak solutions via weighted Sobolev spaces or via Lebesgue spaces with variable exponent; optimal control problems in contact mechanics; mathematical techniques in the study of dissipative dynamic contact problems; asymptotic analysis in contact mechanics, regularity results; convergence results; viscoelastic problems via fractional differential operators/fractional calculus of variations. Chapter 14 presents further plans on the scientific, professional and academic career such as: to do a research activity allowing to continue to publish in international journals of hight level, to participate to international meetings in order to disseminate the best results, to organize scientific meetings, to extend the editorial activities for scientific journals, to continue the collaborations started in the past and to establish new contacts, to apply for national/international/interdisciplinary research projects as manager or member, to publish Lecture Notes and new monographs addressed to the students or researchers, to extend the advising activity to Ph.D. theses.

Acknowledgments The end of this thesis is a special moment for me. At this special moment it is my pleasure to express my gratitude to my mentors Professor Vicenţiu Rădulescu, Professor Mircea Sofonea and Professor Barbara Wohlmuth. Special thanks equally go to Professor Sorin Micu and Professor Constantin Niculescu for encouragement and stimulating discussions over the years. I also express my thanks to my co-authors and my colleagues for a nice collaboration. And many special thanks go to my family for constant support and understanding!

December 2014

The most frequent notation

- $\mathbb{S}^3$ denotes the space of second order symmetric tensors on $\mathbb{R}^3$.
- Every field in $\mathbb{R}^3$ or $\mathbb{S}^3$ is typeset in boldface.
- By $\cdot$ and $\|\cdot\|$ we denote the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively.
- For each $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $\|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$; for each $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$, $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$; the indices i and j run between 1 and d and the summation convention over repeated indices is applied.
- $\boldsymbol{u} = (u_i)$ denotes the displacement field.
- $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$ denotes the normal displacement ($\boldsymbol{n}$ being herein the outward normal vector).
- $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n\, \boldsymbol{n}$ denotes the tangential component of the displacement field.
- $\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the Cauchy stress tensor.
- $\sigma_n = (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{n}$ denotes the normal component of the stress on the boundary.
- $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n\, \boldsymbol{n}$ denotes the tangential component of the stress on the boundary.
- $\overline{\Omega} = \Omega \cup \partial\Omega$; $\Gamma = \partial\Omega$.
- $H^1(\Omega)^d$ ($d \in \{2,3\}$) denotes the standard Sobolev space.
- $L^p(\Omega)^d$ ($d \in \{2,3\}$, $p \geq 1$) denotes the standard Lebesgue space.
- $\gamma : H^1(\Omega) \to L^2(\Omega)$ is Sobolev's trace operator for scalar valued functions.
- $\boldsymbol{\gamma} : H^1(\Omega)^d \to L^2(\Omega)^d$ is Sobolev's trace operator for vector valued functions ($d \in \{2,3\}$).
- For each $\boldsymbol{w} \in H^1(\Omega)^d$, $w_\nu = \boldsymbol{\gamma}\boldsymbol{w} \cdot \boldsymbol{\nu}$ and $\boldsymbol{w}_\tau = \boldsymbol{\gamma}\boldsymbol{w} - w_\nu \boldsymbol{\nu}$ a.e. on Γ ($d \in \{2,3\}$).
- Div denotes the divergence operator for tensor valuated functions.
- div denotes the divergence operator for vector valuated functions.
- $\mathcal{E} = (\mathcal{E}_{ijkl})$ (or $\mathcal{C} = (\mathcal{C}_{ijkl})$) denotes a fourth order elastic tensor.
- $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ is the infinitezimal strain tensor with components $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ for all $i, j \in \{1,2,3\}$.
- If $\mathcal{E} = (\mathcal{E}_{ijl})$ is the piezoelectric tensor, $\mathcal{E}^\top$ denotes the transpose of the tensor $\mathcal{E}$ given by $\mathcal{E}\,\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^\top\, \boldsymbol{v}, \qquad \boldsymbol{\sigma} \in \mathbb{S}^d,\ \boldsymbol{v} \in \mathbb{R}^d$, and $\mathcal{E}^\top = (\mathcal{E}^\top_{ijl}) = (\mathcal{E}_{lij})$ for all $i, j, l \in \{1, ..., d\}$.
- l.s.c = lower semicontinuous

Part I

A variational approach via Lagrange multipliers

Chapter 1

Slip-independent frictional contact problems

This chapter is based on the papers [69, 68]. Firstly, we study the antiplane shear deformation of two elastic bodies in frictional contact on their common boundary. To model the friction, we use Tresca's law. Our study is based on a mixed variational formulation with dual Lagrange multipliers, the well-possednes of this weak formulation being guaranteed by arguments in the saddle point theory. This approach results in an efficient iterative solver for the nonlinear problem with a negligible additional effort compared to solving a linear problem. Nextly, we study the frictional contact between an elasto-piezoelectric body and a rigid foundation. Our study is based on a non-symmetric mixed variational formulation involving dual Lagrange multipliers. The well-posedness of this variational problem is justified by combining a fixed point technique with a saddle point technique.

1.1 An antiplane problem

This section is based on the paper [69]. The mechanical model used in this section involves the particular type of deformation that a solid can undergo, the *antiplane shear deformation*. For a cylindrical body subject to antiplane shear, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. The antiplane shear (or longitudinal shear, generalized shear) may be viewed as complementary to the *plane strain deformation*, and represents the Mode III, fracture mode for crack problems.

1.1.1 The model and its weak solvability

Let us consider two cylinders $\mathcal{B}^m$, $\mathcal{B}^s \subset \mathbb{R}^3$ having the generators parallel to the x_3-axis of a rectangular cartesian coordinate system $Ox_1x_2x_3$. We use a superscript k to indicate that a quantity is related to the cylinder $\mathcal{B}^k$, $k = m, s$. We assume that the bodies are homogeneous, isotropic and elastic media; more precisely, we shall use the constitutive law

$$\boldsymbol{\sigma}^k = \lambda^k \mathrm{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}^k))I + 2\mu^k \boldsymbol{\varepsilon}(\boldsymbol{u}^k) \quad \text{in} \quad \mathcal{B}^k, \tag{1.1}$$

where $\boldsymbol{\sigma}^k = (\sigma_{ij}^k)$ denotes the stress field, $\boldsymbol{\varepsilon}(\boldsymbol{u}^k) = (\varepsilon_{ij}(\boldsymbol{u}^k))$ the linearized strain tensor, $\lambda^k > 0$ and $\mu^k > 0$ are the Lamé coefficients, $\mathrm{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}^k)) = \sum_{i=1}^3 \varepsilon_{ii}(\boldsymbol{u}^k)$ and I is the unit tensor in $\mathbb{R}^3$. Moreover, we assume that the generators are sufficiently long so that end effects in the axial direction are negligible. Let us denote by Ω^k a cross-section, which is a domain in $\mathbb{R}^2$. Thus, $\mathcal{B}^k = \Omega^k \times (-\infty, +\infty)$. For each domain Ω^k, we assume that its boundary Γ^k is Lipschitz continuous and is divided into three disjoint measurable parts Γ_1^k, Γ_2^k and Γ_3^k, with meas $(\Gamma_1^k) > 0$. We assume that the bodies are clamped on Γ_1^k, body forces of density $\boldsymbol{f}_0^k$ act on Ω^k and surface tractions of density $\boldsymbol{f}_2^k$ act on Γ_2^k. Moreover, we assume that the bodies in the initial configuration are in contact on their common boundary part $\Gamma_3 = \Gamma_3^1 = \Gamma_3^2$ and that $\overline{\Gamma}_3$ is a compact subset of $\partial\Omega^k \backslash \overline{\Gamma}_1^k$, $k = m, s$. We load the solid in a special way, as follows,

$$\boldsymbol{f}_0^k = (0, 0, f_0^k), \quad f_0^k = f_0^k(x_1, x_2) : \Omega^k \to \mathbb{R}, \tag{1.2}$$

$$\boldsymbol{f}_2^k = (0, 0, f_2^k), \quad f_2^k = f_2^k(x_1, x_2) : \Gamma_2^k \to \mathbb{R}. \tag{1.3}$$

The unit outward normal on $\Gamma^k \times (-\infty, +\infty)$ is denoted by $\boldsymbol{n}^k$, $\boldsymbol{n}^k = (n_1^k, n_2^k, 0)$ and is defined almost everywhere. We note that on Γ_3,

$$\boldsymbol{n} = \boldsymbol{n}^s = -\boldsymbol{n}^m, \quad \boldsymbol{\sigma}^s \boldsymbol{n}^s = -\boldsymbol{\sigma}^m \boldsymbol{n}^m, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}_\tau^s = -\boldsymbol{\sigma}_\tau^m.$$

For a vector $\boldsymbol{v}^k$, we denote by v_n^k and $\boldsymbol{v}_\tau^k$ its *normal* and *tangential* parts on the boundary, given by $v_n^k = \boldsymbol{v}^k \cdot \boldsymbol{n}^k$ and $\boldsymbol{v}_\tau^k = \boldsymbol{v}^k - v_n^k \boldsymbol{n}^k$, respectively. Furthermore, for a given stress field $\boldsymbol{\sigma}^k$, we denote by σ_n^k and $\boldsymbol{\sigma}_\tau^k$ the *normal* and the *tangential* parts on the boundary, that is $\sigma_n^k = (\boldsymbol{\sigma}^k \boldsymbol{n}^k) \cdot \boldsymbol{n}^k$ and $\boldsymbol{\sigma}_\tau^k = \boldsymbol{\sigma}^k \boldsymbol{n}^k - \sigma_n^k \boldsymbol{n}^k$, respectively.

The body forces (1.2) and the surface tractions (1.3) would be expected to give rise to a deformation of the elastic cylinder $\mathcal{B}^k$, such that the displacement $\boldsymbol{u}^k$ is of the form

$$\boldsymbol{u}^k = (0, 0, u^k), \quad u^k = u^k(x_1, x_2) : \Omega^k \to \mathbb{R}. \tag{1.4}$$

The Cauchy stress vector on $\Gamma^k \times (-\infty, +\infty)$ is given by

$$\boldsymbol{\sigma}^k \boldsymbol{n}^k = (0, 0, \mu^k \partial_n u^k),$$

where, as usual, $\partial_n u^k = \nabla u^k \cdot \boldsymbol{n}^k$. In addition,

$$\mathrm{Div}\, \boldsymbol{\sigma}^k = (0, 0, \mu^k \Delta u^k). \tag{1.5}$$

According to the physical setting, we have

$$\mathrm{Div}\, \boldsymbol{\sigma}^k + \boldsymbol{f}_0^k = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}^k, \tag{1.6}$$

$$\boldsymbol{u}^k = \boldsymbol{0} \quad \text{on} \quad \Gamma_1^k \times (-\infty, +\infty), \tag{1.7}$$

$$\boldsymbol{\sigma}^k \boldsymbol{n}^k = \boldsymbol{f}_2^k \quad \text{on} \quad \Gamma_2^k \times (-\infty, +\infty). \tag{1.8}$$

Finally, we have to describe the frictional contact condition on $\Gamma_3 \times (-\infty, +\infty)$. Since $u_n^s = u_n^m = 0$, on $\Gamma_3 \times (-\infty, +\infty)$ the contact is bilateral. The friction was modeled by using Tresca's law,

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \leq g, \\ \|\boldsymbol{\sigma}_\tau\| < g \;\Rightarrow\; \boldsymbol{u}_\tau^s - \boldsymbol{u}_\tau^m = 0, & \text{on} \quad \Gamma_3 \times (-\infty, +\infty), \\ \|\boldsymbol{\sigma}_\tau\| = g \;\Rightarrow\; \boldsymbol{\sigma}_\tau = -\beta(\boldsymbol{u}_\tau^s - \boldsymbol{u}_\tau^m), \quad \beta > 0, \end{cases} \tag{1.9}$$

where g is the *friction bound.* We note that $\boldsymbol{u}_\tau^k = (0,0,u^k)$ and $\boldsymbol{\sigma}_\tau^k = (0,0,\sigma_\tau^k)$ with $\sigma_\tau^k = \mu^k\,\partial_n u^k$. In addition, we have

$$\partial_n\, u = \mu^s \partial_n\, u^s = -\mu^m \partial_n\, u^m \quad \text{on} \quad \Gamma_3. \tag{1.10}$$

The mathematical description of the mechanical model is the following one.

Problem 1.1. *Find the displacement fields $u^k : \Omega \to \mathbb{R}$, $k = m,\ s$, such that*

$$\mu^k \Delta\, u^k + f_0^k = 0 \qquad in \qquad \Omega^k, \tag{1.11}$$

$$u^k = 0 \qquad on \qquad \Gamma_1^k, \tag{1.12}$$

$$\mu^k \partial_n\, u^k = f_2^k \qquad on \qquad \Gamma_2^k, \tag{1.13}$$

$$\begin{cases} |\partial_n\, u| \leq g, \\ |\partial_n\, u| < g \Rightarrow u^s - u^m = 0, & on \quad \Gamma_3. \\ |\partial_n\, u| = g \Rightarrow \partial_n\, u = -\beta(u^s - u^m), \quad \beta > 0, \end{cases} \tag{1.14}$$

In the study of Problem 1.1 we made the following assumptions.

Assumption 1.1. $f_0^k \in L^2(\Omega^k), \quad f_2^k \in L^2(\Gamma_2^k)$.

Assumption 1.2. $g \in L^2(\Gamma_3), \qquad g \geq 0 \;$ *a.e. on* Γ_3.

In order to write a weak formulation for Problem 1.1, we need the Hilbert space

$$V^k = \left\{ v^k \in H^1(\Omega^k) \,|\, v^k = 0 \quad \text{a. e. on} \quad \Gamma_1^k \right\} \; k \in \{m, s\} \; (v^k = 0 \text{ in the sense of the trace})$$

endowed with the inner product

$$(u^k, v^k)_{V^k} = \int_{\Omega^k} \nabla u^k \cdot \nabla v^k \, dx, \qquad \text{for all } u^k, v^k \in V^k,$$

and the associated norm,

$$\|v^k\|_{V^k} = \|\nabla v^k\|_{L^2(\Omega^k)^2}, \qquad v^k \in V^k.$$

We consider the product space $V = V^m \times V^s$ and let $a : V \times V \to \mathbb{R}$ be the bilinear form

$$a(u, v) = \sum_{k \in \{m,s\}} \mu^k (u^k, v^k)_{V^k}.$$

This form is continuous with the continuity constant $M_a = \mu^m + \mu^s$ and V-elliptic, with the V-ellipticity constant $m_a = \min\{\mu^m, \mu^s\}$.

We define $f \in V$ such that

$$(f, v)_V = \sum_{k \in \{m,s\}} \left(\int_{\Omega^k} f_0^k \, v^k \, dx + \int_{\Gamma_2^k} f_2^k \, v^k \, ds \right).$$

Next, we define a Lagrange multiplier λ in $D = (H^{1/2}(\Gamma_3))'$, as follows:

$$\langle \lambda, w \rangle_{\Gamma_3} = -\int_{\Gamma_3} \partial_n u \, w \, ds \tag{1.15}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_3}$ denotes the duality pairing. Furthermore, we introduce a nonempty closed convex set,

$$\Lambda = \left\{ \zeta \in D \, : \, \langle \zeta, \, w \rangle_{\Gamma_3} \leq \int_{\Gamma_3} g \, |w| \, ds, \quad w \in H^{1/2}(\Gamma_3) \right\}. \tag{1.16}$$

We denote by $[v]$ the *jump on* Γ_3 :

$$[v] = v^s - v^m, \qquad v = (v^m, v^s) \in V.$$

Let us denote by $b : V \times D \to \mathbb{R}$, the bilinear and continuous form

$$b(v, \, \zeta) = \langle \zeta, [v] \rangle_{\Gamma_3}. \tag{1.17}$$

The weak formulation of Problem 1.1 is as follows.

Problem 1.2. *Find $u \in V$ and $\lambda \in \Lambda$ such that*

$$\begin{array}{lcll} a(u, \, v) + b(v, \lambda) & = & (f, v)_V & \textit{for all } v \in V, \\ b(u, \, \zeta - \lambda) & \leq & 0 & \textit{for all } \zeta \in \Lambda. \end{array}$$

The well-possedness of Problem 1.2 is given by the following theorem.

Theorem 1.1. *[Theorem 2.1 in [69]] If Assumptions* 1.1-1.2 *hold true, then there exists a unique solution $(u, \, \lambda) \in V \times \Lambda$ of Problem 1.2. Moreover, if $f_1, \, f_2$ are elements in V corresponding to the data $(f_0, \, f_2)_1, \, (f_0, \, f_2)_2$, respectively, then we have the stability result,*

$$\|u_1 - u_2\|_V + \|\lambda_1 - \lambda_2\|_D \leq \frac{\alpha + m_a + M_a}{\alpha \, m_a} \|f_1 - f_2\|_V, \tag{1.18}$$

where (u_1, λ_1), (u_2, λ_2) are the solutions of Problem 1.2 corresponding to $f_1, \, f_2 \in V$, respectively.

The proof of Theorem 1.1, given in [69], is based on the *saddle point theory* (see for instance [50]).

1.1.2 Nonconforming discretization and optimal a priori error estimates

In this subsection, we give the discretization of Problem 1.2, and we present an optimal a priori error estimate for the discretization error. Let us assume that the bodies Ω^k, $k = m, s$, are polygonal domains. To approximate V, we use lowest order finite elements on simplicial or quadrilateral triangulations. The finite element space associated with the shape regular triangulation $\mathcal{T}_{h,\Omega^k}$ is denoted by $S_1(\Omega^k, \mathcal{T}_{h,\Omega^k})$. The meshsize h is defined by the maximal diameter of the elements in $\mathcal{T}_{h,\Omega^m}$ and $\mathcal{T}_{h,\Omega^s}$. For simplicity, we assume that Γ_1^k, $k = m, s$, and Γ_3 can be written as union of edges, Γ_3 both from the triangulation $\mathcal{T}_{h,\Omega^s}$ of the slave side and from the triangulation $\mathcal{T}_{h,\Omega^m}$ of the master side. Before introducing the discrete spaces, we decompose the set of all vertices into three disjoint sets $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{N}$. By the first subset $\mathcal{S}$, we denote all vertices on $\overline{\Gamma}_3$ of the triangulation $\mathcal{T}_{h,\Omega^s}$ on the slave side, by $\mathcal{M}$ all vertices on $\overline{\Gamma}_3$ of the triangulation $\mathcal{T}_{h,\Omega^m}$ on the master side. The set $\mathcal{N}$ contains all remaining ones. Then we have for the discrete spaces V_h^k

$$V_h^k = \left\{ v_h^k \in S_1\left(\Omega^k,\, \mathcal{T}_{h,\Omega^k}\right) \,:\, v_h^k = 0 \quad \text{on} \quad \Gamma_1^k \right\} \subset V^k,$$

and we define $V_h = V_h^m \times V_h^s$. For the discretization of the Lagrange multiplier space we use dual shape functions, introduced in [163]. In the case of linear or bilinear finite elements in 2D, the dual basis functions are associated with the vertices. We use discontinuous piecewise linear functions having value two at the associated vertex and value minus one at the two neighbor vertices as basis functions. We denote this discrete Lagrange multiplier space by $M_h = \operatorname{span}\{\psi_p,\ p \in \mathcal{S}\}$, where ψ_p denotes the basis function associated with the vertex p. Then the biorthogonality of the basis functions yields

$$\langle \psi_p, \varphi_q \rangle_{\Gamma_3} \;=\; \delta_{pq} \int_{\Gamma_3} \varphi_q \, ds, \qquad p, q \in \mathcal{S}, \tag{1.19}$$

where φ_q are the standard nodal basis functions of $S_1\left(\Omega^s,\, \mathcal{T}_{h,\Omega^s}\right)$ associated with the vertex q. The finite element space V_h can be written in terms of the standard finite element basis $\boldsymbol{\varphi}$ as $V_h = \operatorname{span}\{\varphi_p,\ p \in \mathcal{S} \cup \mathcal{M} \cup \mathcal{N}\}$. Additionally to the basis $\boldsymbol{\varphi}$, we introduce the constrained finite element basis $\hat{\boldsymbol{\varphi}}$, see [165]. To introduce these basis functions, we define the entries of the coupling matrices D and M between the finite element basis functions φ_p and the basis functions for the Lagrange multiplier space ψ_p by

$$\begin{aligned} D[p,q] &= \langle \psi_p, \varphi_q \rangle_{\Gamma_3}, \quad p,\, q \in \mathcal{S}, \\ M[p,q] &= \langle \psi_p, \varphi_q \rangle_{\Gamma_3}, \quad p \in \mathcal{S},\, q \in \mathcal{M}. \end{aligned}$$

Due to the biorthogonality (1.19), the matrix D is diagonal. In terms of $\hat{M} = D^{-1}M$, we obtain the constrained basis $\hat{\boldsymbol{\varphi}}$ of V_h from the nodal basis $\boldsymbol{\varphi}$ of V_h by the transformation

$$\hat{\boldsymbol{\varphi}} = \begin{pmatrix} \hat{\varphi}_{\mathcal{N}} \\ \hat{\varphi}_{\mathcal{M}} \\ \hat{\varphi}_{\mathcal{S}} \end{pmatrix} = \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & \hat{M}^{\top} \\ 0 & 0 & Id \end{pmatrix} \begin{pmatrix} \varphi_{\mathcal{N}} \\ \varphi_{\mathcal{M}} \\ \varphi_{\mathcal{S}} \end{pmatrix} = Q\boldsymbol{\varphi}. \tag{1.20}$$

We note that only basis functions associated with a node $p \in \mathcal{M}$ are changed, and that by definition

$$b(\hat{\varphi}_q, \psi_p) = 0, \quad p \in \mathcal{S},\, q \in \mathcal{M}.$$

For simplicity of notation, we use the same symbol for a function in V_h and M_h as for its algebraic representation with respect to the nodal basis. Let v_h be the algebraic representation of an element $v_h \in V_h$ with respect to the basis $\boldsymbol{\varphi}$ and let $\hat{v}_h$ be the corresponding algebraic representation with respect to the constrained basis $\hat{\boldsymbol{\varphi}}$. Then we have the relation $v_h = Q^{\top}\hat{v}_h$. Now, after an easy computation, taking into account the biorthogonality (1.19), we get

$$b(v_h, \mu_h) = \langle \mu_h, \sum_{p\in\mathcal{S}} v_p\varphi_p - \sum_{q\in\mathcal{M}} v_q\varphi_q\rangle_{\Gamma_3} = \langle \mu_h, \sum_{p\in\mathcal{S}} \hat{v}_p\hat{\varphi}_p\rangle_{\Gamma_3} = \langle \mu_h, \sum_{p\in\mathcal{S}} \hat{v}_p\varphi_p\rangle_{\Gamma_3}.$$

Before introducing the discrete set Λ_h for the admissible Lagrange multiplier, we define for $v_h \in V_h$ the restriction to the slave side of the interface Γ_3 by

$$v_{h,\mathcal{S}} = \sum_{p\in\mathcal{S}} v_p\varphi_p.$$

Now we define the discrete convex set Λ_h by

$$\Lambda_h = \Big\{ \mu_h \in M_h \,:\, \langle \mu_h, v_{h,\mathcal{S}}\rangle_{\Gamma_3} \le \int_{\Gamma_3} g|v_{h,\mathcal{S}}|_h \, ds, \quad v_h \in V_h \Big\}, \tag{1.21}$$

where the mesh dependent absolute value $|v_{h,\mathcal{S}}|_h$ of the function $v_{h,\mathcal{S}}$ is given by

$$|v_{h,\mathcal{S}}|_h = \sum_{p\in\mathcal{S}} |v_p|\varphi_p.$$

We remark that in general $|v_{h,\mathcal{S}}|_h \neq |v_{h,\mathcal{S}}|$. Everywhere below in this subsection, we assume that g is a strictly positive constant. In this case, the convex set Λ_h can be equivalently written as

$$\Lambda_h = \Big\{ \mu_h \in M_h \,:\, \mu_h = \sum_{p\in\mathcal{S}} \gamma_p\psi_p, \quad |\gamma_p| \le g, \quad p \in \mathcal{S} \Big\}. \tag{1.22}$$

The discrete formulation of Problem 1.2 is the following.

Problem 1.3. *Find $u_h \in V_h$ and $\lambda_h \in \Lambda_h$ such that*

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= (f, v_h)_V, \qquad && v_h \in V_h, \\ b(u_h, \zeta_h - \lambda_h) &\leq 0, && \zeta_h \in \Lambda_h. \end{aligned}$$

Using the discrete inf-sup property for the spaces M_h and V_h, see, e.g., [163], we get the existence and the uniqueness of the solution. In order to obtain an optimal a priori error estimate, several lemmas will be proved. We note that $\Lambda_h \not\subset \Lambda$.

Before presenting the first lemma, we have to consider for a function $v_h \in V_h$ the discrete jump $\hat{v}_{h,\mathcal{S}}$ on the interface Γ_3 in the constrained basis and its mesh dependent absolute value by

$$\hat{v}_{h,\mathcal{S}} = \sum_{p \in \mathcal{S}} \hat{v}_p \varphi_p, \qquad |\hat{v}_{h,\mathcal{S}}|_h = \sum_{p \in \mathcal{S}} |\hat{v}_p| \varphi_p,$$

respectively.

Lemma 1.1. *[Lemma 3.1 in [69]] Let (u, λ) be the solution of Problem 1.2 and (u_h, λ_h) the solution of Problem 1.3. Then we have*

$$b(u, \lambda) = \int_{\Gamma_3} g|[u]|\, ds, \qquad b(u_h, \lambda_h) = \int_{\Gamma_3} g|\hat{u}_{h,\mathcal{S}}|_h\, ds.$$

Furthermore, the following result holds.

Lemma 1.2. *[Lemma 3.2 in [69]] Let $(u, \lambda) \in V \times \Lambda$ be the solution of Problem 3.18 and let $(u_h, \lambda_h) \in V_h \times \Lambda_h$ be the solution of the discrete Problem 1.3. Then there exists a positive constant C independent of the meshsize h, such that for all $v_h \in V_h$, $\mu_h \in M_h$,*

$$\|u - u_h\|_V^2 + \|\lambda - \lambda_h\|_M^2 \leq C \left\{ \|u - v_h\|_V^2 + \|\lambda - \mu_h\|_M^2 \right\} + b(u, \lambda_h - \lambda).$$

Let us denote $\overline{\gamma_{sl}} = \operatorname{supp}[u] \subset \Gamma_3$, $\gamma_{st} = \Gamma_3 \backslash \gamma_{sl}$, and we introduce the sets

$$\begin{aligned} \mathcal{W}^* &= \overline{\mathring{\gamma}_{st}} \cap \overline{\gamma_{sl}}, \\ \mathcal{W}^0 &= \Big\{ w \in \Gamma_3 \,:\, [u](w) = 0 \text{ and } \exists\, r > 0 \,:\, [u](w - \varepsilon)\,[u](w + \varepsilon) < 0, \quad 0 < \epsilon < r \Big\}, \\ \mathcal{W} &= \mathcal{W}^* \cup \mathcal{W}^0. \end{aligned}$$

Everywhere below we will use the following assumption.

Assumption 1.3. *The number of points in $\mathcal{W}$ is finite.*

The minimum distance between the elements in $\mathcal{W}$ is denoted by a, i.e.,

$$a = \inf\{|w_j - w_k| \,:\, 1 \leq j \neq k \leq N_w\},$$

where N_w denotes the number of points in $\mathcal{W}$. By Assumption 1.3, $N_w < \infty$ and thus $a > 0$. For $h < \frac{a}{2} =: h_0$, we find between two neighbor points in $\mathcal{W}$ at *least two vertices* in $\mathcal{S}$. Then the following lemma holds.

Lemma 1.3. *[Lemma 3.3 in [69]] Let* $(u, \lambda) \in V \times \Lambda$ *be the solution of Problem 1.2 and let* $(u_h, \lambda_h) \in V_h \times \Lambda_h$ *be the solution of Problem 1.3. Under Assumption 1.3 and the regularity assumption* $u^k \in H^{\frac{3}{2}+\nu}(\Omega^k)$, $0 < \nu \le \frac{1}{2}$, $k = m, s$, *we then have the a priori error estimate*

$$b(u, \lambda_h - \lambda) \;\; \le \;\; Ch^{\frac{1}{2}+\nu} \|\lambda - \lambda_h\|_M \sum_{k=m,s} |u^k|_{\frac{3}{2}+\nu,\Omega^k}$$

for a positive constant C *independent of* $h < h_0$.

Based on the results obtained in Lemma 1.2 and Lemma 1.3 and using the well known approximation properties for the spaces V_h and M_h, by applying Young's inequality, the following theorem holds.

Theorem 1.2 (Theorem 3.4 in [69]). *Let* $(u, \lambda) \in V \times \Lambda$ *be the solution of Problem 1.2 and let* $(u_h, \lambda_h) \in V_h \times \Lambda_h$ *be the solution of Problem 1.3. Under Assumption 1.3 and the regularity assumption* $u^k \in H^{\frac{3}{2}+\nu}(\Omega^k)$, $0 < \nu \le \frac{1}{2}$, $k = m, s$, *we then have the a priori error estimate*

$$\|u - u_h\|_V + \|\lambda - \lambda_h\|_M \;\; \le \;\; Ch^{\frac{1}{2}+\nu} \sum_{k=m,s} |u^k|_{\frac{3}{2}+\nu,\Omega^k}$$

for a positive constant C *independent of the meshsize* $h < h_0$.

Remark 1.1. *The discrete nonlinear problem was solved by using an inexact primal-dual active set strategy in Section 4 of the paper [69]. Numerical examples validating the theoretical result and illustrating the performance of the algorithm are also presented in [69], see Section 5.*

1.2 An elasto-piezoelectric problem

This section is based on the paper [68]. The *piezoelectricity* is the ability of certain crystals to produce a voltage when subjected to mechanical stress. The word is derived from the Greek piezein, which means to squeeze or press. Piezoelectric materials also show the opposite effect, called *converse piezoelectricity*; i.e., the application of an electrical field creates mechanical stresses (distortion) in the crystal. Because the charges inside the crystal are separated, the applied voltage affects different points within the crystal differently, resulting in the distortion. Many materials exhibit the piezoelectric effect (e.g. ceramics: $BaTiO_3$, $KNbO_3$, $LiNbO_3$, $LiTaO_3$, $BiFeO_3$). The first mathematical model of an elastic medium taking linear interaction of electric and mechanical fields into account was constructed by W. Voigt, see [157], and more refined models can be found for example in the works of R. Toupin [153, 154], R. Mindlin [121, 122, 123], S. Kalinski and J. Petikiewicz [80] and T. Ikeda [76].

1.2.1 An abstract auxiliary result

In this subsection we present the results in the study of the following abstract problem.

Problem 1.4. *Given $f \in X$, find $u \in X$ and $\lambda \in Y$ such that $\lambda \in \Lambda \subset Y$ and*

$$\begin{array}{lcll} a(u,v) + b(v,\lambda) & = & (f,\, v)_X & \textit{for all } v \in X, \\ b(u,\mu-\lambda) & \leq & 0 & \textit{for all } \mu \in \Lambda. \end{array}$$

We underline that Problem 1.4 is *not a classical saddle point problem*, because $a(\cdot,\cdot)$ is *non-symmetric*. The study of this problem was made under the following assumptions.

Assumption 1.4. *$(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 1.5. *$a(\cdot,\,\cdot) : X \times X \to \mathbb{R}$ is a non-symmetric bilinear form such that*
(i_1) there exists $M_a > 0 :\ |a(u,v)| \leq M_a \|u\|_X \|v\|_X$ for all $u, v \in X$,
(i_2) there exists $m_a > 0 :\ a(v,v) \geq m_a\, \|v\|_X^2$ for all $v \in X$.

Assumption 1.6. *$b(\cdot,\,\cdot) : X \times Y \to \mathbb{R}$ is a bilinear form such that*
(j_1) there exists $M_b > 0 :\ |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ for all $v \in X$, $\mu \in Y$,
(j_2) there exists $\alpha > 0 :\ \inf\limits_{\mu \in Y, \mu \neq 0_Y}\ \sup\limits_{v \in X, v \neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Assumption 1.7. *Λ is a closed convex subset of Y such that $0_Y \in \Lambda$.*

Let $a_0(u,\,v)$ and $c(u,\,v)$ be the symmetric, respectively the antisymmetric part of $a(u,\,v)$, that is

$$a_0(u,\,v) = \frac{1}{2}(a(u,\,v) + a(v,\,u)), \quad c(u,v) = \frac{1}{2}(a(u,\,v) - a(v,\,u)).$$

For a given $r \in [0,1]$, we introduce the following bilinear form

$$a_r\,(u,\,v) = a_0(u,\,v) + r\, c(u,\,v), \quad u,\, v \in X, \tag{1.23}$$

as a "perturbation" of $a_0(\cdot,\,\cdot)$. We underline that $a_1(u,\,v) = a(u,\,v)$ and for all $r \in [0,1]$ $a_r(u,\,v)$ is X-elliptic with the same ellipticity-constant m_a. Moreover, the bilinear forms $a_0(\cdot,\,\cdot)$ and $c(\cdot,\,\cdot)$ are continuous with the same continuity-constant M_a.

Let us consider the following problem.

Problem 1.5. *For a given $f \in X$, find $u \in X$ and $\lambda \in Y$ such that $\lambda \in \Lambda$ and*

$$\begin{array}{lcll} a_0(u,v) + b(v,\lambda) & = & (f,\, v)_X & \textit{for all } v \in X, \\ b(u,\mu-\lambda) & \leq & 0 & \textit{for all } \mu \in \Lambda. \end{array}$$

Lemma 1.4. *[Lemma 3.4 in [68]] Assumptions 1.4–1.7 hold true. Given $f \in X$, there exists a unique solution of Problem 1.5, $(u,\,\lambda) \in X \times \Lambda$.*

Let $\mathcal{L} : X \times \Lambda \to \mathbb{R}$ be the functional defined as follows:

$$\mathcal{L}(v,\, \mu) = \frac{1}{2} a(v, v) - (f, v)_X + b(v, \mu).$$

Using this definition, an equivalent formulation of Problem 1.5. is the following saddle point problem: find $u \in X$ and $\lambda \in \Lambda$ such that

$$\mathcal{L}(u,\, \mu) \leq \mathcal{L}(u,\, \lambda) \leq \mathcal{L}(v,\, \lambda) \quad v \in X,\, \mu \in \Lambda.$$

We consider now the following "perturbate" problem.

Problem 1.6. *For a given $f \in X$, find $u \in X$ and $\lambda \in Y$ such that $\lambda \in \Lambda$, and*

$$a_r\,(u, v) + b(v, \lambda) \;=\; (f,\, v)_X \quad \text{for all } v \in X, \tag{1.24}$$

$$b(u, \mu - \lambda) \;\leq\; 0 \quad \text{for all } \mu \in \Lambda. \tag{1.25}$$

We have the following lemmas.

Lemma 1.5. *[Lemma 3.2 in [68]] Assume that for every $f \in X$ there exists a unique solution of Problem 1.6, $(u,\, \lambda) \in X \times \Lambda$. If $(u_1,\, \lambda_1)$ and $(u_2,\, \lambda_2)$ are solutions of Problem 1.6 corresponding to two given functions $f_1,\, f_2 \in X$, then*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq \frac{\alpha + m_a + 2\, M_a}{\alpha\, m_a}\, \|f_1 - f_2\|_X.$$

Lemma 1.6. *[Lemma 3.3 in [68]] Let $\tau \in [0, 1]$. Assume that for every $f \in X$ there exists a unique solution of Problem 1.6 with $r = \tau$, $(u,\, \lambda) \in X \times \Lambda$. Then, for every $f \in X$ there exists a unique solution $(u,\, \lambda)$ of Problem 1.6 with $r \in [\tau, \tau + t_0]$, where*

$$t_0 < \frac{\alpha\, m_a}{M_a\, (\alpha + m_a + 2\, M_a)}. \tag{1.26}$$

Applying Lemma 1.6 and Lemma 1.5 we were led to the following result.

Theorem 1.3. *[Theorem 3.1 in [68]] Let $f \in X$. If Assumptions* 1.4–1.7 *hold true, then there exists a unique solution of Problem 1.4, $(u, \lambda) \in X \times \Lambda$. Moreover, if $(u_1,\, \lambda_1)$ and $(u_2,\, \lambda_2)$ are two solutions of Problem 1.4, corresponding to two given functions $f_1,\, f_2 \in X$, then*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq \frac{\alpha + m_a + 2\, M_a}{\alpha\, m_a}\, \|f_1 - f_2\|_X.$$

The proof of Theorem 1.3 can be found in [68]. The main idea of this proof was to use the results known in the saddle point theory, see, e.g., [22, 23, 50, 61], for the symmetric part of $a(\cdot, \cdot)$. The prove was completed by a fixed point technique. The reader can found a version of this fixed point technique in [83], in the framework of the elliptic variational inequalities of the first kind.

1.2.2 The model and its weak solvability

In this subsection we study the weak solvability of an elasto-piezoelectric model in the following physical setting. An elasto-piezoelectric body which occupies the bounded domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$, is in frictional contact with a rigid foundation. We consider two partitions of the boundary $\Gamma = \partial\Omega$: firstly, we consider a partition given by the measurable parts Γ_1, Γ_2 and Γ_3, such that meas $\Gamma_1 > 0$ and $\overline{\Gamma_3}$ is a compact subset of $\partial\Omega\backslash\overline{\Gamma_1}$; secondly, we consider a partition given by the measurable parts Γ_a, Γ_b, such that meas $\Gamma_a > 0$. The unit outward normal to Γ is denoted by $\boldsymbol{n}$ and is assumed to be constant on Γ_3, i.e. Γ_3 is a straight line or a face. We associate the body with a rectangular cartesian coordinate system $Ox_1x_2x_3$ such that $\boldsymbol{e}_1 = \boldsymbol{n}_{\Gamma_3}$. We assume that the body is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω, a surface traction of density $\boldsymbol{f}_2$ acts on Γ_2, a surface electric charge of density q_2 acts on Γ_b, and the electric potential vanishes on Γ_a. Moreover, we assume that on Γ_3 the deformable body is in bilateral contact with the rigid foundation. Herein φ denotes the electric potential.

The equilibrium equations are given by

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{1.27}$$

$$\operatorname{div} \boldsymbol{D} = q_0 \quad \text{in } \Omega, \tag{1.28}$$

where $\boldsymbol{D} = (D_i)$ is the electric displacement field, and q_0 is the volume density of free electric charges.

To describe the behavior of the material, we use the following constitutive law:

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{E}^\top \nabla \varphi \quad \text{in } \Omega, \tag{1.29}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \beta \nabla \varphi \quad \text{in } \Omega, \tag{1.30}$$

where $\mathcal{C} = (\mathcal{C}_{ijls})$ is the elastic tensor, $\mathcal{E} = (\mathcal{E}_{ijl})$ is the piezoelectric tensor, and β is the permittivity tensor. Note that (1.29) represents an electro-elastic constitutive law and (1.30) describes a linear dependence of the electric displacement field on the strain and electric fields. Such kind of electro-mechanic relations can be found in the literature, see, e.g., [157].

To complete the model, we have to prescribe the mechanic and electric boundary conditions. According to the physical setting, we write

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{1.31}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{1.32}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{1.33}$$

$$\boldsymbol{D}\cdot\boldsymbol{n} = q_2 \quad \text{on } \Gamma_b. \tag{1.34}$$

Finally, we describe the frictional bilateral contact using Tresca's law:

$$\begin{cases} u_n = 0, \ \|\boldsymbol{\sigma}_\tau\| \le g, \\ \|\boldsymbol{\sigma}_\tau\| < g \Rightarrow \boldsymbol{u}_\tau = 0, \\ \|\boldsymbol{\sigma}_\tau\| = g \Rightarrow \text{ there exists } \alpha > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\alpha\, \boldsymbol{u}_\tau \end{cases} \quad \text{on } \Gamma_3, \tag{1.35}$$

where the constant $g \ge 0$ represents the *friction bound.* When the strict inequality holds, the material point is in the *sticky* zone; when the equality holds, the material point is in the *slippy* zone. The boundary of these zones is unknown a priori.

To resume, we have the following problem:

Problem 1.7. *Find the displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and the electric potential field* $\varphi : \Omega \to \mathbb{R}$ *such that* (1.27)–(1.35) *hold.*

In the study of Problem 1.7, we made the following assumptions.

Assumption 1.8. $\mathcal{C} = (\mathcal{C}_{ijls}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$; $\mathcal{C}_{ijls} = \mathcal{C}_{ijsl} = \mathcal{C}_{lsij} \in L^\infty(\Omega)$;
There exists $m_{\mathcal{C}} > 0$ *such that* $\mathcal{C}_{ijls}\, \varepsilon_{ij}\, \varepsilon_{ls} \ge m_{\mathcal{C}}\, \|\boldsymbol{\varepsilon}\|^2$, *for all* $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, *a.e. on* Ω.

Assumption 1.9. $\mathcal{E} = (\mathcal{E}_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$; $\mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty(\Omega)$.

Assumption 1.10. $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$; $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$;
There exists $m_\beta > 0$ *such that* $\beta_{ij}(x) D_i D_j \ge m_\beta \|\boldsymbol{D}\|^2, \boldsymbol{D} \in \mathbb{R}^d$, *a.e.* $x \in \Omega$.

Assumption 1.11. $\boldsymbol{f}_0 \in L^2(\Omega)^d$, $\quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d$.

Assumption 1.12. $q_0 \in L^2(\Omega)$, $\quad q_2 \in L^2(\Gamma_b)$.

Let us introduce the following Hilbert spaces:

$$\begin{aligned} \boldsymbol{V} &= \Big\{ \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1 \Big\}, \\ \boldsymbol{V}_n &= \Big\{ \boldsymbol{v} \in \boldsymbol{V} \mid v_n = 0 \text{ on } \Gamma_3 \Big\}, \\ \Phi &= \Big\{ \theta \in H^1(\Omega) \mid \theta = 0 \text{ on } \Gamma_a \Big\}. \end{aligned}$$

We introduce the functional space $\tilde{\boldsymbol{V}} = \boldsymbol{V} \times \Phi$, that is a Hilbert space endowed with the inner product

$$(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}})_{\tilde{V}} = (\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^d} + (\varphi, \theta)_{H^1(\Omega)}, \quad \tilde{\boldsymbol{u}} = (\boldsymbol{u}, \varphi),\, \tilde{\boldsymbol{v}} = (\boldsymbol{v}, \theta) \in \tilde{\boldsymbol{V}};$$

the corresponding norm is denoted by $\|\cdot\|_{\tilde{V}}$. Let $a : \tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}} \to \mathbb{R}$ be the bilinear form given by:

$$\begin{aligned} a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) &= \int_\Omega \mathcal{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\, \varphi\, dx \\ &\quad - \int_\Omega \mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla \theta\, dx + \int_\Omega \beta \nabla \varphi \cdot \nabla \theta\, dx. \end{aligned} \tag{1.36}$$

Moreover, using Riesz's representation theorem, we define $\tilde{\boldsymbol{f}} \in \tilde{\boldsymbol{V}}$ such that for all $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}$,

$$(\tilde{\boldsymbol{f}},\, \tilde{\boldsymbol{v}})_{\tilde{V}} = \int_{\Omega} \boldsymbol{f}_0\, \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2\, \boldsymbol{v}\, d\Gamma - \int_{\Gamma_b} q_2\, \theta\, d\Gamma + \int_{\Omega} q_0\, \theta\, dx.$$

Let $\boldsymbol{D}$ be the dual space of the space $\boldsymbol{S} = \{\boldsymbol{w} = \boldsymbol{v}|_{\Gamma_3} \quad \boldsymbol{v} \in \boldsymbol{V}_n\}$.

We define

$$\boldsymbol{\Lambda} = \left\{ \boldsymbol{\mu} \in \boldsymbol{D} \,|\, \langle \boldsymbol{\mu},\, \boldsymbol{v}|_{\Gamma_3} \rangle_{\Gamma_3} \le \int_{\Gamma_3} g\, \|\boldsymbol{v}\|\, d\Gamma, \qquad \boldsymbol{v} \in \boldsymbol{V}_n \right\}, \tag{1.37}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_3}$ denotes the duality pairing between $\boldsymbol{D}$ and $\boldsymbol{S}$.

We suppose that the stress $\boldsymbol{\sigma}$ is a regular enough function to define $\boldsymbol{\lambda} \in \boldsymbol{D}$ as follows

$$\langle \boldsymbol{\lambda},\, \boldsymbol{\zeta} \rangle_{\Gamma_3} = -\int_{\Gamma_3} \boldsymbol{\sigma}_{\boldsymbol{\tau}} \cdot \boldsymbol{\zeta}\, ds, \qquad \boldsymbol{\zeta} \in \boldsymbol{S}.$$

Furthermore, we introduce a bilinear and continuous form as follows:

$$b : \tilde{\boldsymbol{V}} \times \boldsymbol{D} \to \mathbb{R}, \qquad b(\tilde{\boldsymbol{v}},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v}|_{\Gamma_3} \rangle_{\Gamma_3}. \tag{1.38}$$

The mechanical model leads us to the following variational formulation.

Problem 1.8. *Find $\tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{V}}$ and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that*

$$\begin{aligned} a(\tilde{\boldsymbol{u}},\, \tilde{\boldsymbol{v}}) + b(\tilde{\boldsymbol{v}}, \boldsymbol{\lambda}) &= (\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{v}})_{\tilde{V}}, \quad && \tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}, \\ b(\tilde{\boldsymbol{u}},\, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\le 0, && \boldsymbol{\mu} \in \boldsymbol{\Lambda}. \end{aligned}$$

Theorem 1.4. *[Theorem 2.1 in [68]] If Assumptions* 1.8–1.12 *hold true, then, Problem 1.8 has a unique solution $(\tilde{\boldsymbol{u}},\, \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$. Moreover, if $(\tilde{\boldsymbol{u}}_1,\, \boldsymbol{\lambda}_1)$ and $(\tilde{\boldsymbol{u}}_2,\, \boldsymbol{\lambda}_2)$ are two solutions of Problem 1.8 for two functions $\tilde{\boldsymbol{f}}_1,\, \tilde{\boldsymbol{f}}_2 \in \tilde{\boldsymbol{V}}$, corresponding to data $\{\boldsymbol{f}_0,\, \boldsymbol{f}_2,\, q_0,\, q_2\}_1$, respectively $\{\boldsymbol{f}_0,\, \boldsymbol{f}_2,\, q_0,\, q_2\}_2$, then we have*

$$\|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\|_{\tilde{V}} + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_D \le C\, \|\tilde{\boldsymbol{f}}_1 - \tilde{\boldsymbol{f}}_2\|_{\tilde{V}},$$

where $C > 0$ is a constant that depends of $\mathcal{C}$, $\mathcal{E}$ and β.

The proof of Theorem 1.4, given in [68], is based on the abstract result we presented before, Theorem 1.3.

1.2.3 Discretization and an optimal a priori error estimate

In this subsection, we discuss the discrete 2D case. Let us assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain and that Γ_1, Γ_3 and Γ_a can be written as union of edges of the triangulation. Furthermore, let us denote by $\boldsymbol{\tau}$ a unit vector such that $\boldsymbol{n} \cdot \boldsymbol{\tau} = 0$. We refer the body to a rectangular cartesian coordinate system Ox_1x_2 such that $\boldsymbol{e}_1 = \boldsymbol{n}_{\Gamma_3}$ and $\boldsymbol{e}_2 = \boldsymbol{\tau}_{\Gamma_3}$. To simplify the writing, everywhere below we will write $\boldsymbol{n}$ and $\boldsymbol{\tau}$ instead of $\boldsymbol{n}_{\Gamma_3}$ and $\boldsymbol{\tau}_{\Gamma_3}$, respectively. To approximate $\tilde{\boldsymbol{V}}$, we use standard conforming finite elements of lowest order on quasi-uniform simplicial triangulations, and we denote by $S_1(\Omega, \mathcal{T}_{h,\Omega})$ the finite element space associated with the shape regular triangulation $\mathcal{T}_{h,\Omega}$. The meshsize h is defined by the maximal diameter of the elements in $\mathcal{T}_{h,\Omega}$. Let us consider the discrete spaces

$$\begin{aligned} \boldsymbol{V}_h &= \Big\{\boldsymbol{v}_h \in [S_1(\Omega,\, \mathcal{T}_{h,\Omega})]^2 \,:\, \boldsymbol{v}_{h|_{\Gamma_1}} = \boldsymbol{0}\Big\} \subset \boldsymbol{V}, \\ (\boldsymbol{V}_h)_n &= \Big\{\boldsymbol{v}_h \in \boldsymbol{V}_h \,:\, (\boldsymbol{v}_h)_{n|_{\Gamma_3}} = 0\Big\} \subset \boldsymbol{V}_n, \\ \Phi_h &= \Big\{\theta_h \in S_1(\Omega,\, \mathcal{T}_{h,\Omega}) \,:\, \theta_{h|_{\Gamma_a}} = 0\Big\} \subset \Phi. \end{aligned}$$

Let us denote

$$\tilde{\boldsymbol{V}}_h = \boldsymbol{V}_h \times \Phi_h \subset \tilde{\boldsymbol{V}}$$

and

$$\boldsymbol{M}_h = \Big\{\boldsymbol{\mu}_h \in \boldsymbol{M} \mid \boldsymbol{\mu}_h = \sum_{i=1}^{N_{M_h}} \gamma_i \psi_i \boldsymbol{n} + \sum_{i=1}^{N_{M_h}} \alpha_i \psi_i \boldsymbol{\tau}\Big\},$$

where N_{M_h} is the number of vertices on $\overline{\Gamma_3}$ and for every $i = 1, ..., N_{M_h}$, ψ_i is the i-th. scalar dual basis function of the standard nodal Lagrange finite element basis function and γ_i, α_i are real coefficients. According to [163], we consider the dual basis such that the following biorthogonality relation holds

$$\langle \psi_i, \varphi_j \rangle_{\Gamma_3} \;=\; \delta_{ij} \int_{\Gamma_3} \varphi_j \, ds, \qquad i, j = 1, \ldots, N_{M_h}, \tag{1.39}$$

where φ_m, $m = 1, \ldots, N_{M_h}$, are the standard scalar nodal basis functions of $S_1(\Omega,\, \mathcal{T}_{h,\Omega})$, restricted to Γ_3. Furthermore, every element $\boldsymbol{v}_h$ of $(\boldsymbol{V}_h)_n$ can be written on Γ_3 as a combination of standard basis functions φ_i as follows

$$\boldsymbol{v}_h = \sum_{j=1}^{N_{M_h}} \zeta_j \varphi_j \boldsymbol{\tau}, \quad \zeta_j \in \mathbb{R},\; j = 1, ..., N_{M_h}.$$

Defining a *mesh dependent absolute value* of an element $\boldsymbol{v}_h \in (\boldsymbol{V}_h)_n$ by

$$|\boldsymbol{v}_h|_h = \sum_{j=1}^{N_{M_h}} |\zeta_j| \varphi_j,$$

we set $\boldsymbol{\Lambda}_h$ as follows

$$\boldsymbol{\Lambda}_h = \Big\{\boldsymbol{\mu}_h \in \boldsymbol{M}_h \,|\, \langle \boldsymbol{\mu}_h,\, \boldsymbol{v}_h \rangle_{\Gamma_3} \leq \int_{\Gamma_3} g\, |\boldsymbol{v}_h|_h \, ds, \quad \boldsymbol{v}_h \in (\boldsymbol{V}_h)_n\Big\}.$$

We now consider the following discrete problem.

Problem 1.9. *Find* $\tilde{\boldsymbol{u}}_h \in \tilde{\boldsymbol{V}}_h$ *and* $\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}_h$ *such that*

$$\begin{array}{lll} a(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + b(\tilde{\boldsymbol{v}}_h, \boldsymbol{\lambda}_h) = (\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{v}}_h)_{\tilde{V}}, & & \tilde{\boldsymbol{v}}_h \in \tilde{\boldsymbol{V}}_h \\ b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\mu}_h - \boldsymbol{\lambda}_h) \leq 0, & & \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h. \end{array}$$

Existence and uniqueness of a solution follows from a discrete inf-sup condition for the spaces $\tilde{\boldsymbol{V}}_h$ and $\boldsymbol{M}_h$, see, e.g., [163].

Let us denote by $\mathcal{P}_C = \{p_i \,:\, 1 \leq i \leq N_{M_h}\}$ the set of vertices on $\overline{\Gamma_3}$.

The following result takes place.

Lemma 1.7. *[Lemma 4.2 in [68]] Let* $(\tilde{\boldsymbol{u}} = (\boldsymbol{u}, \varphi), \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ *be the solution of Problem 1.8 and let* $(\tilde{\boldsymbol{u}}_h = (\boldsymbol{u}_h, \varphi_h), \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ *be the solution of Problem 1.9. Then, the following equalities hold*

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) = \int_{\Gamma_3} g|\boldsymbol{u}|ds, \tag{1.40}$$

$$b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) = \int_{\Gamma_3} g|\boldsymbol{u}_h|_h ds. \tag{1.41}$$

Using this lemma we have got the following result.

Lemma 1.8. *[Lemma 4.3 in [68]] Let* $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ *be the solution of Problem 1.8 and let* $(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ *be the solution of Problem 1.9. Then, there exists a positive constant* C *independent of the meshsize* h*, such that for all* $\boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h$, $\boldsymbol{\mu}_h \in \boldsymbol{M}_h$,

$$\begin{aligned} \|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_h\|^2_{\tilde{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|^2_{-\frac{1}{2},\Gamma_3} \leq\ & C\left\{\|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{v}}_h\|^2_{\tilde{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|^2_{-\frac{1}{2},\Gamma_3}\right\} \\ & + b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}). \end{aligned}$$

Let us denote $\overline{\gamma_{sl}} = \operatorname{supp}(\boldsymbol{u}_{|\Gamma_3} \cdot \boldsymbol{\tau})$ and $\gamma_{st} = \Gamma_3 \backslash \gamma_{sl}$.

We made the following assumption.

Assumption 1.13.

- $\overline{\gamma_{st}}$ *is a compact subset of* $\overline{\Gamma_3}$ *such that the number of points in* $\overline{\mathring{\gamma}_{st}} \cap \overline{\gamma_{sl}}$ *is finite;*
- $\overline{\mathring{\gamma}_{st}} = \overline{\gamma_{st}}$.

Let $\mathcal{W}_C = \{w_j \,:\, 1 \leq j \leq N_w\}$ be the set of points in $\overline{\mathring{\gamma}_{st}} \cap \overline{\gamma_{sl}}$. The minimum distance between the elements in $\mathcal{W}_C$ is denoted by a, i.e., $a = \inf\{|w_j - w_k| \,:\, 1 \leq j \neq k \leq N_w\}$, where $|\cdot|$ denotes the Euclidean norm. By Assumption 4.1, $N_w < \infty$ and thus $a > 0$. For $h < \frac{a}{2} =: h_0$, we find between two neighbor points in $\mathcal{W}_C$ at least two vertices in $\mathcal{P}_C$.

Let us denote by I_h the standard interpolation operator restricted on Γ_3, i.e.,

$$I_h \boldsymbol{u} = \sum_{i=1}^{N_{M_h}} \boldsymbol{u}(p_i)\varphi_i,$$

and let us define the following *modified interpolation operator* by

$$(\tilde{I}_h \boldsymbol{u})(p_i) = \begin{cases} \boldsymbol{u}(p_i) & \text{if} \quad \mathrm{supp}\varphi_i \subset \overline{\gamma_{sl}}, \\ \boldsymbol{0} & \text{else}, \end{cases}$$

for each $i = 1, ..., N_{M_h}$.

We underline that, under Assumption 1.13, we can write on Γ_3 the following identities

$$\begin{aligned} |\tilde{I}_h \boldsymbol{u}|_h &= |\tilde{I}_h \boldsymbol{u}|, \\ \mathrm{sgn}(\boldsymbol{u}\cdot\boldsymbol{\tau}) &= \mathrm{sgn}(\tilde{I}_h \boldsymbol{u}\cdot\boldsymbol{\tau}). \end{aligned} \tag{1.42}$$

The following lemma holds.

Lemma 1.9. *[Lemma 4.4 in [68]] Let $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ be the solution of Problem 1.8 and let $(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ be the solution of Problem 1.9. Under the additional regularity assumption $\boldsymbol{u} \in \left[H^{\frac{3}{2}+\nu}(\Omega)\right]^2$, $0 < \nu \leq \frac{1}{2}$, and Assumption 1.13, we then have the estimate*

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}) \;\; \leq \;\; Ch^{\frac{1}{2}+\nu}|\boldsymbol{u}|_{\frac{3}{2}+\nu,\Omega}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_3}$$

for a positive constant C independent of $h < h_0$.

A straightforward consequence of the results obtained in Lemmas 1.8-1.9 is the following theorem.

Theorem 1.5. *[Theorem 4.1 in [68]] Let $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ be the solution of Problem 1.8 and let $(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ be the solution of Problem 1.9. Under the additional regularity assumption $\tilde{\boldsymbol{u}} \in [H^{\frac{3}{2}+\nu}(\Omega)]^3$, $0 < \nu \leq \frac{1}{2}$ and Assumption 1.13, we then have the following optimal a priori error estimate*

$$\|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_h\|_{\tilde{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_3} \;\; \leq \;\; Ch^{\frac{1}{2}+\nu}|\tilde{\boldsymbol{u}}|_{\frac{3}{2}+\nu,\Omega}$$

for a positive constant C that is independent of the meshsize $h < h_0$.

Remark 1.2. *The a priori results can be extended to the 3D case. The results also hold in the multibody case with nonconforming meshes at the contact interface, see e.g. [62, 66] for the necessary techniques. A numerical example was given in Section 5 of the paper [68].*

Chapter 2

Slip-dependent frictional contact problems

This chapter is based on the papers [105, 109, 112]. *A slip-dependent frictional contact law* is a law in which the friction bound depends on the slip. The first mathematical results on contact problem with slip displacements dependent friction in elastostatics were obtained in [72]. For other mathematical results in the study of slip-dependent frictional contact models see, e.g., [36, 65, 97, 103, 145] for a treatment in the frame of quasivariational inequalities or, see e.g, [115, 116, 117] for a treatment in the frame of hemivariational inequalities. In the present work, the interest lies into a variational approach involving dual Lagrange multipliers which allows to apply modern numerical techniques (see e.g. [162]) in order to approximate the weak solution.

2.1 An abstract result

This section presents the results obtained in Section 2 and Section 3 of the paper [105]. In this section we consider an abstract mixed variational problem, the set of the Lagrange multipliers being dependent on the solution.

Problem 2.1. *Given $f \in X$, $f \neq 0_X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda(u) \subset Y$ and*

$$a(u, v) + b(v, \lambda) = (f, v)_X \quad \text{for all } v \in X, \tag{2.1}$$

$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda(u). \tag{2.2}$$

We shall discuss the existence of the solution based on a fixed point technique for weakly sequentially continuous maps.

Let us make the following assumptions.

Assumption 2.1. *$(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 2.2. *$a(\cdot, \cdot) : X \times X \to \mathbb{R}$ is a symmetric bilinear form such that*

(i_1) *there exists* $M_a > 0 : \ |a(u,v)| \leq M_a \|u\|_X \|v\|_X$ *for all* $u, v \in X$,
(i_2) *there exists* $m_a > 0 : \ a(v,v) \geq m_a \, \|v\|_X^2$ *for all* $v \in X$.

Assumption 2.3. $b(\cdot,\cdot) : X \times Y \to \mathbb{R}$ *is a bilinear form such that*
(j_1) *there exists* $M_b > 0 : \ |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ *for all* $v \in X$, $\mu \in Y$,
(j_2) *there exists* $\alpha > 0 : \ \inf_{\mu \in Y, \mu \neq 0_Y} \ \sup_{v \in X, v \neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha$.

Assumption 2.4. *For each* $\varphi \in X$, $\Lambda(\varphi)$ *is a closed convex subset of* Y *such that* $0_Y \in \Lambda(\varphi)$.

Assumption 2.5. *Let* $(\eta_n)_n \subset X$ *and* $(u_n)_n \subset X$ *be two weakly convergent sequences,* $\eta_n \rightharpoonup \eta$ *in* X *and* $u_n \rightharpoonup u$ *in* X, *as* $n \to \infty$.
(k_1) *For each* $\mu \in \Lambda(\eta)$, *there exists a sequence* $(\mu_n)_n \subset Y$ *such that* $\mu_n \in \Lambda(\eta_n)$ *and* $\liminf_{n\to\infty} b(u_n, \mu_n - \mu) \geq 0$.
(k_2) *For each subsequence* $(\Lambda(\eta_{n'}))_{n'}$ *of the sequence* $(\Lambda(\eta_n))_n$, *if* $(\mu_{n'})_{n'} \subset Y$ *such that* $\mu_{n'} \in \Lambda(\eta_{n'})$ *and* $\mu_{n'} \rightharpoonup \mu$ *in* Y *as* $n' \to \infty$, *then* $\mu \in \Lambda(\eta)$.

Theorem 2.1. *[Theorem 2.1 in [105]] If Assumptions* 2.1-2.5 *hold true, then Problem 2.1 has a solution. In addition, if* $(u, \lambda) \in X \times \Lambda(u)$ *is a solution of Problem 2.1, then*

$$(u, \lambda) \in K_1 \times \big(\Lambda(u) \cap K_2\big), \tag{2.3}$$

where

$$K_1 = \{v \in X \mid \|v\|_X \leq \frac{1}{m_a} \|f\|_X\}$$

and

$$K_2 = \{\mu \in Y \mid \|\mu\|_Y \leq \frac{m_a + M_a}{\alpha \, m_a} \|f\|_X\}.$$

The proof of Theorem 2.1, which can be found in [105], is based on the saddle point theory, see [50], and a fixed point result for weakly sequentially continuous maps, see [5].

Theorem 2.1 is a new result which improves and extends the existence results of solutions for mixed problems which are equivalent to saddle point problems, see e.g. [61]. The main difficulty here it was generated by the dependence $\Lambda = \Lambda(u)$. A convergence of Mosco type for the convex sets of Lagrange multipliers it was required; Assumption 2.5 it was crucial.

2.2 An antiplane problem

This section is based on Section 4 of the paper [105]. In this section we apply the abstract result obtained in Section 2.1 to the weak solvability of a slip-dependent frictional antiplane contact problem.

Let us consider the following mechanical model.

Problem 2.2. *Find a displacement field* $u : \bar{\Omega} \to \mathbb{R}$ *such that*

$$\begin{aligned} \operatorname{div}\left(\mu(\boldsymbol{x})\, \nabla u(\boldsymbol{x})\right) + f_0(\boldsymbol{x}) &= 0 && \text{in } \Omega, && (2.4)\\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_1, && (2.5)\\ \mu(\boldsymbol{x})\, \partial_\nu u(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_2, && (2.6) \end{aligned}$$

$$\left.\begin{array}{l} |\mu(\boldsymbol{x})\, \partial_\nu\, u(\boldsymbol{x})| \le g(\boldsymbol{x}, |u(\boldsymbol{x})|), \\ \mu(\boldsymbol{x})\, \partial_\nu u(\boldsymbol{x}) = -g(\boldsymbol{x}, |u(\boldsymbol{x})|)\, \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \quad \text{if } u(\boldsymbol{x}) \neq 0 \end{array}\right\} \quad \text{on } \Gamma_3. \qquad (2.7)$$

Herein $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz continuous boundary Γ partitioned in three measurable parts Γ_1, Γ_2, Γ_3 such that the Lebesgue measure of Γ_i is strictly positive, for every $i \in \{1, 2, 3\}$. Problem 2.2 models the antiplane shear deformation of an elastic, isotropic, nonhomogeneous cylindrical body, in frictional contact on Γ_3 with a rigid foundation. Referring the body to a cartesian coordinate system $Ox_1x_2x_3$ such that the generators of the cylinder are parallel with the axis Ox_3, the domain $\Omega \subset Ox_1x_2$ denotes the cross section of the cylinder. The function $\mu = \mu(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ denotes a coefficient of the material (one of *Lamé's coefficients*), the functions $f_0 = f_0(x_1, x_2) : \Omega \to \mathbb{R}$, $f_2 = f_2(x_1, x_2) : \Gamma_2 \to \mathbb{R}$ are related to the density of the volume forces and the density of the surface traction, respectively and $g : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, the *friction bound*. Here $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\nu_i = \nu_i(x_1, x_2)$, for each $i \in \{1, 2\}$, represents the outward unit normal vector to the boundary of Ω and $\partial_\nu\, u = \nabla u \cdot \boldsymbol{\nu}$.

The unknown of the problem is the function $u = u(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ that represents the third component of the displacement vector $\boldsymbol{u}$. We recall that, in the antiplane physical setting, the displacement vectorial field has the particular form $\boldsymbol{u} = (0, 0, u(x_1, x_2))$. Once the field u is determined, the stress tensor $\boldsymbol{\sigma}$ can be computed:

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \mu \dfrac{\partial u}{\partial x_1} \\ 0 & 0 & \mu \dfrac{\partial u}{\partial x_2} \\ \mu \dfrac{\partial u}{\partial x_1} & \mu \dfrac{\partial u}{\partial x_2} & 0 \end{pmatrix}.$$

The mechanical problem has the following structure: (2.4) represents the equilibrium equation, (2.5) is the displacement boundary condition, (2.6) is the traction boundary condition and (2.7) is a frictional contact condition. The condition (2.7) is Tresca's law of dry friction with slip-dependent friction bound g. To give an example of such a function g we can consider

$$g(\boldsymbol{x}, r) = k(1 + \delta e^{-r}); \qquad k, \delta > 0. \qquad (2.8)$$

The slip-dependent friction law (2.7) with the friction bound g given by (2.8) describes the slip weakening phenomenon which appears in the study of geophysical problems, see for example

[29, 30, 74, 75, 137]. For details concerning the frictional antiplane model we send the reader to [145] and to the references therein.

We are interested on the weak solvability of Problem 2.2 under the following assumptions.

Assumption 2.6. $f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2)$.

Assumption 2.7. $\mu \in L^\infty(\Omega), \quad \mu(x) \geq \mu^* > 0$ *a.e. in* Ω.

Assumption 2.8. *There exists* $L_g > 0$ *such that*
$|g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \leq L_g\, |r_1 - r_2| \;\; r_1, r_2 \in \mathbb{R}_+,$ *a.e.* $\boldsymbol{x} \in \Gamma_3$;
The mapping $\boldsymbol{x} \mapsto g(\boldsymbol{x}, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in \mathbb{R}$;
The mapping $\boldsymbol{x} \mapsto g(\boldsymbol{x}, 0)$ *belongs to* $L^2(\Gamma_3)$.

Let us describe the functional setting. To start, we introduce the space

$$X = \Big\{ v \in H^1(\Omega) \,|\, \gamma v = 0 \quad \text{a.e. on} \quad \Gamma_1 \Big\}. \tag{2.9}$$

The space X is a Hilbert space endowed with the inner product given by

$$(u, v)_X = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{for all } u, v \in X,$$

and the associated norm

$$\|v\|_X = \|\nabla v\|_{L^2(\Omega)^2} \quad \text{for all } v \in X.$$

Let $a : X \times X \to \mathbb{R}$ be the bilinear form

$$a(u, v) = \int_\Omega \mu \nabla u \cdot \nabla v \, dx \quad \text{for all } u, v \in X \tag{2.10}$$

and $f \in X$ defined as follows

$$(f, v)_X = \int_\Omega f_0 \, v \, dx + \int_{\Gamma_3} f_2 \, \gamma v \, d\Gamma. \tag{2.11}$$

Let $\Gamma_3 \subset \Gamma$ such that $\Gamma_3 \cap \Gamma_1 = \varnothing$. We consider the space

$$S = \{ \widetilde{v} = \gamma v|_{\Gamma_3} \quad v \in X \} \tag{2.12}$$

endowed with the Sobolev-Slobodeckii norm

$$\|\widetilde{v}\|_{\Gamma_3} = \Big(\int_{\Gamma_3} \int_{\Gamma_3} \frac{(\widetilde{v}(\boldsymbol{x}) - \widetilde{v}(\boldsymbol{y}))^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^2} \, ds_x \, ds_y \Big)^{1/2} \quad \text{for all } \widetilde{v} \in S.$$

We can introduce now a second Hilbert space, the dual of the space S,

$$Y = S'. \tag{2.13}$$

Also, we can define a second bilinear form $b : X \times Y \to \mathbb{R}$,

$$b(v,\, \zeta) = \langle \zeta, \gamma v|_{\Gamma_3} \rangle, \tag{2.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the spaces Y and S.

We define a Lagrange multiplier $\lambda \in Y$,

$$\langle \lambda, z \rangle = -\int_{\Gamma_3} \mu\, \partial_\nu u\, z\, d\Gamma \quad \text{for all } z \in S, \tag{2.15}$$

where the space Y is defined in (2.13) and the space S is defined in (2.12).

Furthermore, for each $\varphi \in X$, we introduce a subset of the space Y,

$$\Lambda(\varphi) = \Big\{ \zeta \in Y \,:\, \langle \zeta,\, \gamma w|_{\Gamma_3} \rangle \leq \int_{\Gamma_3} g(\boldsymbol{x}, |\gamma\varphi(\boldsymbol{x})|)\, |\gamma w(\boldsymbol{x})|\, d\Gamma \quad \text{for all } w \in X \Big\}. \tag{2.16}$$

Problem 2.2 has the following weak formulation.

Problem 2.3. *Find $u \in X$ and $\lambda \in \Lambda(u) \subset Y$ such that*

$$a(u,\, v) + b(v, \lambda) = (f, v)_X \quad \textit{for all } v \in X; \tag{2.17}$$

$$b(u, \zeta - \lambda) \leq 0 \quad \textit{for all } \zeta \in \Lambda(u). \tag{2.18}$$

Each solution of Problem 2.3 is called *weak solution* of Problem 2.2.

Notice that for each $\mu \in Y$ we have

$$\begin{aligned} \|\mu\|_Y &= \sup_{\gamma w|_{\Gamma_3} \in S,\, \gamma w|_{\Gamma_3} \neq 0_S} \frac{< \mu, \gamma w|_{\Gamma_3} >}{\|\gamma w|_{\Gamma_3}\|_{\Gamma_3}} \\ &\leq c \sup_{v \in X,\, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X}, \end{aligned}$$

where $c > 0$.

Theorem 2.2. *[Theorem 4.3 in [105]] If Assumptions* 2.6*-*2.8 *hold true, then Problem* 2.2 *has a weak solution. In addition, if (u, λ) is a weak solution of Problem* 2.2*, then $(u, \lambda) \in K_1 \times \big(\Lambda(u) \cap K_2\big)$, where $K_1 = \{v \in X \,|\, \|v\|_X \leq \frac{1}{m_a}\|f\|_X\}$, $K_2 = \{\mu \in Y \,|\, \|\mu\|_Y \leq \frac{m_a + M_a}{\alpha\, m_a}\|f\|_X\}$, X given by* (2.9)*, Y given by* (2.13)*, f given by* (2.11)*, $M_a = \|\mu\|_{L^\infty(\Omega)}$, $m_a = \mu^*$ and $\alpha = \frac{1}{c}$.*

The proof of Theorem 2.2 was based on the previous abstract result, Theorem 2.1. The main difficult part of the proof consists in the verification of Assumption 2.5. The crucial point was the construction of an appropriate sequence $(\mu_n)_n \subset Y$:

$$\begin{aligned} \langle \mu_n, \zeta \rangle &= \int_{\Gamma_3} g(\boldsymbol{x}, |\gamma\eta_n(\boldsymbol{x})|)\, \mathrm{sgn}\, \gamma u_n(\boldsymbol{x}) \zeta(\boldsymbol{x})\, d\Gamma \\ &\quad - \int_{\Gamma_3} g(\boldsymbol{x}, |\gamma\eta(\boldsymbol{x})|) |\gamma u_n(\boldsymbol{x})|\, d\Gamma \\ &\quad + \langle \mu, \gamma u_n|_{\Gamma_3} \rangle \quad \text{for all } \zeta \in S. \end{aligned}$$

For details, see [105].

2.3 A 3D slip-dependent frictional contact problem

This section, devoted to the weak solvability of a 3D slip-dependent frictional contact problem, is based on the papers [109, 112]. The model we focus on was previously analyzed into the framework of quasi-variational inequalities in [36]. The novelty herein consists in the variational approach we use. Thus, we propose a mixed variational formulation in a form of a generalized saddle point problem, the set of the Lagrange multipliers being solution-dependent.

The physical setting is as follows. We consider a deformable body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$ with smooth (say Lipschitz continuous) boundary Γ partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal vector to Γ is denoted by $\boldsymbol{\nu}$ and is defined almost everywhere. The body is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. On Γ_3 the body is in slip-dependent frictional contact with a rigid foundation.

The 3D slip-dependent frictional contact model is mathematically described as follows.

Problem 2.4. *Find* $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$ *such that*

$$Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \qquad in\ \Omega, \tag{2.19}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) \qquad in\ \Omega, \tag{2.20}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \qquad on\ \Gamma_1, \tag{2.21}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \qquad on\ \Gamma_2, \tag{2.22}$$

$$u_\nu(\boldsymbol{x}) = 0 \qquad on\ \Gamma_3, \tag{2.23}$$

$$\begin{aligned} &\|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \leq g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|),\\ &\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -g(\boldsymbol{x}, \|\boldsymbol{u}_\tau(\boldsymbol{x})\|)\, \frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|}\\ &\quad if\ \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} \qquad on\ \Gamma_3. \end{aligned} \tag{2.24}$$

Problem 2.4 has the following structure: (2.19) represents the equilibrium equation, (2.20) represents the constitutive law for linearly elastic materials, (2.21) represents the homogeneous displacements boundary condition, (2.22) represents the traction boundary condition and (2.23)-(2.24) model the bilateral contact with friction, the friction law involving a slip-dependent friction bound g.

For more details on this model see e.g. [36] and the references therein.

In order to weakly solve Problem 2.4 we made the following assumptions.

Assumption 2.9. $\mathcal{E} = (\mathcal{E}_{ijls}) : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$,

- $\mathcal{E}_{ijls} = \mathcal{E}_{ijsl} = \mathcal{E}_{lsij} \in L^\infty(\Omega)$,
- *There exists* $m_\mathcal{E} > 0$ *such that* $\mathcal{E}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \geq m_\mathcal{E}\, \|\boldsymbol{\varepsilon}\|^2$, $\boldsymbol{\varepsilon} \in \mathbb{S}^3$, *a.e. in* Ω.

Assumption 2.10. $\boldsymbol{f}_0 \in L^2(\Omega)^3, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3.$

Assumption 2.11. $g : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$,

- *there exists* $L_g > 0 : \, |g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \leq L_g \, |r_1 - r_2| \quad r_1, r_2 \in \mathbb{R}_+, \;$ *a.e.* $\boldsymbol{x} \in \Gamma_3$;
- *the mapping* $\boldsymbol{x} \mapsto g(\boldsymbol{x}, \, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in \mathbb{R}_+$;
- *the mapping* $\boldsymbol{x} \mapsto g(\boldsymbol{x}, 0)$ *belongs to* $L^2(\Gamma_3)$.

Let us introduce the following Hilbert space.

$$V = \{\boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{\gamma v} = 0 \text{ on } \Gamma_1, \, v_\nu = 0 \text{ on } \Gamma_3\}. \tag{2.25}$$

Define $\boldsymbol{f} \in V$ using Riesz's representation theorem,

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x}) \, d\Gamma \tag{2.26}$$

for all $\boldsymbol{v} \in V$.

Also, we introduce the space

$$S = \{\boldsymbol{\gamma w}|_{\Gamma_3} \quad \boldsymbol{w} \in V\}, \tag{2.27}$$

where $\boldsymbol{\gamma w}|_{\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{\gamma w} \in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3; \mathbb{R}^3)$ where $H^{1/2}(\Gamma_3; \mathbb{R}^3)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^3$. It is known that S can be organized as a real Hilbert space, see for instance [1, 92]. We use the Sobolev-Slobodeckii norm

$$\|\boldsymbol{\zeta}\|_S = \Big(\int_{\Gamma_3} \int_{\Gamma_3} \frac{\|\boldsymbol{\zeta}(\boldsymbol{x}) - \boldsymbol{\zeta}(\boldsymbol{y})\|^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^3} \, ds_x \, ds_y \Big)^{1/2}.$$

Let us introduce now the following real Hilbert space,

$$D = S' \text{ (the dual of the space } S). \tag{2.28}$$

The duality paring between D and S will be denoted by $\langle \cdot, \cdot \rangle$.

For each $\boldsymbol{\varphi} \in V$ we define

$$\boldsymbol{\Lambda}(\boldsymbol{\varphi}) = \{\boldsymbol{\mu} \in D \,|\, \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle \leq \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\varphi}_\tau(\boldsymbol{x})\|) \|\boldsymbol{v}_\tau(\boldsymbol{x})\| \, d\Gamma \quad \boldsymbol{v} \in V\}.$$

Let us define a Lagrange multiplier $\boldsymbol{\lambda} \in D$,

$$\langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle = - \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{\zeta}(\boldsymbol{x}) \, d\Gamma \tag{2.29}$$

for all $\boldsymbol{\zeta} \in S$.

Notice that $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{u})$.

We also define

$$a : V \times V \to \mathbb{R} \quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx; \tag{2.30}$$

$$b : V \times D \to \mathbb{R} \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle. \tag{2.31}$$

Therefore, Problem 2.4 has the following weak formulation.

Problem 2.5. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{u}) \subset D$ such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \textit{for all } \boldsymbol{v} \in V \tag{2.32}$$

$$b(\boldsymbol{u}, \boldsymbol{\zeta} - \boldsymbol{\lambda}) \leq 0 \quad \textit{for all } \boldsymbol{\zeta} \in \Lambda(\boldsymbol{u}). \tag{2.33}$$

Each solution of Problem 2.5 is called *weak solution* of Problem 2.4.

Theorem 2.3. *[An existence result (Theorem 2 in [112])] If Assumptions 2.9 -2.11 hold true, then Problem 2.4 has a weak solution.*

The idea of the proof was to use the abstract result, Theorem 2.1; for details of the proof of Theorem 2.3 we send the reader to the paper [112] or to the conference paper [109] (the paper [112] is an revised/extended version of the conference paper [109]). However, it is worth to mention here the crucial point of the proof: to construct an appropriate sequence $(\boldsymbol{\mu}_n)_n$ in order to verify Assumption 2.5. Let us give an example: for each $n \geq 1$,

$$< \boldsymbol{\mu}_n, \boldsymbol{\zeta} >= \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\eta}_{n\tau}(\boldsymbol{x})\|)\boldsymbol{\psi}(\boldsymbol{u}_{n\tau}(\boldsymbol{x})) \cdot \boldsymbol{\zeta}(\boldsymbol{x})\, d\Gamma$$
$$- \int_{\Gamma_3} g(\boldsymbol{x}, \|\boldsymbol{\eta}_{\tau}(\boldsymbol{x})\|)\|\boldsymbol{u}_{n\tau}(\boldsymbol{x})\|\, d\Gamma + \langle \boldsymbol{\mu}, \boldsymbol{\gamma u}_n|_{\Gamma_3} \rangle,$$

for all $\boldsymbol{\zeta} \in S$, where

$$\boldsymbol{\psi}(\boldsymbol{r}) = \begin{cases} \dfrac{\boldsymbol{r}}{\|\boldsymbol{r}\|} & \text{if } \boldsymbol{r} \neq \boldsymbol{0}; \\ \boldsymbol{0} & \text{if } \boldsymbol{r} = \boldsymbol{0}. \end{cases}$$

Notice that the form $a(\cdot, \cdot)$ defined in (2.30) verifies Assumption 2.2 with

$$M_a = \|\mathcal{E}\|_\infty \text{ and } m_a = m_{\mathcal{E}}, \tag{2.34}$$

where

$$\|\mathcal{E}\|_\infty = \max_{0 \leq i,j,k,l \leq d} \|E_{ijkl}\|_{L^\infty(\Omega)}.$$

Also, we note that for each $\boldsymbol{\mu} \in D$, there exists $c > 0$ such that

$$\|\boldsymbol{\mu}\|_D \leq c \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V},$$

and we can take

$$\alpha = \frac{1}{c}. \tag{2.35}$$

Let us introduce now

$$\boldsymbol{K}_1 = \{\boldsymbol{v} \in V \mid \|\boldsymbol{v}\|_V \leq \frac{1}{m_a} \|\boldsymbol{f}\|_V\}; \tag{2.36}$$

$$\boldsymbol{K}_2 = \{\boldsymbol{\mu} \in D \mid \|\boldsymbol{\mu}\|_D \leq \frac{m_a + M_a}{\alpha\, m_a} \|\boldsymbol{f}\|_V\}. \tag{2.37}$$

Theorem 2.4. *[A boundedness result (Theorem 3 in [112])] If $(\boldsymbol{u}, \boldsymbol{\lambda})$ is a weak solution of Problem 2.4, then*

$$(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{K}_1 \times \big(\Lambda(\boldsymbol{u}) \cap \boldsymbol{K}_2\big)$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are given by (2.36)-(2.37), *V given by* (2.25), *D given by* (2.28), *$\boldsymbol{f}$ given by* (2.26), *m_a and M_a being the constants in* (2.34) *and α being the constant in* (2.35).

The proof of Theorem 2.4 uses the abstract result, Theorem 2.1.

Chapter 3

Contact problems for nonlinearly elastic materials

This chapter is based on the papers [99, 100, 104, 107]. In this chapter we discuss a class of problems which model the contact between nonlinearly elastic bodies and rigid foundations, under the small deformation hypothesis, for static processes. The contact between the body and the foundation can be frictional bilateral or frictionless unilateral. For every mechanical problem we discuss a weak formulation consisting of a system of a nonlinear variational equation and a variational inequality, involving dual Lagrange multipliers. The weak solvability of the models is based on the saddle point theory and fixed point techniques.

3.1 Problems governed by strongly monotone and Lipschitz continuous operators

This section presents some results obtained in the papers [99, 104] drawing the attention to the weak solvability via dual Lagrange multipliers for a class of contact problems leading to mixed variational problems governed by strongly monotone and Lipschitz continuous operators.

3.1.1 Abstract results

In this subsection we present results obtained in Section 5 of the paper [99] and some results obtained in Section 2 of the paper [104], focusing on the following abstract problem.

Problem 3.1. *Given f, $h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that*

$$(Au, v)_X + b(v, \lambda) = (f, v)_X \qquad \text{for all } v \in X, \tag{3.1}$$

$$b(u, \mu - \lambda) \leq b(h, \mu - \lambda) \qquad \text{for all } \mu \in \Lambda. \tag{3.2}$$

The study was made under the following hypotheses.

Assumption 3.1. *$(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 3.2. *$A : X \to X$ is a nonlinear operator such that:*
there exists $m_A > 0 : (Au - Av,\, u - v)_X \geq m_A \|u - v\|_X^2$ for all $u, v \in X$,
there exists $L_A > 0 : \|Au - Av\|_X \leq L_A \|u - v\|_X$ for all $u, v \in X$.

Assumption 3.3. *$b : X \times Y \to R$ is a bilinear form such that:*
there exists $M_b > 0 : |b(v, \mu)| \leq M_b \|v\|_X \|\mu\|_Y$ for all $v \in X$, $\mu \in Y$,
there exists $\alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha$.

Assumption 3.4. *$\Lambda \subset Y$ is a closed convex set such that $0_Y \in Y$.*

Under these assumptions, Problem 3.1 is not a saddle point problem. This is a new variational problem, *a mixed variational problem governed by a nonlinear operator A.*

The following existence and uniqueness result holds.

Theorem 3.1. *If Assumptions* 3.1–3.4 *hold true, then there exists a unique solution of Problem* 3.1*, $(u, \lambda) \in X \times \Lambda$.*

The proof of Theorem 3.1 is based on Theorem 5.2 in [99] if Λ is an unbounded set and on Theorem 2.1 in [104] if Λ is a bounded set.

3.1.2 Contact models

This subsection, based on Section 3, Section 4 and Section 6 in [99] and, on a part of Section 3 in [104], presents results in the weak solvability of frictionless unilateral or frictional bilateral contact problems, for nonlinearly elastic materials, by using a technique involving dual Lagrange multipliers and applying the abstract results presented in Section 3.1.1.

Physical setting and mathematical description of the models

We consider a body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, with the boundary partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal vector to Γ is denoted by $\boldsymbol{n}$ and is defined almost everywhere. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. On Γ_3 the body can be in contact with a rigid foundation.

In order to describe the behavior of the materials, we use the constitutive law,

$$\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \tag{3.3}$$

where $\mathcal{F}$ denotes a nonlinear elastic operator. This kind of constitutive law can be found in the literature, see for example [59] and the references therein. As an example, we may consider

$$\boldsymbol{\sigma} = \lambda_0 (tr\boldsymbol{\varepsilon}) I_3 + 2\mu_0 \boldsymbol{\varepsilon} + \beta(\boldsymbol{\varepsilon} - P_K \boldsymbol{\varepsilon}) \tag{3.4}$$

where λ_0 and μ_0 denote Lame's constants, $tr\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}_{kk}$, $I_3 = (\delta_{ij})$ is the unit in $\mathbb{S}^3$, K denotes a closed convex subset of $\mathbb{S}^3$ that contains the zero element $0_{\mathbb{S}^3}$, $P_K : \mathbb{S}^3 \to K$ is the projection operator onto K, and β is a strictly positive constant. A second example is the following constitutive law,

$$\boldsymbol{\sigma} = k(tr\boldsymbol{\varepsilon})I_3 + \psi(\|\boldsymbol{\varepsilon}^D\|^2)\boldsymbol{\varepsilon}^D, \tag{3.5}$$

where $k > 0$ is a coefficient of the material, $\psi : \mathbb{R} \to \mathbb{R}$ is a constitutive function and $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \frac{1}{3}(tr\boldsymbol{\varepsilon})I_3$, is the deviator of the tensor $\boldsymbol{\varepsilon}$.

Assuming that on Γ_3 the body is in frictional bilateral contact with a rigid foundation, we use Tresca's law to state the following mechanical problem.

Problem 3.2. *Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that*

$$\begin{aligned}
\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 &= \boldsymbol{0} && \text{in } \Omega,\\
\boldsymbol{\sigma} &= \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_1,\\
\boldsymbol{\sigma}\boldsymbol{n} &= \boldsymbol{f}_2 && \text{on } \Gamma_2,
\end{aligned}$$

$$(C)\begin{cases} u_n = 0, \ \|\boldsymbol{\sigma}_\tau\| \le \zeta, \\ \text{if } \|\boldsymbol{\sigma}_\tau\| < \zeta \text{ then } \boldsymbol{u}_\tau = 0, \\ \text{if } \|\boldsymbol{\sigma}_\tau\| = \zeta \text{ then there exists } \psi > 0 : \ \boldsymbol{\sigma}_\tau = -\psi\, \boldsymbol{u}_\tau \end{cases} \qquad \text{on } \Gamma_3,$$

where $\zeta > 0$ denotes the friction bound.

If we assume that on Γ_3, the body can be in frictionless unilateral contact with a rigid foundation, we can model the contact by Signorini's condition with zero gap, yielding the second problem.

Problem 3.3. *Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that*

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{3.6}$$

$$\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega, \tag{3.7}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{3.8}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{3.9}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \ \sigma_n \le 0, \ u_n \le 0, \ \sigma_n\, u_n = 0 \quad \text{on } \Gamma_3. \tag{3.10}$$

Finally, if we model the contact on Γ_3 by Signorini's condition with non zero gap, we have to replace (3.10) with the following contact condition,

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \ \sigma_n \le 0, \ u_n - g \le 0, \ \sigma_n\,(u_n - g) = 0 \qquad \text{on } \Gamma_3, \tag{3.11}$$

where $g : \Gamma_3 \to \mathbb{R}$ is the gap between the deformable body and the foundation, measured along the outward normal $\boldsymbol{n}$. Thus, we can formulate the third problem.

Problem 3.4. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ *such that* (3.6)-(3.9) *and* (3.11) *hold.*

Additional details on this subsection, including a description of the physical significance for the contact conditions (C), (3.10) and (3.11), can be found for instance in [59].

Once the displacement field $\boldsymbol{u}$ is determined, the stress tensor $\boldsymbol{\sigma}$ can be obtained via relation (3.3).

Hypotheses and weak formulations

Herein we state the hypotheses and present the weak formulations with dual Lagrange multipliers for each of the models described in the previous section.

Assumption 3.5. $\mathcal{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$;
there exists $M > 0$ *such that* $\|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \leq M\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ *for all* $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3$, *a.e. in* Ω;
there exists $m > 0$ *such that for all* $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3$, *and almost everywhere in* Ω :
$(\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$;
for all $\boldsymbol{\varepsilon} \in \mathbb{S}^3$, $\boldsymbol{x} \to \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ *is Lebesgue measurable in* Ω;
$\boldsymbol{x} \to \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^3})$ *belongs to* $L^2(\Omega)^{3\times 3}$.

Referring to (3.4), we note that, using the property of the non-expansivity of the projection map, it can be proved that the map

$$\mathcal{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3; \quad \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = \lambda_0(tr\boldsymbol{\varepsilon})I_3 + 2\mu_0\boldsymbol{\varepsilon} + \beta(\boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon})$$

satisfies Assumption 3.5. Moreover, referring to (3.5), under appropriate assumptions on the constitutive function ψ, see [59] p.125, the map

$$\mathcal{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3; \quad \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = k(tr\boldsymbol{\varepsilon})I_3 + \psi(\|\boldsymbol{\varepsilon}^D\|^2)\boldsymbol{\varepsilon}^D,$$

satisfies Assumption 3.5.

Moreover, we made the following assumptions.

Assumption 3.6. $\boldsymbol{f}_0 \in L^2(\Omega)^3, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3.$

Assumption 3.7. *There exists* $g_{ext} : \Omega \to \mathbb{R}$ *such that* $g_{ext} \in H^1(\Omega)$, $\gamma g_{ext} = 0$ *almost everywhere on* Γ_1, $\gamma g_{ext} \geq 0$ *almost everywhere on* $\Gamma \setminus \Gamma_1$, $g = \gamma g_{ext}$ *almost everywhere on* Γ_3.

Assumption 3.8. *The unit outward normal to* Γ_3, *denoted by* $\boldsymbol{n}_3$, *is a constant vector.*

Weak formulation of Problem 3.2. Let us introduce the space

$$V_1 = \{\boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{v} = 0 \text{ a.e. on } \Gamma_1,\, v_\nu = 0 \text{ a.e. on } \Gamma_3\}.$$

We define an operator $A : V_1 \to V_1$ such that, for each $\boldsymbol{u} \in V_1$, $A\boldsymbol{u}$ is the element of V_1 that satisfies,

$$(A\boldsymbol{u},\, \boldsymbol{v})_{V_1} = \int_\Omega \mathcal{F}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \qquad \text{for all } \boldsymbol{v} \in V_1. \tag{3.12}$$

Also, we define $\boldsymbol{f} \in V_1$ such that,

$$(\boldsymbol{f},\, \boldsymbol{v})_{V_1} = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma}\boldsymbol{v}\, da \qquad \text{for all } \boldsymbol{v} \in V_1.$$

Let D^T be the dual of the space $\boldsymbol{\gamma}(V_1) = \{\boldsymbol{\gamma}\boldsymbol{v} \quad \boldsymbol{v} \in V_1\}$. We define $\boldsymbol{\lambda} \in D^T$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{\gamma}\boldsymbol{v} \rangle_T = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, da, \qquad \text{for all } \boldsymbol{\gamma}\boldsymbol{v} \in \boldsymbol{\gamma}(V_1),$$

where $\langle \cdot, \cdot \rangle_T$ denotes the duality pairing between D^T and $\boldsymbol{\gamma}(V_1)$. Furthermore, we define a bilinear form as follows,

$$b : V_1 \times D^T \to \mathbb{R}, \quad b(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\boldsymbol{v} \rangle_T, \qquad \text{for all } \boldsymbol{v} \in V_1,\, \boldsymbol{\mu} \in D^T. \tag{3.13}$$

Let us introduce the following subset of D^T,

$$\Lambda = \Big\{ \boldsymbol{\mu} \in D^T \;:\; \langle \boldsymbol{\mu},\, \boldsymbol{\gamma}\boldsymbol{v} \rangle_T \le \int_{\Gamma_3} \zeta \|\boldsymbol{v}_\tau\| d\Gamma \quad \text{for all } \boldsymbol{\gamma}\boldsymbol{v} \in \boldsymbol{\gamma}(V_1) \Big\}. \tag{3.14}$$

We have the following weak formulation of Problem 3.2.

Problem 3.5. *Find $\boldsymbol{u} \in V_1$ and $\boldsymbol{\lambda} \in \Lambda$, such that*

$$\begin{aligned} (A\boldsymbol{u}, \boldsymbol{v})_{V_1} + b(\boldsymbol{v}, \boldsymbol{\lambda}) &= (\boldsymbol{f},\, \boldsymbol{v})_{V_1} && \textit{for all } \boldsymbol{v} \in V_1, \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\le 0 && \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

A solution of Problem 3.5 is called a *weak solution* to Problem 3.2.

Weak formulation of Problem 3.3. We introduce the space

$$V = \{\boldsymbol{v} \in H^1(\Omega)^3 \,|\, \boldsymbol{v} = 0 \text{ a.e. on } \Gamma_1\}.$$

We define an operator $A : V \to V$ such that, for each $\boldsymbol{u} \in V$, $A\boldsymbol{u}$ is the element of V that satisfies,

$$(A\boldsymbol{u},\, \boldsymbol{v})_V = \int_\Omega \mathcal{F}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \qquad \text{for all } \boldsymbol{v} \in V. \tag{3.15}$$

Next, we define $\boldsymbol{f} \in V$ such that

$$(\boldsymbol{f},\, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, da \qquad \text{for all } \boldsymbol{v} \in V. \tag{3.16}$$

Let D^S be the dual of the space $\boldsymbol{\gamma}(V)$ and let us denote by $\langle \cdot, \cdot \rangle_S$ the duality pairing between D^S and $\boldsymbol{\gamma}(V)$. We define $\boldsymbol{\lambda} \in D^S$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{\gamma v} \rangle_S = - \int_{\Gamma_3} \sigma_n\, v_n\, da \qquad \text{for all } \boldsymbol{\gamma v} \in \gamma(V). \tag{3.17}$$

In addition, we define a bilinear form as follows

$$b : V \times D^S \to \mathbb{R}, \quad b(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle_S, \quad \text{for all } \boldsymbol{v} \in V,\, \boldsymbol{\mu} \in D^S. \tag{3.18}$$

We introduce the following subset of D^S,

$$\Lambda = \Big\{ \boldsymbol{\mu} \in D^S \,:\, \langle \boldsymbol{\mu},\, \boldsymbol{\gamma v} \rangle_S \leq 0 \quad \text{for all } \boldsymbol{\gamma v} \in \mathcal{K} \Big\}, \tag{3.19}$$

where

$$\mathcal{K} = \{ \boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V) \,:\, v_n \leq 0 \;\; \text{almost everywhere on} \;\; \Gamma_3 \}. \tag{3.20}$$

We arrive to the following weak formulation of Problem 3.3.

Problem 3.6. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$, such that*

$$\begin{aligned} (A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) &= (\boldsymbol{f},\, \boldsymbol{v})_V && \textit{for all } \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq 0 && \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

A solution of Problem 3.6 is called a *weak solution* to Problem 3.3.

Weak formulation of Problem 3.4. We can keep (3.15)-(3.20). Thus, we can write the following weak formulation of Problem 3.4.

Problem 3.7. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$, such that*

$$\begin{aligned} (A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) &= (\boldsymbol{f},\, \boldsymbol{v})_V && \textit{for all } \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq b(g_{ext}\boldsymbol{n}_3, \boldsymbol{\mu} - \boldsymbol{\lambda}) && \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

A solution of Problem 3.7 is called a *weak solution* to Problem 3.4.

Weak solvability of the models

The well-posedness of Problem 3.5 is given by the following theorem.

Theorem 3.2. *[Theorem 6.1 in [99]] If Assumptions* 3.5-3.6 *and* 3.8 *hold true, then Problem* 3.5 *has a unique solution* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V_1 \times \Lambda$. *Moreover, if* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem* 3.5 *corresponding to the data* $\boldsymbol{f} \in V_1$ *and* $\boldsymbol{f}^* \in V_1$, *there exists* $C_T > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_{V_1} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D^T} \leq C_T \|\boldsymbol{f} - \boldsymbol{f}^*\|_{V_1}. \tag{3.21}$$

The well-posedness of Problem 3.6 is given by the following theorem.

Theorem 3.3. *[Theorem 6.2 in [99]]If Assumptions* 3.5-3.6 *and* 3.8 *hold true, then Problem* 3.6 *has a unique solution* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$. *Moreover, if* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem* 3.6 *corresponding to the data* $\boldsymbol{f} \in V$ *and* $\boldsymbol{f}^* \in V$, *there exists* $C_1^S > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_{V} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D^S} \leq C_1^S \|\boldsymbol{f} - \boldsymbol{f}^*\|_{V}. \tag{3.22}$$

Finally, we discuss the well-posedness of Problem 3.7.

Theorem 3.4. *[Theorem 6.3 in [99]]If Assumptions* 3.5-3.8 *hold true, then Problem* 3.7 *has a unique solution* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$. *Moreover, if* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem* 3.7 *corresponding to the data* $(\boldsymbol{f},\, g_{ext}\boldsymbol{n}_3) \in V \times V$ *and* $(\boldsymbol{f}^*,\, g^*_{ext}\boldsymbol{n}_3) \in V \times V$, *there exists* $C_2^S > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_{V} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D^S} \leq C_2^S \Big(\|\boldsymbol{f} - \boldsymbol{f}^*\|_{V} + \|g_{ext}\boldsymbol{n}_3 - g^*_{ext}\boldsymbol{n}_3\|_{V} \Big). \tag{3.23}$$

The proofs of Theorems 3.2, 3.3 and 3.4 are based on Theorem 3.1; for details see [99] and [104].

3.2 Problems governed by proper convex l.s.c functionals

The results we present in this section were obtained in the papers [100, 104]. This section focuses on the weak solvability of a class of contact models, under the small deformations hypothesis, for static processes, for materials whose behavior is described by a constitutive law stated in a form of a subdifferential inclusion. The weak solvability of the models is based on weak formulations with dual Lagrange multipliers.

3.2.1 An abstract result

This subsection, based on Section 4 in [100] and on a part of Section 2 in [104], delivers abstract results in the study of the following problem.

Problem 3.8. *Find $u \in X$ and $\lambda \in \Lambda$ such that*

$$J(v) - J(u) + b(v-u, \lambda) \geq (f, v-u)_X \quad \text{for all } v \in X$$
$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda.$$

This is a new variational problem, *a mixed variational problem governed by a functional J.* The analysis of this problem was made under the following hypotheses.

Assumption 3.9. $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ *and* $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ *are Hilbert spaces.*

Assumption 3.10. $J : X \to [0, \infty)$ *is a convex and lower semicontinuous functional such that there exist* m_1, $m_2 > 0 : m_1\|v\|_X^2 \geq J(v) \geq m_2\|v\|_X^2$ *for all* $v \in X$.

Assumption 3.11. $b : X \times Y \to \mathbb{R}$ *is a bilinear form such that:*
i) there exists $M_b > 0 : |b(v, \mu)| \leq M_b\|v\|_X\|\mu\|_Y$ *for all* $v \in X$, $\mu \in Y$;
ii) there exists $\alpha > 0 : \inf\limits_{\mu \in Y, \mu \neq 0_Y} \sup\limits_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X\|\mu\|_Y} \geq \alpha.$

Assumption 3.12. Λ *is a closed convex subset of* Y *that contains* 0_Y.

Theorem 3.5. *If Assumptions* 3.9-3.12 *hold true, then Problem* 3.8 *has at least one solution.*

For the proof of Theorem 3.5 see the proof of Theorem 3 in [100] if Λ is unbounded, and the proof of Theorem 2.3 in [104] if Λ is bounded, respectively.

In order to study the uniqueness and the stability of the solution, the following additional assumption it was used.

Assumption 3.13. $J : X \to [0, \infty)$ *is a Gâteaux differentiable functional such that,*

there exists $L > 0 : \|\nabla J(u) - \nabla J(v)\|_X \leq L\|u - v\|_X$ *for all* $u, v \in X$,
there exists $m > 0 : (\nabla J(u) - \nabla J(v), u - v)_X \geq m\|u - v\|_X^2$ *for all* $u, v \in X$.

Notice that a pair (u, λ) is a solution of Problem 3.8 if and only if

$$(P) : \begin{cases} (\nabla J(u), v)_X + b(v, \lambda) = (f, v)_X & \text{for all } v \in X, \\ b(u, \mu - \lambda) \leq 0 & \text{for all } \mu \in \Lambda. \end{cases}$$

Theorem 3.6. *Under Assumptions* 3.9-3.13, *Problem* 3.8 *has a unique solution, which depends Lipschitz continuously on the data* f.

For the proof of Theorem 3.6 see the proof of Theorem 4 in [100] if Λ is unbounded, and the proof of Theorem 2.4 in [104] if Λ is bounded, respectively.

3.2.2 3D contact models

This subsection is based on Section 3 and Section 5 in [100] and, on a part of Section 3 in [104]. In this subsection we apply the abstract results we have presented in the previous subsection, to the weak solvability of two classes of contact problems.

A frictionless unilateral contact model

This model was analyzed in the paper [100]. The physical setting is as follows. We consider a body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, with the boundary partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal to Γ is denoted by $\boldsymbol{\nu}$ and is defined almost everywhere. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body can be in contact with a rigid foundation. In order to describe the behavior of the materials, we use a nonlinear constitutive law expressed by the subdifferential of a proper, convex, lower semicontinuous functional and the contact will be modelled using Signorini's condition with zero gap.

Problem 3.9. *Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, such that*

$$\begin{aligned}
&Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} && in\ \Omega,\\
&\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) && in\ \Omega,\\
&\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} && on\ \Gamma_1,\\
&\boldsymbol{\sigma}\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) && on\ \Gamma_2,\\
&\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = \boldsymbol{0},\ u_\nu(\boldsymbol{x}) \le 0, \sigma_\nu(\boldsymbol{x}) \le 0, \sigma_\nu(\boldsymbol{x}) u_\nu(\boldsymbol{x}) = 0 && on\ \Gamma_3.
\end{aligned}$$

The study was made under the following assumptions.

Assumption 3.14. $\boldsymbol{f}_0 \in L^2(\Omega)^3; \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3.$

Assumption 3.15. $\omega : \mathbb{S}^3 \to [0, \infty)$ *is a convex, lower semicontinuous functional such that there exist* $\alpha_1, \alpha_2 > 0 : \alpha_1 \|\boldsymbol{\varepsilon}\|^2 \ge \omega(\boldsymbol{\varepsilon}) \ge \alpha_2 \|\boldsymbol{\varepsilon}\|^2$ *for all* $\boldsymbol{\varepsilon} \in \mathbb{S}^3$.

To give an example of such a function ω we can consider

$$\omega : \mathbb{S}^3 \to [0, \infty), \quad \omega(\boldsymbol{\varepsilon}) = \frac{1}{2}\mathcal{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \frac{\beta}{2}\|\boldsymbol{\varepsilon} - P_K \boldsymbol{\varepsilon}\|^2 \tag{3.24}$$

where $\mathcal{A}$ is a fourth order symmetric tensor satisfying the ellipticity condition, β is a strictly positive constant, $K \subset \mathbb{S}^3$ denotes a closed, convex set containing the element $0_{\mathbb{S}^3}$ and $P_K : \mathbb{S}^3 \to K$ is the projection operator; see e.g. [59].

Let us introduce the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \;:\; \boldsymbol{\gamma}\,\boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\},$$

and

$$L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\mu} = (\mu_{ij}) \,:\, \mu_{ij} \in L^2(\Omega),\, \mu_{ij} = \mu_{ji} \text{ for all } i,j \in \{1,2,3\}\}.$$

We define a functional as follows,

$$W : L^2_s(\Omega)^{3\times 3} \to [0,\infty), \quad W(\boldsymbol{\tau}) = \int_\Omega \omega(\boldsymbol{\tau}(\boldsymbol{x}))dx.$$

Next, we define the functional

$$J : V \to [0,\infty), \quad J(\boldsymbol{v}) = W(\boldsymbol{\varepsilon}(\boldsymbol{v})). \tag{3.25}$$

Also, we define $\boldsymbol{f} \in V$ as follows,

$$(\boldsymbol{f},\boldsymbol{v})_V = \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v} da + \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} dx \quad \text{for all } v \in V.$$

Let us denote by D the dual of the space $\boldsymbol{\gamma}(V)$. We define the following subset of D,

$$\Lambda = \{\boldsymbol{\mu} \in D \,:\, \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}\rangle \le 0, \text{for all } \boldsymbol{\gamma v} \in \mathcal{K}\}\,, \tag{3.26}$$

where

$$\mathcal{K} = \{\boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V) \,:\, v_\nu \le 0 \text{ a.e. on } \Gamma_3\};$$

$\langle \cdot,\cdot \rangle$ denotes the duality pairing between D and $\boldsymbol{\gamma}(V)$.

In addition, we define the bilinear form

$$b : V \times D \to R, \quad b(\boldsymbol{v},\boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}\rangle \qquad \text{for all } \boldsymbol{v} \in V, \mu \in D \tag{3.27}$$

and the Lagrange multiplier $\lambda \in D$,

$$\langle \lambda, \boldsymbol{\gamma v}\rangle = -\int_{\Gamma_3} \sigma_\nu v_\nu \, d\Gamma, \qquad \text{for all } \boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V).$$

Problem 3.9 has the following weak formulation.

Problem 3.10. *Find* $\boldsymbol{u} \in V$ *and* $\boldsymbol{\lambda} \in \Lambda$ *such that*

$$J(\boldsymbol{v}) - J(\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\lambda}) \ge (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V \quad \textit{for all } \boldsymbol{v} \in V$$

$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}) \le 0 \qquad\qquad \textit{for all } \boldsymbol{\mu} \in \Lambda.$$

Based on the previous abstract result, the following theorems take place.

Theorem 3.7 (An existence result (Theorem 5 in [100])). *If Assumptions* 3.14 *and* 3.15 *hold true, then Problem* 3.10 *has at least one solution.*

Let us make now the following additional assumption.

Assumption 3.16. *ω is a Gâteaux differentiable functional such that:*
there exists $L > 0$: $\|\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})\| \leq L\|\boldsymbol{\varepsilon} - \boldsymbol{\tau}\|$ for all $\boldsymbol{\varepsilon}, \boldsymbol{\tau} \in \mathbb{S}^3$;
there exists $m > 0$: $(\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})) \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\tau}) \geq m\|\boldsymbol{\varepsilon} - \boldsymbol{\tau}\|^2$ for all $\boldsymbol{\varepsilon}, \boldsymbol{\tau} \in \mathbb{S}^3$.

An example of such a function is ω in (3.24).

Theorem 3.8 (An existence, uniqueness and stability result, (Theorem 6 in [100])). *If Assumptions* 3.14, 3.15 *and* 3.16 *hold true, then Problem* 3.10 *has a unique solution. Moreover, if $(\boldsymbol{u}_1, \boldsymbol{\lambda}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{\lambda}_2)$ are two solutions of Problem* 3.10 *corresponding to the data $\boldsymbol{f}_1$, $\boldsymbol{f}_2 \in V$, then there exists $C > 0$ such that*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_D \leq C\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_V.$$

The proof of Theorem 3.8, gave in [100], is based on the previous abstract result, Theorem 3.6.

A frictional contact model

The model we discuss now was analyzed in the paper [104]. The physical setting is the following. A body occupies the bounded domain $\Omega \subset \mathbb{R}^3$, with the boundary partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal vector to Γ is denoted by $\boldsymbol{\nu}$ and is defined almost everywhere. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. On Γ_3 the body is in frictional contact with a foundation.

According to the previous physical setting we state the following boundary value problem.

Problem 3.11. *Find $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$ such that*

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{3.28}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \qquad \text{in } \Omega, \tag{3.29}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{3.30}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{3.31}$$

$$-\sigma_\nu = F,\ \|\boldsymbol{\sigma}_\tau\| \leq k|\sigma_\nu|,\ \boldsymbol{\sigma}_\tau = -k|\sigma_\nu|\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \qquad \text{on } \Gamma_3, \tag{3.32}$$

where $\omega : \mathbb{S}^3 \to [0,\infty)$ is a constitutive function, $F : \Gamma_3 \to \mathbb{R}_+$ is the prescribed normal stress and $k : \Gamma_3 \to \mathbb{R}_+$ is the coefficient of friction.

Problem 3.11 has the following structure: (3.28) represents the equilibrium equation, (3.29) represents the constitutive law, (3.30) represents the displacements boundary condition, (3.31) represents the traction boundary condition and (3.32) models the frictional contact with prescribed normal stress. For details on this model we send the reader to, e.g., [147].

We made the following assumptions.

Assumption 3.17. *$\omega : \mathbb{S}^3 \to [0,\infty)$ is a convex, lower semicontinuous functional. In addition, there exist $\alpha_1,\ \alpha_2 > 0$ such that $\alpha_1\|\boldsymbol{\varepsilon}\|^2 \geq \omega(\boldsymbol{\varepsilon}) \geq \alpha_2\|\boldsymbol{\varepsilon}\|^2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^3$.*

Assumption 3.18. *The density of the volume forces verifies $\boldsymbol{f}_0 \in L^2(\Omega)^3$ and the density of the tractions verifies $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.*

Assumption 3.19. *The prescribed normal stress verifies $F \in L^2(\Gamma_3)$ and $F(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.*

Assumption 3.20. *The coefficient of friction verifies $k \in L^\infty(\Gamma_3)$ and $k(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.*

Let us replace now (3.32) with the following condition

$$u_\nu = 0,\ \|\boldsymbol{\sigma}_\tau\| \leq \zeta,\ \boldsymbol{\sigma}_\tau = -\zeta\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq \boldsymbol{0}. \tag{3.33}$$

This condition is a frictional bilateral contact condition where $\zeta : \Gamma_3 \to \mathbb{R}_+$ denotes the friction bound.

Now, a second model can be formulated as follows.

Problem 3.12. *Find $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$ such that (3.28)-(3.31) and (3.33) hold true.*

We shall study Problem 3.12 under Assumptions 3.17-3.18, and in addition we shall make the following assumption.

Assumption 3.21. *The friction bound verifies $\zeta \in L^2(\Gamma_3)$ and $\zeta(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.*

Weak solvability of Problem 3.11

Let us introduce two functional spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \ :\ \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\},$$

and

$$L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\mu} = (\mu_{ij}) \ :\ \mu_{ij} \in L^2(\Omega),\, \mu_{ij} = \mu_{ji} \text{ for all } i,j \in \{1,2,3\}\}$$

We now introduce the functional

$$W : L^2_s(\Omega)^{3\times 3} \to [0,\infty), \quad W(\boldsymbol{\tau}) = \int_\Omega \omega(\boldsymbol{\tau}(\boldsymbol{x}))\,dx. \tag{3.34}$$

Using the functional W we introduce a new one

$$J : V \to [0, \infty), \quad J(\boldsymbol{v}) = W(\boldsymbol{\varepsilon}(\boldsymbol{v})). \tag{3.35}$$

We define $\boldsymbol{f} \in V$ such that, for all $\boldsymbol{v} \in V$,

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x}) v_\nu(\boldsymbol{x})\, d\Gamma.$$

Next, let D be the dual of the Hilbert space

$$S = \{\widetilde{\boldsymbol{v}} = \boldsymbol{\gamma v}_{|\Gamma_3} \quad \boldsymbol{v} \in V\}.$$

We define $\boldsymbol{\lambda} \in D$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x})\, d\Gamma \qquad \text{for all } \boldsymbol{w} \in S,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D and S. Furthermore, we define a bilinear form as follows,

$$b : V \times D \to \mathbb{R}, \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}_{|\Gamma_3} \rangle, \quad \text{for all } \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D.$$

Let us introduce the following subset of D,

$$\Lambda = \left\{ \boldsymbol{\mu} \in D \,:\, \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle \leq \int_{\Gamma_3} k\, F\, \|\boldsymbol{v}_\tau\|\, d\Gamma \quad \boldsymbol{v} \in V \right\}. \tag{3.36}$$

We are led to the following weak formulation of Problem 3.11.

Problem 3.13. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$, such that*

$$\begin{aligned} J(\boldsymbol{v}) - J(\boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) &\geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V && \textit{for all } \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq 0 && \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

A solution of Problem 3.13 is called a *weak solution* of Problem 3.11.

Theorem 3.9. *[Theorem 3.1 in [104]] If Assumptions* 3.17-3.20 *hold true, then Problem* 3.13 *has at least one solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$. If, in addition, Assumption* 3.16 *is fulfilled, then Problem* 3.13 *has a unique solution; moreover, there exists $C > 0$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_V + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_D \leq C \|\boldsymbol{f} - \boldsymbol{f}^*\|_V, \tag{3.37}$$

where $(\boldsymbol{u}, \boldsymbol{\lambda})$ and $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ are two solutions of Problem 3.13 *corresponding to the data $\boldsymbol{f} \in V$ and $\boldsymbol{f}^* \in V$.*

The proof of Theorem 3.9, given in [104], is based on Theorem 3.6.

Weak solvability of Problem 3.12
Herein we use the Hilbert space

$$V_1 = \Big\{ \boldsymbol{v} \in H^1(\Omega)^3 \mid \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1,\, v_\nu = 0 \text{ a.e. on } \Gamma_3 \Big\}.$$

We define $\boldsymbol{f}_1 \in V_1$ such that, for all $\boldsymbol{v} \in V_1$,

$$(\boldsymbol{f}_1,\, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma.$$

Next, we define the functional

$$J_1 : V_1 \to [0, \infty), \quad J_1(\boldsymbol{v}) = W(\boldsymbol{\varepsilon}(\boldsymbol{v})),$$

where W is the functional defined in (3.34).
Let D_1 be the dual of the Hilbert space

$$S_1 = \{\widetilde{\boldsymbol{v}} = \boldsymbol{\gamma v}_{|\Gamma_3} \quad \boldsymbol{v} \in V_1\}.$$

We define $\boldsymbol{\lambda} \in D_1$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x})\, d\Gamma, \qquad \text{for all } \boldsymbol{w} \in S_1,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D_1 and S_1. Furthermore, we define a bilinear form as follows,

$$b : V_1 \times D_1 \to \mathbb{R}, \quad b_1(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}_{|\Gamma_3} \rangle, \quad \text{for all } \boldsymbol{v} \in V_1,\, \boldsymbol{\mu} \in D_1.$$

Let us introduce the following subset of D_1,

$$\Lambda_1 = \Big\{ \boldsymbol{\mu} \in D_1 \,:\, \langle \boldsymbol{\mu},\, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle \le \int_{\Gamma_3} \zeta(\boldsymbol{x})\, \|\boldsymbol{v}_\tau\|\, d\Gamma \quad \boldsymbol{v} \in V_1 \Big\}.$$

Clearly, $\boldsymbol{\lambda} \in \Lambda_1$. Furthermore,

$$\begin{aligned} b_1(\boldsymbol{u}, \boldsymbol{\lambda}) &= \int_{\Gamma_3} \zeta(\boldsymbol{x}) \|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma, \\ b_1(\boldsymbol{u}, \boldsymbol{\mu}) &\le \int_{\Gamma_3} \zeta(\boldsymbol{x}) \|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma \qquad \text{for all } \boldsymbol{\mu} \in \Lambda_1. \end{aligned}$$

Consequently, we are led to the following weak formulation of Problem 3.12.

Problem 3.14. *Find $\boldsymbol{u} \in V_1$ and $\boldsymbol{\lambda} \in \Lambda_1$, such that*

$$\begin{aligned} J_1(\boldsymbol{v}) - J_1(\boldsymbol{u}) + b_1(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\lambda}) &\geq (\boldsymbol{f}_1, \boldsymbol{v}-\boldsymbol{u})_V \qquad && \text{for all } \boldsymbol{v} \in V_1, \\ b_1(\boldsymbol{u},\boldsymbol{\mu}-\boldsymbol{\lambda}) &\leq 0 && \text{for all } \boldsymbol{\mu} \in \Lambda_1. \end{aligned}$$

Theorem 3.10. *[Theorem 3.2 in [104]] If Assumptions* 3.17-3.18 *and* 3.21 *hold true, then Problem* 3.14 *has a unique solution; moreover, there exists $C > 0$ such that*

$$\|\boldsymbol{u}-\boldsymbol{u}^*\|_{V_1} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}^*\|_{D_1} \leq C\|\boldsymbol{f}-\boldsymbol{f}^*\|_{V_1}$$

where $(\boldsymbol{u},\boldsymbol{\lambda})$ and $(\boldsymbol{u}^*,\boldsymbol{\lambda}^*)$ are two solutions of Problem 3.14 *corresponding to the data $\boldsymbol{f} \in V_1$ and $\boldsymbol{f}^* \in V_1$.*

A solution of Problem 3.14 is called a *weak solution* of Problem 3.12.

The proof of Theorem 3.10, given in [104], is based on Theorem 3.6.

3.3 Problems governed by a nonlinear, hemicontinuous, generalized monoton operator

This section, based on the paper [107], focuses on a new theoretical result which will allow to explore contact models for a class of nonlinearly elastic materials leading to mixed variational problems governed by nonlinear, hemicontinuous, generalized monoton operators. The key herein is not the saddle point theory; the key here is a fixed point theorem for set valued mapping.

3.3.1 An abstract result

This subsection is based on the Sections 2-4 of the paper [107]. In this subsection we focus on the following mixed variational problem.

Problem 3.15. *Given $f \in X'$, find $(u,\lambda) \in X \times \Lambda$ such that*

$$(Au, v)_{X',X} + b(v,\lambda) = (f, v)_{X',X} \qquad \text{for all } v \in X, \tag{3.38}$$

$$b(u, \mu-\lambda) \leq 0 \qquad \text{for all } \mu \in \Lambda. \tag{3.39}$$

Here and everywhere below X' denotes the dual of the space X and Λ is a subset of a space Y.

Assumption 3.22. $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ *and* $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ *are two real reflexive Banach spaces.*

Assumption 3.23. Λ *is a closed convex bounded subset of Y such that $0_Y \in \Lambda$.*

Assumption 3.24. *There exists a functional $h : X \to \mathbb{R}$ such that:*

- (i_1) *$h(tw) = t^r h(w)$ for all $t > 0$, $w \in X$ and $r > 1$;*
- (i_2) *$(Av - Au, v-u)_{X',X} \geq h(v-u)$ for all $u, v \in X$;*
- (i_3) *If $(x_n)_n \subset X$ is a sequence such that $x_n \rightharpoonup x$ in X as $n \to \infty$, then $h(x) \leq \limsup_{n\to\infty} h(x_n)$.*

Notice that (i_1) and (i_2) in Assumption 3.24 express a generalized monotonicity property for the operator $A : X \to X'$. According to the literature, the operator A is a relaxed $h-$monotone operator, see for example [42] and the references therein.

Assumption 3.25. *The operator $A : X \to X'$ is hemicontinuous, i.e., for all $u, v \in X$, the mapping $f : \mathbb{R} \to (-\infty, +\infty)$, $f(t) = (A(u + tv), v)_{X',X}$ is continuous at 0.*

Assumption 3.26. $\dfrac{(Au, u)_{X',X}}{\|u\|_X} \to \infty$ *as* $\|u\|_X \to \infty$.

Assumption 3.27. *The form $b : X \times Y \to \mathbb{R}$ is bilinear. In addition,*

• *for each sequence $(u_n)_n \subset X$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$ we have $b(u_n, \mu) \to b(u, \mu)$ as $n \to \infty$, for all $\mu \in \Lambda$.*

• *for each sequence $(\lambda_n)_n \subset Y$ such that $\lambda_n \rightharpoonup \lambda$ in Y as $n \to \infty$, we have $b(v, \lambda_n) \to b(v, \lambda)$ as $n \to \infty$, for all $v \in X$.*

Under Assumptions 3.22-3.27, Problem 3.15 has at least one solution. Assumptions 3.22-3.27 impose a new technique in order to handle Problem 3.15, namely a fixed point technique involving a set valued mapping, instead of a saddle point technique. Let us mention here the main tool.

Theorem 3.11. *Let $\mathcal{K} \neq \emptyset$ be a convex subset of a Hausdorff topological vector space $\mathcal{E}$. Let $F : \mathcal{K} \to 2^{\mathcal{K}}$ be a set valued map such that*

(h_1) for each $u \in \mathcal{K}$, $F(u)$ is a nonempty convex subset of $\mathcal{K}$;

(h_2) for each $v \in \mathcal{K}$, $F^{-1}(v) = \{\, u \in \mathcal{K} :\ v \in F(u)\}$ contains an open set O_v which may be empty;

(h_3) $\bigcup_{v \in \mathcal{K}} \mathcal{O}_v = \mathcal{K}$;

(h_4) there exists a nonempty set $\mathcal{V}_0$ contained in a compact convex subset $\mathcal{V}_1$ of $\mathcal{K}$ such that $\mathcal{D} = \bigcap_{v \in \mathcal{V}_0} \mathcal{O}_v^c$ is either empty or compact.

Then, there exists $u_0 \in \mathcal{K}$ such that $u_0 \in F(u_0)$.

We note that $2^{\mathcal{K}}$ denotes the family of all subsets of $\mathcal{K}$, and $\mathcal{O}_v^c$ is the complement of $\mathcal{O}_v$ in $\mathcal{K}$. For a proof of this theorem we refer to [152].

Let us construct a bounded convex closed nonempty subset of X as follows,

$$K_n = \{v \in X \,:\, \|v\|_X \leq n\}$$

where n is an arbitrarily fixed positive integer. We consider the following auxiliary problem.

Problem 3.16. *Given $f \in X'$, find $(u_n, \lambda_n) \in K_n \times \Lambda$ such that*

$$(Au_n, v - u_n)_{X',X} + b(v, \lambda_n) - b(u_n, \mu) \geq (f, v - u_n)_{X',X} \qquad (3.40)$$
$$\text{for all } (v, \mu) \in K_n \times \Lambda.$$

Lemma 3.1. *[Lemma 1 in [107]] A pair* $(u_n, \lambda_n) \in K_n \times \Lambda$ *is a solution of Problem* 3.16 *if and only if it verifies*

$$(Av, v - u_n)_{X',X} + b(v, \lambda_n) - b(u_n, \mu) \geq (f, v - u_n)_{X',X} + h(v - u_n) \qquad \textit{for all } (v, \mu) \in K_n \times \Lambda. \tag{3.41}$$

Let us define a set valued map $F : K_n \times \Lambda \to 2^{K_n \times \Lambda}$ as follows,

$$F(u, \lambda) = \left\{(v, \mu) \in K_n \times \Lambda \,:\, (Au, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) < (f,\, v - u)_{X',X}\right\}.$$

Arguing by contradiction, using the map $F(\cdot, \cdot)$ and Theorem 3.11, the following existence result was delivered.

Theorem 3.12. *[Theorem 2 in [107]] If Assumptions* 3.22-3.25 *and Assumption* 3.27 *hold true, then Problem* 3.16 *has at least one solution* $(u_n, \lambda_n) \in K_n \times \Lambda$.

Based on Theorem 3.12 we have got the following existence result.

Theorem 3.13. *[Theorem 3 in [107]] If Assumptions* 3.22-3.27 *hold true, then Problem* 3.15 *has at least one solution.*

The proofs of Lemma 3.1, Theorems 3.12-3.13 can be found in [107].

Let us present an example of spaces X, Y, subset Λ, operator A and form $b(\cdot, \cdot)$ which verify Assumptions 3.22-3.27.

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. Let p be a real number such that $\infty > p \geq 4$. We define a subspace of $W^{1,p}(\Omega)$ as follows,

$$X = \{v \,:\, v \in W^{1,p}(\Omega),\, \gamma v = 0 \text{ a.e. on } \Gamma_D\} \tag{3.42}$$

where Γ_D is a part of Γ with positive Lebesgue measure and $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is the Sobolev trace operator. It is known that the space X is a Banach space endowed with the norm

$$\|u\|_X = \|\nabla u\|_{L^p(\Omega)^N}.$$

Let p' be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We now consider Γ_C a part of Γ such that $meas(\Gamma_C) > 0$ and $\Gamma_C \cap \Gamma_D = \emptyset$. Then, we can take

$$Y = L^{p'}(\Gamma_C). \tag{3.43}$$

Next, we define a subset of Y as follows,

$$\Lambda = \{\mu \in Y \,:\, \langle \mu, \gamma v_{|_{\Gamma_C}} \rangle \leq \int_{\Gamma_C} g|\gamma v(\boldsymbol{x})|\, d\Gamma \quad \text{for all } v \in X\}, \tag{3.44}$$

where g is a positive real number.

We define $A : X \to X'$ as follows: for each $u \in X$, Au is the element of X' such that

$$(Au, v)_{X',X} = \int_\Omega \mu \|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, dx \quad \text{for all } v \in X \tag{3.45}$$

where μ is a positive real number. The operator A is hemicontinuous and relaxed $h-$monotone with $h \equiv 0$ being in fact Lipschitz continuous and monotone. Besides, for each $u \in X$, $u \neq 0_X$, we have

$$\frac{(Au, u)_{X',X}}{\|u\|_X} = \mu \|u\|_X^{p-1}.$$

Finally, we define $b : X \times L^{p'}(\Gamma_C) \to \mathbb{R}$ as follows

$$b(v, \mu) = \langle \mu, \gamma v_{|\Gamma_C} \rangle, \tag{3.46}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $L^{p'}(\Gamma_C)$ and $L^p(\Gamma_C)$.

3.3.2 An antiplane frictional contact problem

This subsection is based on Section 5 of the paper [107]. Herein we apply the abstract existence result, Theorem 3.13, to the weak solvability of the following boundary value problem.

Problem 3.17. *Find $u : \bar{\Omega} \to \mathbb{R}$ such that*

$$\begin{aligned} \operatorname{div}(\mu \|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x})) + f_0(\boldsymbol{x}) &= 0 && \text{in } \Omega, && (3.47)\\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_D, && (3.48)\\ \mu \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_N, && (3.49) \end{aligned}$$

$$\left.\begin{array}{l} |\mu \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})| \leq g, \\ \mu \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x}) = -g \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \quad \text{if } u(\boldsymbol{x}) \neq 0 \end{array}\right\} \text{on } \Gamma_C. \tag{3.50}$$

This problem models the antiplane shear deformation of a nonlinearly elastic cylindrical body, in frictional contact on Γ_C with a rigid foundation. See [145] for details on the antiplane contact models. We also refer to the works [114, 115, 116, 117] for a treatement of some antiplane contact problems in a general setting of the hemivariational inequalities.

Herein $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ partitioned in three measurable parts Γ_D, Γ_N, Γ_C with positive Lebesgue measures. Referring the body to a cartesian coordinate system $Ox_1x_2x_3$ such that the generators of the cylinder are parallel with the axis Ox_3, the domain $\Omega \subset Ox_1x_2$ denotes the cross section of the cylinder. The functions $f_0 = f_0(x_1, x_2) : \Omega \to \mathbb{R}$, $f_2 = f_2(x_1, x_2) : \Gamma_N \to \mathbb{R}$ are related to the density of the volume forces and the density of the surface traction, respectively, and $g > 0$ is the friction bound. The vector $\boldsymbol{\nu} = (\nu_1, \nu_2)$,

$\nu_i = \nu_i(x_1, x_2)$, for each $i \in \{1, 2\}$, represents the outward unit normal vector to the boundary of Ω and $\partial_\nu u = \nabla u \cdot \boldsymbol{\nu}$. The behavior of the nonlinearly elastic material is described by the following constitutive law:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = k \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\, \boldsymbol{I}_3 + \mu \|\boldsymbol{\varepsilon}^D(\boldsymbol{u}(x))\|^{p-2} \boldsymbol{\varepsilon}^D(\boldsymbol{u}(\boldsymbol{x})) \tag{3.51}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, tr is the trace of a cartesian tensor of second order, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{I}_3$ is the identity tensor, $k, \mu > 0$ are material parameters, p is a constant such that $4 \leq p < \infty$, $\boldsymbol{\varepsilon}^D$ denotes the *deviator* of the tensor $\boldsymbol{\varepsilon}$, defined by $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \frac{1}{3} (\operatorname{tr} \boldsymbol{\varepsilon})\, \boldsymbol{I}_3$. The constitutive law (3.51) is a Hencky-type constitutive law.

The mechanical problem has the following structure: (3.47) represents the equilibrium equation, (3.48) is the displacement boundary condition, (3.49) is the traction boundary condition and (3.50) is Tresca's law of dry friction; see e.g. [145, 147] for more details on frictional laws.

We shall study Problem 3.17 assuming that

$$f_0 \in L^{p'}(\Omega), \quad f_2 \in L^{p'}(\Gamma_N). \tag{3.52}$$

We define $f \in X'$ as follows

$$(f, v)_{X',X} = \int_\Omega f_0(\boldsymbol{x})\, v(\boldsymbol{x})\, dx + \int_{\Gamma_D} f_2(\boldsymbol{x})\, \gamma v(\boldsymbol{x})\, d\Gamma \quad \text{ for all } v \in X. \tag{3.53}$$

Next, we define a Lagrange multiplier $\lambda \in Y$ as follows

$$\langle \lambda, z \rangle = -\int_{\Gamma_C} \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, z(\boldsymbol{x})\, d\Gamma \quad \text{ for all } z \in L^p(\Gamma_C), \tag{3.54}$$

where Y is the space defined in (3.43).

Problem 3.17 has the following weak formulation.

Problem 3.18. *Find $u \in X$ and $\lambda \in \Lambda \subset Y$ such that*

$$(Au,\, v)_{X',X} + b(v, \lambda) = (f, v)_{X',X} \quad \textit{for all } v \in X. \tag{3.55}$$

and

$$b(u, \zeta - \lambda) \leq 0 \quad \textit{for all } \zeta \in \Lambda. \tag{3.56}$$

Theorem 3.14. *[Theorem 4 in [107]]If $4 \leq p < \infty$, $k, \mu, g > 0$, $f_0 \in L^{p'}(\Omega)$, and $f_2 \in L^{p'}(\Gamma_N)$, then Problem* 3.18 *has at least one solution.*

The proof of Theorem 3.14, given in [107], is based on Theorem 3.13.

As each solution of Problem 3.18 is called *weak solution* of Problem 3.17, Theorem 3.14 ensures us that Problem 3.17 has at least one weak solution.

Chapter 4

Viscoelastic frictional contact problems

This chapter is based on the papers [101, 111]. We discuss antiplane models which describe the contact between a deformable cylinder and a rigid foundation, under the small deformation hypothesis, for quasistatic processes. The behavior of the material is modelled using viscoelastic constitutive laws and the frictional contact is modelled using Tresca's law. We draw the attention to the weak solvability of the models based on a weak formulation with dual Lagrange multipliers in the case of viscoelastic materials with long memory as well as in the case of viscoelastic materials with short memory. The results we have got are based on new abstract results in the study of new classes of mixed variational problems: a class of time dependent mixed variational problems and a class of evolutionary mixed variational problems.

4.1 The case of viscoelasticity with long-memory term

In this section we present the results obtained in the paper [101], discussing the weak solvability of a contact model for viscoelastic materials with long memory, by using arguments which involve dual Lagrange multipliers; for a classical approach of such kind of models we refer to, e.g. [143]. The weak solvability of the proposed model through an approach with Lagrange multipliers is related to the solvability of a new abstract variational problem.

4.1.1 An abstract result

In this subsection we shall present an abstract result obtained under the following assumptions.

Assumption 4.1. *$(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 4.2. *$A : X \to X$ is an operator such that:*
there exists $m_A > 0 : (Au - Av,\, u - v)_X \geq m_A \|u - v\|_X^2$ for all $u, v \in X$,
there exists $L_A > 0 : \|Au - Av\|_X \leq L_A \|u - v\|_X$ for all $u, v \in X$.

Assumption 4.3. $b : X \times Y \to R$ *is a bilinear form such that:*

there exists $M_b > 0 : |b(v, \mu)| \le M_b \|v\|_X \|\mu\|_Y$ *for all* $v \in X$, $\mu \in Y$,

there exists $\alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \ge \alpha.$

Assumption 4.4. $\Lambda \subset Y$ *is a closed convex set such that* $0_Y \in Y$.

Assumption 4.5. $f \in C([0, T]; X)$.

Assumption 4.6. $B \in C([0, T]; \mathcal{L}(X))$.

Problem 4.1. *Given* $f : [0, T] \to X$*, find* $u : [0, T] \to X$ *and* $\lambda : [0, T] \to Y$ *such that, for every* $t \in [0, T]$*, we have* $\lambda(t) \in \Lambda$ *and*

$$(Au(t), v)_X + (\int_0^t B(t-s)u(s)ds, v)_X + b(v, \lambda(t)) = (f(t), v)_X \quad \textit{for all } v \in X \tag{4.1}$$

$$b(u(t), \mu - \lambda(t)) \le 0 \quad \textit{for all } \mu \in \Lambda. \tag{4.2}$$

This is a new variational problem, *a time dependent mixed variational problem with long-memory* (a mixed variational problem governed by an integral term).

Let $\eta \in C([0, T], X)$ and $t \in [0, T]$. We consider the following auxiliary problem.

Problem 4.2. *Find* $u_\eta(t) \in X$ *and* $\lambda_\eta(t) \in \Lambda$ *such that*

$$\begin{aligned} (Au_\eta(t), v)_X + b(v, \lambda_\eta(t)) &= (f(t) - \eta(t), v)_X \quad \textit{for all } v \in X \\ b(u_\eta(t), \mu - \lambda_\eta(t)) &\le 0 \quad \textit{for all } \mu \in \Lambda. \end{aligned}$$

Problem 4.2 has a unique solution, $(u_\eta(t), \lambda_\eta(t)) \in X \times \Lambda$. In addition,

$$u_\eta \in C([0, T], X) \text{ and } \lambda_\eta \in C([0, T], Y). \tag{4.3}$$

Let us define

$$\mathcal{T} : C([0, T], X) \to C([0, T], X), \qquad (\mathcal{T}\eta)(t) = \int_0^t B(t-s)u_\eta(s)ds.$$

The operator $\mathcal{T}$ is a contraction.

Let $\eta^* \in C([0, T]; X)$ be the unique fixed point of the operator $\mathcal{T}$ and $(u_{\eta^*}, \lambda_{\eta^*})$ be the solution of Problem 4.2 for $\eta = \eta^*$. Let $t \in [0, T]$. Notice that the pair $(u_{\eta^*}(t), \lambda_{\eta^*}(t))$ verifies (4.1) and (4.2). It was proved the following theorem.

Theorem 4.1. *If Assumptions* 4.1 − 4.6 *hold true, then there exists a unique solution of Problem* 4.1*,* (u, λ)*, such that* $(u(t), \lambda(t)) \in X \times \Lambda$ *for all* $t \in [0, T]$ *and*

$$u \in C([0, T]; X),\ \lambda \in C([0, T]; Y).$$

Moreover, given $f_1,\ f_2 \in C([0,T];X)$*, there exists* $C>0$ *such that*

$$\|u_1-u_2\|_{C([0,T];X)}+\|\lambda_1-\lambda_2\|_{C([0,T];Y)}\le C\|f_1-f_2\|_{C([0,T];X)},$$

(u_1,λ_1) *and* (u_2,λ_2) *being the solutions of Problem 4.1 corresponding to the data* f_1 *and* f_2*, respectively.*

The proof of Theorem 4.1 can be found in [101] if Λ is unbounded (see Theorem 2 in [101]), and it follows from Theorems 2.1 and 2.2 in [104], combined with the Banach's fixed point theorem, if Λ is a bounded set.

4.1.2 A mechanical model and its weak solvability

In this subsection we discuss the weak solvability of the following contact model.

Problem 4.3. *Find* $u:\overline{\Omega}\times[0,T]\to\mathbb{R}$ *such that, for every* $t\in[0,T]$,

$$\begin{aligned}
\operatorname{div}\Big(a(\boldsymbol{x},\nabla u(\boldsymbol{x},t))+\int_0^t\theta(\boldsymbol{x},t-s)\nabla u(\boldsymbol{x},s)ds\Big)+f_0(\boldsymbol{x},t)=0 &\quad \textit{in } \Omega\\
u(\boldsymbol{x},t)=0 &\quad \textit{on } \Gamma_1\\
a(\boldsymbol{x},\nabla u(\boldsymbol{x},t))\cdot\boldsymbol{\nu}(\boldsymbol{x})+\int_0^t\theta(\boldsymbol{x},t-s)\partial_\nu u(\boldsymbol{x},s)ds=f_2(\boldsymbol{x},t) &\quad \textit{on } \Gamma_2
\end{aligned}$$

$$\left.\begin{aligned}
&|a(\boldsymbol{x},\nabla u(\boldsymbol{x},t))\cdot\boldsymbol{\nu}(\boldsymbol{x})+\int_0^t\theta(\boldsymbol{x},t-s)\partial_\nu u(\boldsymbol{x},s)ds|\le g(\boldsymbol{x});\\
&|a(\boldsymbol{x},\nabla u(\boldsymbol{x},t))\cdot\boldsymbol{\nu}(\boldsymbol{x})+\int_0^t\theta(\boldsymbol{x},t-s)\partial_\nu u(\boldsymbol{x},s)ds|<g(\boldsymbol{x})\\
&\Rightarrow u(\boldsymbol{x},t)=0;\\
&|a(\boldsymbol{x},\nabla u(\boldsymbol{x},t))\cdot\boldsymbol{\nu}+\int_0^t\theta(\boldsymbol{x},t-s)\partial_\nu u(\boldsymbol{x},s)ds|=g(\boldsymbol{x})\\
&\Rightarrow\ \textit{there exists } \beta>0 \textit{ such that}\\
&a(\boldsymbol{x},\nabla u(\boldsymbol{x},t))\cdot\boldsymbol{\nu}(\boldsymbol{x})+\int_0^t\theta(\boldsymbol{x},t-s)\partial_\nu u(\boldsymbol{x},s)ds\\
&=-\beta u(\boldsymbol{x},t)
\end{aligned}\right\}\ \textit{on}\,\Gamma_3, \tag{4.4}$$

where $\Omega\subset\mathbb{R}^2$ is an open, bounded, connected subset, with Lipschitz continuous boundary Γ partitioned in three measurable parts $\Gamma_1,\Gamma_2,\Gamma_3$ such that the Lebesgue measure of Γ_1 is positive. This problem models the *antiplane shear deformation* of a cylindrical body in bilateral frictional contact on Γ_3 with a rigid foundation. The domain Ω denotes the cross section of the cylinder, the unknown $u=u(x_1,x_2):\bar{\Omega}\times[0,T]\to\mathbb{R}$ represents the 3^{rd} component of the displacement vector, $a:\bar{\Omega}\times\mathbb{R}^2\to\mathbb{R}^2$ is a constitutive function, $\theta:\bar{\Omega}\times[0,T]\to\mathbb{R}$ is a coefficient of relaxation,

$g : \Gamma_3 \to \mathbb{R}$ is the *friction bound* and the functions $f_0 : \Omega \times [0,T] \to \mathbb{R}$, $f_2 : \Gamma_2 \times [0,T] \to \mathbb{R}$ are related to the density of the volume forces and the density of the surface traction, respectively. Here $\boldsymbol{\nu}$ is the unit outward normal vector on the boundary Γ, defined almost everywhere, and $\partial_\nu u = \nabla u \cdot \boldsymbol{\nu}$. Notice that $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\nu_i = \nu_i(x_1, x_2)$, for each $i \in \{1, 2\}$.

Assumption 4.7.

- *there exists* $L_a > 0 : \ \|a(\boldsymbol{x}, \boldsymbol{\xi}_1) - a(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \leq L_a \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|$ *for all* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^2$, *a.e.* $\boldsymbol{x} \in \Omega$;
- *there exists* $M_a > 0 : \ (a(\boldsymbol{x}, \boldsymbol{\xi}_1) - a(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq M_a \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2$, *for all* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^2$, *a.e.* $\boldsymbol{x} \in \Omega$;
- *For each* $\boldsymbol{\xi} \in \mathbb{R}^2$, $\boldsymbol{x} \to a(\boldsymbol{x}, \boldsymbol{\xi})$ *is measurable in* Ω;
- *The mapping* $\boldsymbol{x} \to a(\boldsymbol{x}, \mathbf{0}) \in L^2(\Omega)^2$.

Let us give three examples of constitutive laws related to three examples of such functions a.

Example 4.1. *We can describe the behavior of the material with the following constitutive law*

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{x}, t) = \lambda(\boldsymbol{x})(tr\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t)))\mathbb{I}_{\mathbb{S}^3} + 2\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t)) + \int_0^t \theta(\boldsymbol{x}, t-s)\boldsymbol{\varepsilon}(u(\boldsymbol{x}, s))ds \\ + \int_0^t \zeta(\boldsymbol{x}, t-s) tr(\boldsymbol{\varepsilon}(u(\boldsymbol{x}, s)))\mathbb{I}_{\mathbb{S}^3} ds,\end{aligned}$$

where λ and μ are coefficients of the material, $\mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) = \varepsilon_{kk}(\boldsymbol{u})$, $\mathbb{I}_{\mathbb{S}^3}$ *is the unit tensor and θ, ζ are coefficients of relaxation. In the antiplane context, the equilibrium equation reduces to*

$$\mathrm{div}\left(\mu(\boldsymbol{x})\nabla u(\boldsymbol{x}, t) + \int_0^t \theta(\boldsymbol{x}, t-s)\nabla u(\boldsymbol{x}, s)ds\right) + f_0(\boldsymbol{x}, t) = 0 \ \ in\ \Omega \times (0, T),$$

see for example [145]. In this situation we define

$$a(\boldsymbol{x}, \boldsymbol{\xi}) = \mu(\boldsymbol{x})\, \boldsymbol{\xi}.$$

Assuming $\mu \in L^\infty(\Omega)$, $\mu(\boldsymbol{x}) \geq \mu^* > 0$ a.e. $\boldsymbol{x} \in \Omega$, then Assumption 4.7 is fulfilled.

Example 4.2. *Let us describe the behavior of the material with the viscoelastic constitutive law*

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{x}, t) = \lambda(\boldsymbol{x})(tr\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t)))\mathbb{I}_{\mathbb{S}^3} + 2\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t)) \\ +2\beta(\boldsymbol{x})(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t)) - P_{\mathcal{K}}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}, t))) \\ + \int_0^t \theta(\boldsymbol{x}, t-s)\boldsymbol{\varepsilon}(u(\boldsymbol{x}, s))ds \\ + \int_0^t \zeta(\boldsymbol{x}, t-s)\, \mathrm{tr}(\boldsymbol{\varepsilon}(u(\boldsymbol{x}, s)))\mathbb{I}_{\mathbb{S}^3} ds,\end{aligned}$$

where λ, μ and β are coefficients of the material, $\mathcal{K}$ is the non-empty, closed and convex von Mises set

$$\mathcal{K} = \{\boldsymbol{\sigma} \in \mathbb{S}^3 \,|\, \frac{1}{2}\boldsymbol{\sigma}^D \cdot \boldsymbol{\sigma}^D \leq k^2, \quad k > 0\} \tag{4.5}$$

and $P_{\mathcal{K}} : \mathbb{S}^3 \to \mathcal{K}$ represents the projection operator on $\mathcal{K}$. We recall that $\boldsymbol{\sigma}^D$ is the deviator of $\boldsymbol{\sigma}$, i.e., $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{3}(tr\,\boldsymbol{\sigma})I_{\mathbb{S}^3}$.

The equilibrium equation reduces to the following scalar equation

$$\begin{aligned}\operatorname{div}\Big((\mu(\boldsymbol{x}) + \beta(\boldsymbol{x}))\nabla u(\boldsymbol{x},t) - 2\beta(\boldsymbol{x})P_{\widetilde{K}}\frac{1}{2}\nabla u(\boldsymbol{x},t) + \int_0^t \theta(\boldsymbol{x}, t-s)\nabla u(\boldsymbol{x},s)ds\Big)\\ +f_0(\boldsymbol{x},t) = 0 \quad \textit{in} \quad \Omega \times (0,T),\end{aligned}$$

where $\widetilde{K} = \overline{B(\mathbf{0}_{\mathbb{R}^2}, k)}$, ($k$ given by (4.5)) and $P_{\widetilde{K}} : \mathbb{R}^2 \to \widetilde{K}$ is the projection operator on $\widetilde{K}$.

We define

$$a(\boldsymbol{x}, \boldsymbol{\xi}) = [\mu(\boldsymbol{x}) + \beta(\boldsymbol{x})]\boldsymbol{\xi} - 2\beta(\boldsymbol{x})P_{\widetilde{K}}\frac{1}{2}\boldsymbol{\xi}.$$

Let us assume that $\mu \in L^\infty(\Omega)$, $\mu(\boldsymbol{x}) \geq \mu^* > 0$ a.e. $\boldsymbol{x} \in \Omega$, and $\beta \in L^\infty(\Omega)$. Taking into account the non-expansivity of the projection map $P_{\widetilde{K}}$, Assumption 4.7 is verified.

Example 4.3. *The behavior of the material is described now as follows,*

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{x},t) &= k_0(\operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t)))\mathbb{I}_{\mathbb{S}^3} + \psi(|\boldsymbol{\varepsilon}^D(\boldsymbol{u}(\boldsymbol{x},t))|^2)\boldsymbol{\varepsilon}^D(\boldsymbol{u}(\boldsymbol{x},t))\\ &+ \int_0^t \theta(\boldsymbol{x},t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},s))ds + \int_0^t \zeta(\boldsymbol{x},t-s)tr(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},s)))\mathbb{I}_{\mathbb{S}^3}ds,\end{aligned}$$

where $k_0 > 0$ is a coefficient of the material, $\boldsymbol{\varepsilon}^D(\boldsymbol{u})$ is the deviatoric part of $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\psi : \mathbb{R} \to \mathbb{R}$ is a constitutive function. In the antiplane context the equilibrium equation reduces to

$$\begin{aligned}\operatorname{div}\left(\frac{1}{2}\psi\left(\frac{1}{2}|\nabla u(\boldsymbol{x},t)|^2_{\mathbb{R}^2}\right)\nabla u(\boldsymbol{x},t) + \int_0^t \theta(\boldsymbol{x},t-s)\nabla u(\boldsymbol{x},s)ds\right)\\ +f_0(\boldsymbol{x},t) = 0 \quad \textit{in} \quad \Omega \times (0,T).\end{aligned}$$

Thus, we can consider

$$a(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2}\psi\left(\frac{1}{2}|\boldsymbol{\xi}|^2_{\mathbb{R}^2}\right)\boldsymbol{\xi}.$$

Assume that $\psi : \mathbb{R} \to \mathbb{R}$ is a piecewise continuously differentiable function such that there exist positive constants c_1, c_2, d_1 and d_2 which verify the following inequalities

$$\psi(\xi^2) \leq d_1, \quad -c_1 \leq \psi'(\xi^2) \leq 0, \quad c_2 \leq \psi(\xi^2) + 2\psi'(\xi^2)\xi \leq d_2;$$

see [59], page 125. In this case Assumption 4.7 is fulfilled, too.

In addition to Assumption 4.7, we made the following assumptions.

Assumption 4.8. $\theta \in C([0,T]; L^\infty(\Omega))$.

Assumption 4.9. $f_0 \in C([0,T]; L^2(\Omega))$; $\quad f_2 \in C([0,T]; L^2(\Gamma_2))$.

Assumption 4.10. $g \in L^2(\Gamma_3)$, $\quad g(x) \geq 0$ *a.e. on* Γ_3.

Let us introduce the space $V = \{v \in H^1(\Omega) \quad v = 0 \quad \text{a.e. on } \Gamma_1\}$ ($v = 0$ in the sense of the trace). We define $A : V \to V$ as follows: for each $u \in V$, Au is the unique element of V such that

$$(Au, v)_V = \int_\Omega a(\boldsymbol{x}, \nabla u(\boldsymbol{x}, t)) \cdot \nabla v(\boldsymbol{x})\, dx.$$

Besides, we define a function f as follows,

$$f : [0,T] \to V,\ (f(t), v)_V = \int_\Omega f_0(t) v\, dx + \int_{\Gamma_2} f_2(t) v\, dx, \qquad \text{for all } v \in V.$$

In addition, we define an operator $B : [0,T] \to \mathcal{L}(V)$ such that, for $t \in [0,T]$ and $u \in V$, $B(t)u$ is the element of V which verifies

$$(B(t)u, v)_V = \int_\Omega \theta(t) \nabla u \cdot \nabla v\, dx \qquad \text{for all } v \in V.$$

We note that

$$f \in C([0,T]; V)$$
$$B \in C([0,T]; \mathcal{L}(V)).$$

Let $D = (\gamma(V))'$ be the dual of the space $\gamma(V) = \{w = v|_\Gamma \quad v \in V\}$. For every $t \in [0,T]$ we define $\lambda(t) \in D$ as follows

$$\langle \lambda(t), \gamma w \rangle = -\int_{\Gamma_3} a(\boldsymbol{x}, \nabla u(\boldsymbol{x}, t)) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \gamma w(\boldsymbol{x}) d\Gamma$$
$$-\int_{\Gamma_3} \int_0^t \theta(t-s)(\boldsymbol{x}) \partial_\nu u(\boldsymbol{x}, s) ds \gamma w(\boldsymbol{x}) d\Gamma \quad \text{for all } \gamma w \in \gamma(V),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $(\gamma(V))'$ and $\gamma(V)$. Also, we introduce $\Lambda \subset (\gamma(V))'$ as follows,

$$\Lambda = \{\mu \in D \,|\, \langle \mu, \gamma w \rangle \leq \int_{\Gamma_3} g(\boldsymbol{x}) |\gamma w(\boldsymbol{x})| d\Gamma \quad \text{for all } \gamma w \in \gamma(V)\}. \tag{4.6}$$

Next, we define

$$b : V \times D \to \mathbb{R} \quad b(v, \mu) = \langle \mu, \gamma v \rangle. \tag{4.7}$$

We have the following weak formulation of Problem 4.3.

Problem 4.4. *Given $f:[0,T]\to V$, find $u:[0,T]\to V$ and $\lambda:[0,T]\to D$ such that, for every $t\in[0,T]$, we have $\lambda(t)\in\Lambda$ and*

$$\begin{aligned}(Au(t),v)_V+(\int_0^t B(t-s)u(s)ds,v)_V+b(v,\lambda(t)) &= (f(t),v)_V \quad \text{for all } v\in V\\ b(u(t),\mu-\lambda(t)) &\le 0 \qquad\qquad \text{for all } \mu\in\Lambda.\end{aligned}$$

Theorem 4.2. *[Theorem 1 in [101]] If Assumptions 4.7-4.10 are fulfilled, then Problem 4.4 has a unique solution (u,λ) with the regularity*

$$u\in C([0,T];V),\ \lambda\in C([0,T];D).$$

Moreover, given f^1, $f^2\in C([0,T];V)$, there exists $C>0$ such that

$$\|u^1-u^2\|_{C([0,T];V)}+\|\lambda^1-\lambda^2\|_{C([0,T];D)}\le C\|f^1-f^2\|_{C([0,T];V)}, \tag{4.8}$$

(u^1,λ^1) and (u^2,λ^2) being the solutions of Problem 4.4 corresponding to the data f^1 and f^2, respectively.

The proof of Theorem 4.2 is based on the abstract result, Theorem 4.1.

4.2 The case of viscoelasticity with short-memory term

This section is based on the paper [111]. We discuss herein an abstract mixed variational problem which consists of a system of an evolutionary variational equation in a Hilbert space X and an evolutionary inequality in a subset of a second Hilbert space Y, associated with an initial condition. The existence and the uniqueness of the solution is proved based on a fixed point technique. The continuous dependence on the data was also investigated. The abstract results we obtain can be applied to the mathematical treatment of a class of frictional contact problems for viscoelastic materials with short memory. In this section we consider an antiplane model for which we deliver a mixed variational formulation with friction bound dependent set of Lagrange multipliers. After proving the existence and the uniqueness of the weak solution, we study the continuous dependence on the initial data, on the densities of the volume forces and surface tractions. Moreover, we prove the continuous dependence of the solution on the friction bound.

4.2.1 An abstract result

Let T be a positive real number. In this subsection we study the following abstract problem.

Problem 4.5. *Given $f:[0,T]\to X$, $g\in W$ and $u_0\in X$, find $u:[0,T]\to X$ and $\lambda:[0,T]\to \Lambda(g)\subset Y$ such that for each $t\in(0,T)$, we have*

$$\begin{aligned} a(\dot u(t),v)+e(u(t),v)+b(v,\lambda(t)) &= (f(t),v)_X \quad \text{for all } v\in X, && (4.9)\\ b(\dot u(t),\mu-\lambda(t)) &\le 0 \quad \text{for all } \mu\in\Lambda(g), && (4.10)\\ u(0) &= u_0. && (4.11)\end{aligned}$$

This is *an evolutionary mixed variational problem with short-memory term.*

Problem 4.5 was studied under the following assumptions.

Assumption 4.11. *$(X,(\cdot,\cdot)_X,\|\cdot\|_X)$, $(Y,(\cdot,\cdot)_Y,\|\cdot\|_Y)$ and $(W,(\cdot,\cdot)_W,\|\cdot\|_W)$ are three Hilbert spaces.*

Assumption 4.12. *$a(\cdot,\cdot):X\times X\to\mathbb{R}$ is a symmetric bilinear form such that*
(a_1) *there exists $M_a>0: |a(u,v)|\le M_a\|u\|_X\|v\|_X$ for all $u,v\in X$;*
(a_2) *there exists $m_a>0: a(v,v)\ge m_a\|v\|_X^2$ for all $v\in X$.*

Assumption 4.13. *$e(\cdot,\cdot):X\times X\to\mathbb{R}$ is a symmetric bilinear form such that*
(e_1) *there exists $M_e>0: |e(u,v)|\le M_e\|u\|_X\|v\|_X$ for all $u,v\in X$;*
(e_2) *there exists $m_e>0: e(v,v)\ge m_e\|v\|_X^2$ for all $v\in X$.*

Assumption 4.14. *$b(\cdot,\cdot):X\times Y\to\mathbb{R}$ is a bilinear form such that*
(b_1) *there exists $M_b>0: |b(v,\mu)|\le M_b\|v\|_X\|\mu\|_Y$ for all $v\in X$, $\mu\in Y$;*
(b_2) *there exists $\alpha>0: \inf_{\mu\in Y,\mu\neq 0_Y}\sup_{v\in X, v\neq 0_X}\frac{b(v,\mu)}{\|v\|_X\|\mu\|_Y}\ge\alpha$.*

Assumption 4.15. *$f\in C([0,T];X)$.*

Assumption 4.16. *For each $\varphi\in W$, $\Lambda(\varphi)$ is a closed convex subset of Y such that $0_Y\in\Lambda(\varphi)$.*

Assumption 4.17. *If $(\eta_n)_n\subset W$ and $(w_n)_n\subset X$ are two sequences such that $\eta_n\to\eta$ in W and $w_n\rightharpoonup w$ in X, as $n\to\infty$, then:*
(k_1) *for each $\mu\in\Lambda(\eta)\subset Y$, there exists a sequence $(\mu_n)_n\subset Y$ such that $\mu_n\in\Lambda(\eta_n)$ for all $n\ge 1$, and $\limsup_{n\to\infty} b(w_n,\mu-\mu_n)\le 0$;*
(k_2) *For each subsequence $(\Lambda(\eta_{n'}))_{n'}$ of the sequence $(\Lambda(\eta_n))_n$, if $(\mu_{n'})_{n'}\subset Y$ such that $\mu_{n'}\in\Lambda(\eta_{n'})$ and $\mu_{n'}\rightharpoonup\mu$ in Y as $n'\to\infty$, then $\mu\in\Lambda(\eta)$.*

Let $\eta\in C([0,T];X)$ be given and let us consider the following intermediate problem.

Problem 4.6. *Given $f:[0,T]\to X$ and $g\in W$ find $w_\eta:[0,T]\to X$ and $\lambda_\eta:[0,T]\to\Lambda(g)\subset Y$ so that, for each $t\in[0,T]$, we have*

$$\begin{aligned} a(w_\eta(t),v)+e(\eta(t),v)+b(v,\lambda_\eta(t)) &= (f(t),v)_X \quad \text{for all } v\in X, && (4.12)\\ b(w_\eta(t),\mu-\lambda_\eta(t)) &\le 0 \quad \text{for all } \mu\in\Lambda(g). && (4.13)\end{aligned}$$

The existence and the uniqueness of the solution of this problem is provided by the following lemma.

Lemma 4.1. *[Lemma 2 in [111]] Problem 4.6 has a unique solution with the regularity*

$$w_\eta \in C([0,T];X), \quad \lambda_\eta \in C([0,T];Y). \tag{4.14}$$

Let us associate with Problem 4.6 the following functional.

$$\mathcal{L}^t_\eta : X \times \Lambda(g) \to \mathbb{R}, \quad \mathcal{L}^t_\eta(v,\, \mu) = \frac{1}{2}a(v,v) - (f_\eta(t),v)_X + b(v,\mu). \tag{4.15}$$

It was proved that a pair $(w_\eta(t), \lambda_\eta(t))$ verifies (4.12) and (4.13) if and only if it is a solution of the following saddle point problem.

$$\left.\begin{aligned} &\text{Find } (w_\eta(t), \lambda_\eta(t)) \in X \times \Lambda(g) \text{ so that} \\ &\mathcal{L}^t_\eta(w_\eta(t),\, \mu) \le \mathcal{L}^t_\eta(w_\eta(t),\, \lambda_\eta(t)) \le \mathcal{L}^t_\eta(v,\, \lambda_\eta(t)) \quad \text{for all } v \in X,\ \mu \in \Lambda(g). \end{aligned}\right\} \tag{4.16}$$

Following [50, 61], it was proved that the Problem 4.16 has a solution.

Let us consider the operator $\mathcal{T} : C([0,T];X) \to C([0,T];X)$ defined as follows: for each $\eta \in C([0,T];X)$,

$$\mathcal{T}\eta(t) = \int_0^t w_\eta(s)\,ds + u_0 \quad \text{for all } t \in [0,T]. \tag{4.17}$$

Lemma 4.2. *[Lemma 3 in [111]] The operator $\mathcal{T}$ has a unique fixed point $\eta^* \in C([0,T];\Lambda(g))$.*

The main abstract result in this subsection is the following one.

Theorem 4.3. *[Theorem 4 in [111]] If Assumptions* 4.11-4.17 *hold true, then Problem* 4.5 *has a unique solution with the regularity*

$$u \in C^1([0,T];X), \quad \lambda \in C([0,T];\Lambda(g)).$$

The proof of Theorem 4.3, which can be found in [111], is based on the saddle point theory and Banach's fixed point theorem.

In addition to this theorem it is worth to mention the following three results.

Let us start with the following stability properties.

Proposition 4.1. *[Proposition 5 in [111]] If Assumptions 4.11-4.17 hold true, then:*

(p_1) given $f \in C([0,T];X)$, $g \in W$ and two initial data u_0^1, $u_0^2 \in X$, there exists $c_1 > 0$ such that

$$\|u_1 - u_2\|_{C^1([0,T];X)} \le c_1 \|u_0^1 - u_0^2\|_X, \tag{4.18}$$

where u_1, u_2 are the corresponding solutions of Problem 4.5;

(p_2) given $g \in W$, $f_1, f_2 \in X$ and two initial data u_0^1, $u_0^2 \in X$, there exists $c_2 > 0$ such that

$$\|u_1 - u_2\|_{C^1([0,T];X)} \le c_2(\|f_1 - f_2\|_X + \|u_0^1 - u_0^2\|_X), \tag{4.19}$$

where u_1, u_2 are the corresponding solutions of Problem 4.5.

Next, we mention a boundedness property.

Proposition 4.2. *[Proposition 6 in [111]] If Assumptions* 4.11-4.17 *hold true, then there exist two positive friction bound independent constants K_1 and K_2 such that*

$$\|u\|_{C^1([0,T];X)} \leq K_1; \tag{4.20}$$
$$\|\lambda\|_{C([0,T];Y)} \leq K_2, \tag{4.21}$$

where (u, λ) is the solution of Problem 4.5*.*

Finally, let us indicate some convergence properties.

Proposition 4.3. *[Proposition 7 in [111]] If Assumptions* 4.11-4.17 *hold true and $(g_n)_n \subset W$ is a sequence such that $g_n \to g$ in W as $n \to \infty$, then for all $t \in [0, T]$,*

$$u_n(t) \to u(t) \ \text{in } X \ \text{as } n \to \infty; \tag{4.22}$$
$$\dot{u}_n(t) \to \dot{u}(t) \ \text{in } X \ \text{as } n \to \infty; \tag{4.23}$$
$$\lambda_n(t) \to \lambda(t) \ \text{in } Y \ \text{as } n \to \infty, \tag{4.24}$$

where (u, λ) and (u_n, λ_n) denote the solutions of Problem 4.5 *associated with the data $(f, g, u_0) \in C([0,T]; X) \times W \times X$ and $(f, g_n, u_0) \in C([0,T]; X) \times W \times X$, $n \geq 1$.*

Propositions 4.1, 4.2 and 4.3 have been proved in [111].

4.2.2 A mechanical model and its weak solvability

In this subsection we discuss the weak solvability of the following model.

Problem 4.7. *Find a displacement field $u : \bar{\Omega} \times [0, T] \to \mathbb{R}$ such that, for all $t \in (0, T)$, we have*

$$div\, (\theta(\boldsymbol{x})\, \nabla \dot{u}(\boldsymbol{x}, t) + \mu(\boldsymbol{x})\, \nabla u(\boldsymbol{x}, t)) + f_0(\boldsymbol{x}, t) = 0 \quad \text{in } \Omega, \tag{4.25}$$

$$u(\boldsymbol{x}, t) = 0 \quad \text{on } \Gamma_1, \tag{4.26}$$

$$\theta(\boldsymbol{x})\, \partial_\nu \dot{u}(\boldsymbol{x}, t) + \mu(\boldsymbol{x})\, \partial_\nu u(\boldsymbol{x}, t) = f_2(\boldsymbol{x}, t) \quad \text{on } \Gamma_2, \tag{4.27}$$

$$\left.\begin{aligned} &|\theta(\boldsymbol{x})\, \partial_\nu \dot{u}(\boldsymbol{x}, t) + \mu(\boldsymbol{x})\, \partial_\nu u(\boldsymbol{x}, t)| \leq g(\boldsymbol{x}), \\ &\theta(\boldsymbol{x})\, \partial_\nu \dot{u}(\boldsymbol{x}, t) + \mu(\boldsymbol{x})\, \partial_\nu u(\boldsymbol{x}, t) = -g(\boldsymbol{x})\, \tfrac{\dot{u}(\boldsymbol{x}, t)}{|\dot{u}(\boldsymbol{x}, t)|} \ \text{if } \dot{u}(\boldsymbol{x}, t) \neq 0 \end{aligned}\right\} \quad \text{on } \Gamma_3, \tag{4.28}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{4.29}$$

Herein $[0, T]$ is the time interval and $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz continuous boundary. The boundary will be denoted by Γ and will be partitioned in three measurable parts

Γ_1, Γ_2, Γ_3 such that the Lebesgue measure of Γ_1 is positive. Problem 4.7 models the antiplane shear deformation of a viscoelastic, isotropic, nonhomogeneous cylindrical body in frictional contact on Γ_3 with a rigid foundation. Referring the body to a cartesian coordinate system $Ox_1x_2x_3$ such that the generators of the cylinder are parallel with the axis Ox_3, the domain $\Omega \subset Ox_1x_2$ denotes the cross section of the cylinder. The function $\theta = \theta(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ is the viscoelastic coefficient, $\mu = \mu(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ denotes a coefficient of the material (one of *Lamé's coefficients*), the functions $f_0 = f_0(x_1, x_2, t) : \Omega \times (0, T) \to \mathbb{R}$, $f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times (0, T) \to \mathbb{R}$ are related to the density of the volume forces and the density of the surface traction, respectively and $g : \Gamma_3 \to \mathbb{R}_+$ is the friction bound, a given function. Here $\boldsymbol{\nu} = (\nu_1, \nu_2)$ ($\nu_i = \nu_i(x_1, x_2)$, for each $i \in \{1, 2\}$), represents the outward unit normal vector to the boundary Γ and $\partial_\nu u = \nabla u \cdot \boldsymbol{\nu}$.

The unknown of the problem is the function $u = u(x_1, x_2, t) : \bar{\Omega} \times [0, T] \to \mathbb{R}$ which represents the third component of the displacement vector $\boldsymbol{u}$.

In the study of Problem 4.7 we assume that the elasticity and the viscosity coefficients fulfill the following assumptions.

Assumption 4.18. $\mu \in L^\infty(\Omega)$, *and there exists* $\mu^* > 0$ *such that* $\mu(\boldsymbol{x}) \geq \mu^*$ *a.e.* $\boldsymbol{x} \in \Omega$.

Assumption 4.19. $\theta \in L^\infty(\Omega)$, *and there exists* $\theta^* > 0$ *such that* $\theta(\boldsymbol{x}) \geq \theta^*$ *a.e.* $\boldsymbol{x} \in \Omega$.

Assumption 4.20. $f_0 \in C([0, T]; L^2(\Omega)), \quad f_2 \in C([0, T]; L^2(\Gamma_2))$.

Assumption 4.21. $g \in L^2(\Gamma_3)$ *such that* $g(\boldsymbol{x}) \geq 0$ *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Finally, we made the following assumption for the initial displacement.

Assumption 4.22. $u_0 \in X$.

Let us introduce the Hilbert space

$$X = \{\, v \in H^1(\Omega) \mid \gamma v = 0 \text{ a.e. on } \Gamma_1 \}. \tag{4.30}$$

We define the bilinear forms $a : X \times X \to \mathbb{R}$ and $e : X \times X \to \mathbb{R}$ by equalities

$$a(u, v) = \int_\Omega \theta \nabla u \cdot \nabla v \, dx \quad \text{for all } u,\, v \in V; \tag{4.31}$$

$$e(u, v) = \int_\Omega \mu \nabla u \cdot \nabla v \, dx \quad \text{for all } u,\, v \in V. \tag{4.32}$$

Let $t \in [0, T]$. We define $f(t) \in X$ as follows

$$(f(t), v)_X = \int_\Omega f_0(t)\, v \, dx + \int_{\Gamma_2} f_2(t)\, \gamma v \, d\Gamma \quad \text{for all } v \in X. \tag{4.33}$$

We consider the space

$$S = \{\widetilde{v} = \gamma v|_{\Gamma_3} \quad v \in X\} \tag{4.34}$$

and we denote its dual by D. Also, we define a bilinear form $b : V \times D \to \mathbb{R}$ as follows

$$b(v,\, \zeta) = \langle \zeta, \gamma v|_{\Gamma_3} \rangle, \tag{4.35}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the spaces D and S.

Next, for each $\varphi \in L^2(\Gamma_3)$ we define $\Lambda(\varphi)$ as follows:

$$\Lambda(\varphi) = \Big\{ \zeta \in D \,:\, \langle \zeta,\, \gamma w|_{\Gamma_3} \rangle \leq \int_{\Gamma_3} |\varphi(\boldsymbol{x})|\, |\gamma w(\boldsymbol{x})|\, d\Gamma \quad \text{for all } w \in X \Big\}. \tag{4.36}$$

Let us define now a Lagrange multiplier λ, such that at each $t \in [0, T]$, $\lambda(t) \in Y$ and

$$\langle \lambda(t), z \rangle = -\int_{\Gamma_3} (\theta \partial_\nu \dot{u}(t) + \mu\, \partial_\nu u(t))\, z\, d\Gamma \quad \text{for all } z \in S, \tag{4.37}$$

where S is defined in (4.34).

We delivered the following mixed variational formulation of Problem 4.7.

Problem 4.8. *Find $u : [0, T] \to X$ and $\lambda : [0, T] \to \Lambda(g) \subset D$ such that, for all $t \in (0, T)$,*

$$\begin{array}{rcll} a(\dot{u}(t),\, v) + e(u(t), v) + b(v, \lambda(t)) & = & (f(t), v)_X & \text{for all } v \in X, \\ b(\dot{u}(t),\, \zeta - \lambda(t)) & \leq & 0 & \text{for all } \zeta \in \Lambda(g), \\ u(0) & = & u_0. & \end{array}$$

Theorem 4.4. *[Theorem 8 in [111]] If Assumptions 4.18–4.22 hold true, then Problem 4.8 has a unique solution (u, λ) with the regularity*

$$u \in C^1([0, T]; X), \quad \lambda \in C([0, T]; \Lambda(g)).$$

The proof of Theorem 4.4 is based on the previous abstract result, Theorem 4.3, see [111] for details. In addition, the following propositions hold true.

Proposition 4.4. *[Proposition 9 in [111]] if Assumptions 4.18–4.22 hold true, then:*

(i_1) given $f_0 \in C([0, T]; L^2(\Omega))$, $f_2 \in C([0, T]; L^2(\Gamma_2))$, $g \in L^2(\Gamma_3)$ and two initial data u_0^1, $u_0^2 \in X$, there exists $c_1 > 0$ such that

$$\|u_1 - u_2\|_{C^1([0,T];X)} \leq c_1 \|u_0^1 - u_0^2\|_X \tag{4.38}$$

where u_1, u_2 are the corresponding solutions of Problem 4.8.

(i_2) given f_0^1, $f_0^2 \in L^2(\Omega)$, f_2^1, $f_2^2 \in L^2(\Gamma_2)$, $g \in L^2(\Gamma_3)$ and two initial data u_0^1, $u_0^2 \in X$, there exists $c_2 > 0$ such that

$$\|u_1 - u_2\|_{C^1([0,T];X)} \leq c_2(\|f_0^1 - f_0^2\|_{L^2(\Omega)} + \|f_2^1 - f_2^2\|_{L^2(\Gamma_2)} + \|u_0^1 - u_0^2\|_X) \tag{4.39}$$

where u_1, u_2 are the corresponding solutions of Problem 4.8.

Besides, we have the following boundedness result.

Proposition 4.5. *[Proposition 10 in [111]] If Assumptions* 4.18–4.22 *hold true, then the solution* (u, λ) *of Problem* 4.8 *is bounded.*

Finally, we have the continuous dependence of the weak solution on the friction bound.

Proposition 4.6. *[Proposition 11 in [111]] If Assumptions* 4.18–4.22 *hold true, then if* $(g_n)_n \subset L^2(\Gamma_3)$ *is a sequence of friction bounds such that* $g_n \geq 0$ *a.e. on* Γ_3, *for all* $n \geq 1$, *and* $g_n \to g$ *in* $L^2(\Gamma_3)$ *as* $n \to \infty$, *we have, for all* $t \in [0, T]$,

$$\begin{aligned} u_n(t) &\to u(t) \text{ in } X \text{ as } n \to \infty; \\ \dot{u}_n(t) &\to \dot{u}(t) \text{ in } X \text{ as } n \to \infty; \\ \lambda_n(t) &\to \lambda(t) \text{ in } Y \text{ as } n \to \infty, \end{aligned}$$

where (u, λ) *and* (u_n, λ_n) *are solutions of Problem* 4.8 *associated with the friction bounds* g *and* g_n, *for all* $n \geq 1$.

The proof of these three propositions are based on the previous abstract results, Proposition 4.1, Proposition 4.2 and Proposition 4.3.

Chapter 5

Frictionless contact problems

This chapter is based on the papers [98, 70, 11]. Firstly, we focus on a mechanical model which describes the frictionless unilateral contact between an electro-elastic body and a rigid electrically nonconductive foundation. For this model, a mixed variational formulation is provided. Using elements of the saddle point theory and a fixed point technique, an abstract result is proved. Based on this abstract result, the existence of a unique weak solution of the mechanical problem is established.

Next, we analyze the frictionless unilateral contact between an electro-elastic body and a rigid electrically conductive foundation. On the potential contact zone, we use the Signorini condition with non-zero gap and an electric contact condition with a conductivity depending on the Cauchy vector. We provide a weak variationally consistent formulation and show existence, uniqueness and stability of the solution. Our analysis is based on a fixed point theorem for weakly sequentially continuous maps.

Finally, we consider a mathematical model which describes the frictionless contact between a viscoplastic body and an obstacle, the so-called foundation. The process is quasistatic and the contact is modeled with normal compliance and unilateral constraint. We provide a mixed variational formulation of the model which involves dual Lagrange multipliers, then we prove its unique weak solvability. We also prove an estimate which allows us to deduce the continuous dependence of the weak solution with respect to both the normal compliance function and the penetration bound.

5.1 The case of electro-elastic materials

In this section, based on the papers [98, 70], we discuss the weak solvability via dual Lagrange multipliers of a class of electro-elastic contact models for linearly elastic materials.

5.1.1 The case of nonconductive foundation

In this subsection, devoted to the case of nonconductive foundation, we present results obtained in the paper [98]. Let us start with an abstract auxiliary result.

Assumption 5.1. $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ *and* $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ *are two Hilbert spaces.*

Assumption 5.2. $a(\cdot, \cdot) : X \times X \to \mathbb{R}$, *is a non-symmetric form such that*
there exists $M_a > 0 : |a(u, v)| \leq M_a \|u\|_X \|v\|_X$ *for all* $u, v \in X$;
there exists $m_a > 0$ *such that* $a(v, v) \geq m_a \|v\|_X^2$ *for all* $v \in X$.

Assumption 5.3. $b(\cdot, \cdot) : X \times Y \to \mathbb{R}$ *is a bilinear form such that*
there exists $M_b > 0 : |b(v, \mu)| \leq M_b \|v\|_X \|\mu\|_Y$, *for all* $v \in X$, $\mu \in Y$,
there exists $\alpha > 0$ *such that* $\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Assumption 5.4. $\Lambda \subset Y$ *is a closed, convex set that contains* 0_Y.

Let us state the following abstract problem.

Problem 5.1. *Given* f, $g \in X$, *find* $u \in X$ *and* $\lambda \in \Lambda$ *such that*

$$
\begin{aligned}
a(u, v) + b(v, \lambda) &= (f, v)_X, && \text{for all } v \in X, \\
b(u, \mu - \lambda) &\leq b(g, \mu - \lambda), && \text{for all } \mu \in \Lambda.
\end{aligned}
$$

We emphasize that the bilinear form $a(\cdot, \cdot)$ is *non-symmetric*. Consequently, Problem 5.1 is *not a classical saddle point problem*. Moreover, we are interested here in the case $g \neq 0_X$. An analysis of the particular case $g = 0_X$ can be found in [68]; see also Subsection 1.2.1 in the present manuscript.

Refereing to Problem 5.1, the following theorem takes place.

Theorem 5.1. *[Theorem 2 in [98]] Let* $f, g \in X$. *If Assumptions* 5.1-5.4 *hold true, then there exists a unique solution of Problem 5.1,* $(u, \lambda) \in X \times \Lambda$. *Moreover, if* (u_1, λ_1) *and* (u_2, λ_2) *are two solutions of Problem 5.1, corresponding to the data* $f_1, g_1 \in X$ *and* $f_2, g_2 \in X$, *then there exists* $K = K(\alpha, m_a, M_a, M_b) > 0$ *such that*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq K(\|f_1 - f_2\|_X + \|g_1 - g_2\|_X).$$

We proceed with the analysis of a mechanical model. We consider an elasto-piezoelectric body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, in contact with a rigid electrically nonconductive foundation. We assume that the boundary Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$ and $\overline{\Gamma}_3$ is a compact subset of $\Gamma \setminus \overline{\Gamma}_1$. Let us denote by $\boldsymbol{n}_3$ the restriction of $\boldsymbol{n}$ to Γ_3. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ act on Γ_2. Moreover, we assume that Γ_3 is the potential contact zone and we denote by $g : \Gamma_3 \to \mathbb{R}$ the *gap function.* By *gap* in a given point of Γ_3 we understand the distance between the deformable body and the foundation measured along of the outward normal $\boldsymbol{n}$. Let us consider a second partition of the boundary Γ in two disjoint measurable parts Γ_a and Γ_b such that $\text{meas}(\Gamma_a) > 0$ and $\Gamma_b \supseteq \Gamma_3$. On Γ_a the electrical potential vanishes and on Γ_b we assume electric charges of density q_2. Since the foundation is electrically nonconductive, and assuming that the gap zone is also electrically nonconductive, q_2 must vanish on Γ_3. By q_0 we will denote the density of the free electric charges on Ω.

Let us write the universal equilibrium equations

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{5.1}$$

$$\text{div}\,\boldsymbol{D} = \boldsymbol{q}_0 \quad \text{in} \quad \Omega, \tag{5.2}$$

and the constitutive law,

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{E}^\top \nabla\varphi \quad \text{in} \quad \Omega, \tag{5.3}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\beta}\nabla\varphi \quad \text{in} \quad \Omega, \tag{5.4}$$

where $\mathcal{C} = (\mathcal{C}_{ijls})$ is the elasticity tensor, $\mathcal{E} = (\mathcal{E}_{ijl})$ is the piezoelectric tensor and $\boldsymbol{\beta}$ is the permittivity tensor.

We prescribe the mechanical and the electrical boundary conditions, according to the physical setting.

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \qquad\qquad \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{5.5}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \qquad\qquad \boldsymbol{D}\cdot\boldsymbol{n} = q_2 \quad \text{on} \quad \Gamma_b. \tag{5.6}$$

To model the contact process, we use the Signorini condition with non-zero gap. In addition, we assume that the contact is frictionless. Consequently, we can express mathematically the frictionless contact condition as follows,

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \qquad \sigma_n \leq 0, \qquad u_n \leq g, \qquad \sigma_n(u_n - g) = 0, \qquad \text{on } \Gamma_3. \tag{5.7}$$

Knowing the displacement field $\boldsymbol{u}$ and the electric field φ we can compute the stress tensor $\boldsymbol{\sigma}$ and the electric displacement $\boldsymbol{D}$ using (5.3) and (5.4), respectively. Therefore, the displacement field $\boldsymbol{u}$ and the electric field φ are called the main unknowns.

To resume, we consider the following problem.

Problem 5.2. *Find the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and the electric potential field $\varphi : \Omega \to \mathbb{R}$ such that* (5.1)-(5.7) *hold.*

Assumption 5.5. $\mathcal{C} = (\mathcal{C}_{ijls}) : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3, \quad \mathcal{C}_{ijls} = \mathcal{C}_{ijsl} = \mathcal{C}_{lsij} \in L^\infty(\Omega);$
there exists $m_\mathcal{C} > 0$ such that $\mathcal{C}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \geq m_\mathcal{C}\,\|\varepsilon\|^2$, for all $\boldsymbol{\varepsilon} \in \mathbb{S}^3$, a.e. on Ω

Assumption 5.6. $\mathcal{E} = (\mathcal{E}_{ijk}) : \Omega \times \mathbb{S}^3 \to \mathbb{R}^3, \quad \mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty(\Omega)$.

Assumption 5.7. $\boldsymbol{\beta} = (\boldsymbol{\beta}_{ij}) : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\beta}_{ij} = \boldsymbol{\beta}_{ji} \in L^\infty(\Omega)$;
there exists $m_\beta > 0$ *such that* $\boldsymbol{\beta}_{ij} E_i E_j \geq m_\beta \|E\|^2$, *for all* $E \in \mathbb{R}^3$, *a.e. on* Ω

Assumption 5.8. $\boldsymbol{f}_0 \in L^2(\Omega)^3, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3, \quad q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b)$.

Assumption 5.9. *There exists* $g_{ext} : \Omega \to \mathbb{R}$ *such that* $g_{ext} \in H^1(\Omega)$, $g_{ext} = 0$ *on* Γ_1, $\quad g_{ext} \geq 0$ *on* $\Gamma \setminus \Gamma_1$, $g = g_{ext}$ *on* Γ_3.

Assumption 5.10. *The unit outward normal to* Γ_3 *denoted by* $\boldsymbol{n}_3$ *is assumed to be constant.*

Based on these assumptions, we present a mixed variational formulation of this mechanical problem, using the Hilbert spaces,

$$\begin{aligned} V &= \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}, \\ \mathcal{W} &= \{\psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\}, \\ \tilde{V} &= V \times \mathcal{W}. \end{aligned}$$

We consider the inner products $(\cdot,\cdot)_V : V \times V \to \mathbb{R}$, $(\cdot,\cdot)_{\mathcal{W}} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ and $(\cdot,\cdot)_{\tilde{V}} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ defined as follows,

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad (\varphi,\psi)_{\mathcal{W}} = (\nabla\varphi, \nabla\psi)_H$$

and

$$(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}})_{\tilde{V}} = (\boldsymbol{u},\boldsymbol{v})_V + (\varphi,\psi)_{\mathcal{W}}. \tag{5.8}$$

Let us consider $a : \tilde{V} \times \tilde{V} \to \mathbb{R}$ the bilinear form,

$$\begin{aligned} a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) &= \int_\Omega \mathcal{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\varphi\, dx \\ &\quad - \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\psi\, dx + \int_\Omega \boldsymbol{\beta} \nabla\varphi \cdot \nabla\psi\, dx \end{aligned} \tag{5.9}$$

for all $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, \varphi)$, $\tilde{\boldsymbol{v}} = (\boldsymbol{v}, \psi) \in \tilde{V}$. Also, we define $\tilde{\boldsymbol{f}} \in \tilde{V}$ such that for all $\tilde{\boldsymbol{v}} = (\boldsymbol{v}, \psi) \in \tilde{V}$,

$$(\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{v}})_{\tilde{V}} = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da - \int_{\Gamma_b} q_2\, \psi\, da + \int_\Omega q_0\, \psi\, dx. \tag{5.10}$$

We define a dual Lagrange multiplier $\boldsymbol{\lambda} \in D = \left(H^{1/2}(\Gamma_3)^3\right)'$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle_{\Gamma_3} = -\int_{\Gamma_3} \sigma_n\, v_n ds, \qquad \text{for all } \boldsymbol{v} \in H^{1/2}(\Gamma_3)^3, \tag{5.11}$$

where $H^{1/2}(\Gamma_3)^3$ denotes the space of restrictions to Γ_3 of the traces of all functions belonging to V and $\langle\cdot,\cdot\rangle_{\Gamma_3}$ denotes the duality pairing between D and $H^{1/2}(\Gamma_3)^3$. Moreover, we define a bilinear form $b : \tilde{V} \times D \to \mathbb{R}$, as follows

$$b(\tilde{\boldsymbol{v}},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v}\rangle_{\Gamma_3}, \qquad \text{for all } \tilde{\boldsymbol{v}} = (\boldsymbol{v}, \psi) \in \tilde{V},\, \boldsymbol{\mu} \in D. \tag{5.12}$$

Furthermore, we introduce a set as follows,

$$\Lambda = \Big\{ \boldsymbol{\mu} \in D \,:\, \langle \boldsymbol{\mu},\, \boldsymbol{v}\rangle_{\Gamma_3} \leq 0 \quad \text{for all } \boldsymbol{v} \in K \Big\}, \tag{5.13}$$

where

$$K = \{\boldsymbol{v} \in H^{1/2}(\Gamma_3)^3 \,:\, v_n \leq 0 \;\text{ on }\; \Gamma_3\}.$$

We have the following weak formulation of Problem 5.2.

Problem 5.3. *Find $\tilde{\boldsymbol{u}} \in \tilde{V}$ and $\boldsymbol{\lambda} \in \Lambda$, such that*

$$\begin{aligned} a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) + b(\tilde{\boldsymbol{v}}, \boldsymbol{\lambda}) &= (\tilde{\boldsymbol{f}},\, \tilde{\boldsymbol{v}})_V, &\qquad &\textit{for all } \tilde{\boldsymbol{v}} \in \tilde{V},\\ b(\tilde{\boldsymbol{u}}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq b(\tilde{\boldsymbol{g}}_{ext}, \boldsymbol{\mu} - \boldsymbol{\lambda}), &\qquad &\textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

Theorem 5.2. *[Theorem 1 in [98]] If Assumptions* 5.5-5.10 *hold true, then Problem 5.3 has a unique solution $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{V} \times \Lambda$. Moreover, if $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda})$ and $(\tilde{\boldsymbol{u}}^*, \boldsymbol{\lambda}^*)$ are two solutions of Problem 5.3 corresponding to the data $(\tilde{\boldsymbol{f}},\, \tilde{\boldsymbol{g}}_{ext}) \in \tilde{V} \times \tilde{V}$ and $(\tilde{\boldsymbol{f}}^*,\, \tilde{\boldsymbol{g}}^*_{ext}) \in \tilde{V} \times \tilde{V}$, respectively, then*

$$\|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}^*\|_{\tilde{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_D \leq C\Big(\|\tilde{\boldsymbol{f}} - \tilde{\boldsymbol{f}}^*\|_{\tilde{V}} + \|\tilde{\boldsymbol{g}}_{ext} - \tilde{\boldsymbol{g}}^*_{ext}\|_{\tilde{V}}\Big),$$

where $C = C(\mathcal{C},\, \mathcal{E}, \boldsymbol{\beta}, \alpha, M_b) > 0$.

The proof of Theorem 5.2 can be found in [98].

5.1.2 The case of conductive foundation

This subsection, based on the paper [70] is dedicated to the analysis of a new contact model involving piezoelectric materials. We consider an elasto-piezoelectric body which occupies the bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ with smooth enough boundary Γ. Also, we consider two partitions of the boundary Γ. The first one is Γ_1, Γ_2 and Γ_3 such that meas$(\Gamma_1) > 0$. The second one is Γ_a, Γ_b and Γ_3 such that meas$(\Gamma_a) > 0$. The partition Γ_1, Γ_2 and Γ_3 applies to the mechanical boundary conditions whereas the partition Γ_a, Γ_b and Γ_3 to the electrical boundary conditions. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω, and a surface traction of density $\boldsymbol{f}_2$ acts on Γ_2.

Moreover, we assume that on Γ_3 the body can be in contact with a rigid electrically conductive foundation. We denote the gap by g. On Γ_a the electrical potential vanishes, and on Γ_b we assume electric charges of density q_b. By q_0 we denote the density of the free electric charges on Ω.

The model under consideration is obtained from the equilibrium equations

$$\mathrm{Div}\ \boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0} \quad \text{in} \quad \Omega, \tag{5.14}$$

$$\mathrm{div}\ \boldsymbol{D} = q_0 \quad \text{in} \quad \Omega, \tag{5.15}$$

the constitutive laws

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{E}^\top \nabla\varphi \quad \text{in} \quad \Omega, \tag{5.16}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\beta}\nabla\varphi \quad \text{in} \quad \Omega, \tag{5.17}$$

the mechanical and the electrical boundary conditions

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \qquad \boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{5.18}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \qquad \boldsymbol{D}\cdot\boldsymbol{\nu} = q_b \quad \text{on} \quad \Gamma_b, \tag{5.19}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0}, \qquad \sigma_\nu \le 0, \qquad u_\nu \le g, \qquad \sigma_\nu(u_\nu - g) = 0 \qquad \text{on } \Gamma_3, \tag{5.20}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = -k(\sigma_\nu)(\varphi - \varphi_0), \tag{5.21}$$

where $g : \Gamma_3 \to \mathbb{R}_+$ is the gap function and $-k(\sigma_\nu) \ge 0$ is the conductivity.

The electric contact condition on Γ_3 is described by a nonlinear Robin type condition for φ which couples the mechanical stress with the electrical field.

The primary variables are the displacement field $\boldsymbol{u}$ and the electric field φ; the stress tensor $\boldsymbol{\sigma}$ and the electric displacement $\boldsymbol{D}$ can be computed from $\boldsymbol{u}$ and φ by the constitutive relations (5.16) and (5.17).

To resume, we consider the following problem.

Problem 5.4. *Find the displacement field* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^d$ *and the electric potential field* $\varphi : \bar{\Omega} \to \mathbb{R}$ *such that* (5.14)-(5.21) *hold.*

Assumption 5.11. *(Elasticity tensor)*

- $\mathcal{C} = (\mathcal{C}_{ijls}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$,
- $\mathcal{C}_{ijls} = \mathcal{C}_{ijsl} = \mathcal{C}_{lsij} \in L^\infty(\Omega)$,
- *There exists* $m_{\mathcal{C}} > 0$ *such that* $\mathcal{C}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \ge m_{\mathcal{C}}\,\|\boldsymbol{\varepsilon}\|^2$, $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, *a.e. in* Ω.

Assumption 5.12. *(Piezoelectric tensor)*

- $\mathcal{E} = (\mathcal{E}_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$,
- $\mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty(\Omega)$.

Assumption 5.13. *(Permittivity tensor)*

- $\boldsymbol{\beta} = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \quad \beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$,
- *There exists* $m_\beta > 0$ *such that* $\beta_{ij} E_i E_j \geq m_\beta \|E\|^2$, $E \in \mathbb{R}^d$, *a.e. in* Ω.

Concerning the mechanical and the electrical data we assume

Assumption 5.14. $\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d, \quad q_0 \in L^2(\Omega), \quad q_b \in L^2(\Gamma_b)$.

To simplify the presentation, we assume that $\varphi_0 = 0$ and $g = 0$. The general situation can be transferred to this case by a transformation $\boldsymbol{u} - \boldsymbol{g}_{ext} \hookrightarrow \boldsymbol{u}$ and $\varphi - \varphi_{ext} \hookrightarrow \varphi$, where $\boldsymbol{g}_{ext} \in \boldsymbol{H}^1(\Omega)$ and $\varphi_{ext} \in H^1(\Omega)$ are extensions of the data $g\boldsymbol{\nu}$ and φ_0.

We are interested in a variationally consistent formulation of this mechanical problem using Lagrange multipliers.

The set of admissible functions for the displacement field is

$$X = \{\, \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}.$$

For the electric potential, we have the admissible set

$$Y = \{\psi \in H^1(\Omega) \mid \gamma\psi = 0 \text{ a.e. on } \Gamma_a\}.$$

The restriction of $\boldsymbol{\nu}$ to Γ_3 is denoted by $\boldsymbol{\nu}_3$, and we restrict ourselves to geometries such that the following assumption is satisfied.

Assumption 5.15. *The vector* $\boldsymbol{\nu}_3$ *is constant on* Γ_3.

Let us introduce the space

$$S = \{w \,|\, w = v_\nu = \boldsymbol{\gamma v}_{|\Gamma_3} \cdot \boldsymbol{\nu}_3, \quad \boldsymbol{v} \in X\}.$$

The dual space of S is denoted by Z and $\langle \cdot, \cdot \rangle$ stands for the duality paring between Z and S.

Let us define the bilinear forms,

$$a : X \times X \to \mathbb{R} \quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad \boldsymbol{u}, \boldsymbol{v} \in X, \tag{5.22}$$

$$e : X \times Y \to \mathbb{R} \quad e(\boldsymbol{v}, \psi) = \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\psi\, dx, \quad \boldsymbol{v} \in X, \psi \in Y, \tag{5.23}$$

$$c : Y \times Y \to \mathbb{R} \quad c(\varphi, \psi) = \int_\Omega \boldsymbol{\beta}\, \nabla\varphi \cdot \nabla\psi\, dx, \quad \varphi, \psi \in Y. \tag{5.24}$$

Moreover, we define $\boldsymbol{f} \in X$ and $q \in Y$ such that

$$(\boldsymbol{f}, \boldsymbol{v})_X = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, d\Gamma, \quad \boldsymbol{v} \in X, \qquad (q, \psi)_Y = \int_\Omega q_0\, \psi\, dx - \int_{\Gamma_b} q_b\, \gamma\psi\, d\Gamma, \quad \psi \in Y.$$

In order to obtain a well defined formulation, the conductivity operator has to have suitable properties. Moreover, the proof of possible existence and uniqueness results depends crucially on the properties of the conductivity operator.

We recall that $f : E_C \to E_C$ is called a weakly sequentially continuous map if, for all sequences $(x_n)_n \subset E_C$, such that $x_n \rightharpoonup x$ in E then $f(x_n) \rightharpoonup f(x)$ in E. Let X be a real reflexive Banach space, then an operator $A : X \to X'$ is called *completely continuous* if, for all sequences $(u_n)_n \subset X$ such that $u_n \rightharpoonup u$ in X then $Au_n \to Au$ in X'. We also recall the following embedding results.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$, $d \geq 1$.*

(i) If $1 < p < d$ then for $1 \leq q < \frac{(d-1)p}{d-p}$, the operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is completely continuous.

(ii) If $p \geq d$ then for any $q \in [1, \infty)$, the operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is completely continuous.

For a proof of this result, we refer the reader to [87].

We set $K(\mu) = -k(-\mu)$ and require the following assumption to be true.

Assumption 5.16. *(Conductivity operator I)*

- *$K : Z \to L^{d-1+\epsilon}(\Gamma_3)$, for $\epsilon > 0$ fixed;*
- *For each $\mu \in Z$, $K(\mu) \geq 0$;*
- *If $(\mu_n)_n \subset Z$ and $\mu \in Z$ such that $\mu_n \rightharpoonup \mu$ in Z as $n \to \infty$, then $K(\mu_n) \rightharpoonup K(\mu)$ in $L^{d-1+\epsilon}(\Gamma_3)$ as $n \to \infty$.*

Example 5.1. *Let $\mathcal{R} : Z \to L^{d-1+\epsilon}(\Gamma_3)$ be a linear continuous map and $k^* > 0$. Then, we set*

$$K(\mu) = k^* |\mathcal{R}\mu|. \tag{5.25}$$

Example 5.2. *Let $\varphi_{\boldsymbol{x}}(\boldsymbol{y}) = \varphi(\boldsymbol{y} - \boldsymbol{x})$ with φ being a mollifier such that $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi(\boldsymbol{x}) = \varphi(-\boldsymbol{x})$, $\varphi \geq 0$, $\int_{\mathbb{R}^d} \varphi(\boldsymbol{x})\, dx = 1$, $\operatorname{supp}\varphi$ is compact. Then, we define*

$$K(\mu)(\boldsymbol{x}) = \frac{k^* |\langle \mu, \varphi_{\boldsymbol{x}} \rangle|}{1 + \gamma |\langle \mu, \varphi_{\boldsymbol{x}} \rangle|}, \quad k^* > 0, \gamma \geq 0. \tag{5.26}$$

Now we define a functional $j : Z \times Y \times Y \to \mathbb{R}$ and a bilinear form $b : X \times Z \to \mathbb{R}$ by

$$j(\mu, \varphi, \psi) = \int_{\Gamma_3} K(\mu)\, \gamma\varphi\, \gamma\psi\, d\Gamma \quad \mu \in Z, \varphi, \psi \in Y, \qquad b(\boldsymbol{v}, \mu) = \langle \mu, v_\nu \rangle, \quad \boldsymbol{v} \in X, \mu \in Z \tag{5.27}$$

and note that both are well-defined under our assumptions. Introducing the dual cone Λ

$$\Lambda = \Big\{ \mu \in Z \;:\; \langle \mu, v \rangle \geq 0 \quad v \in S, v \geq 0 \} \tag{5.28}$$

and the Lagrange multiplier $\lambda = -\sigma_\nu|_{\Gamma_3}$, the weak formulation of Problem 5.4 is the following one.

Problem 5.5. *Find* $(\boldsymbol{u}, \varphi, \lambda) \in X \times Y \times \Lambda$ *such that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + e(\boldsymbol{v},\varphi) + b(\boldsymbol{v},\lambda) &= (\boldsymbol{f},\boldsymbol{v})_X, && \boldsymbol{v} \in X, \\ c(\varphi,\psi) - e(\boldsymbol{u},\psi) + j(\lambda,\varphi,\psi) &= (q,\psi)_Y, && \psi \in Y, \\ b(\boldsymbol{u},\mu-\lambda) &\leq 0, && \mu \in \Lambda. \end{aligned}$$

Notice that the spaces $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ and $(Z, (\cdot, \cdot)_Z, \|\cdot\|_Z)$ are Hilbert spaces and Λ is a closed, convex cone. The form $a(\cdot, \cdot) : X \times X \to \mathbb{R}$ is a symmetric bilinear form such that

- (a_1) there exists $M_a > 0 : |a(\boldsymbol{u},\boldsymbol{v})| \leq M_a \|\boldsymbol{u}\|_X \|\boldsymbol{v}\|_X, \quad \boldsymbol{u},\boldsymbol{v} \in X;$
- (a_2) $a(\boldsymbol{v},\boldsymbol{v}) \geq m_{\mathcal{C}} \|\boldsymbol{v}\|_X^2, \quad \boldsymbol{v} \in X.$

We can take $M_a = d\,\|\mathcal{C}\|_\infty$ where $\|\mathcal{C}\|_\infty = \max_{i,j,k,l} \|\mathcal{C}_{ijkl}\|_{L^\infty(\Omega)}$.

Also, it is worth to mention that $c(\cdot, \cdot) : Y \times Y \to \mathbb{R}$ is a symmetric bilinear form such that

- (c_1) there exists $M_c > 0 : |c(\boldsymbol{u},\boldsymbol{v})| \leq M_c \|\boldsymbol{u}\|_Y \|\boldsymbol{v}\|_Y, \quad \boldsymbol{u},\boldsymbol{v} \in Y;$
- (c_2) $c(\boldsymbol{v},\boldsymbol{v}) \geq m_\beta \|\boldsymbol{v}\|_Y^2, \quad \boldsymbol{v} \in Y.$

We can take $M_c = d\,\|\beta\|_\infty$ where $\|\beta\|_\infty = \max_{i,j} \|\beta_{ij}\|_{L^\infty(\Omega)}$.

Moreover, the form $b : X \times Z \to \mathbb{R}$ is a bilinear form such that

- (b_1) there exists $M_b > 0 : |b(\boldsymbol{v},\, \mu)| \leq M_b \|\boldsymbol{v}\|_X \|\mu\|_Z \quad \boldsymbol{v} \in X, \mu \in Z;$
- (b_2) there exists $\alpha > 0 : \displaystyle\inf_{\mu \in Z, \mu \neq 0} \sup_{\boldsymbol{v} \in X, \boldsymbol{v} \neq 0} \frac{b(\boldsymbol{v},\mu)}{\|\boldsymbol{v}\|_X \|\mu\|_Z} \geq \alpha.$

Next, $e : X \times Y \to \mathbb{R}$ is a bilinear form and there exists $M_e > 0$ such that $|e(\boldsymbol{v},\, \varphi)| \leq M_e \|\boldsymbol{v}\|_X \|\varphi\|_Y$ for all $\boldsymbol{v} \in X, \varphi \in Y$. We can take $M_e = d\,\|\mathcal{E}\|_\infty$ where $\|\mathcal{E}\|_\infty = \max_{i,j,k} \|\mathcal{E}_{ijk}\|_{L^\infty(\Omega)}$. The functional $j(\cdot, \cdot, \cdot)$ verifies the following properties:

- (j_1) for each fixed $\zeta \in Z$, $j(\zeta, \cdot, \cdot)$ is a continuous bilinear form on $Y \times Y$;
- (j_2) $j(\mu, \psi, \psi) \geq 0 \quad \mu \in Z, \psi \in Y;$
- (j_3) if $(\zeta_n)_n \subset Z, \zeta_n \rightharpoonup \zeta$ in Z as $n \to \infty$ and $(\varphi_n)_n \subset Y$, $\varphi_n \rightharpoonup \varphi$, in Y as $n \to \infty$ then: $j(\zeta_n, \varphi_n, \psi) \to j(\zeta, \varphi, \psi)$ as $n \to \infty$ for all $\psi \in Y$, and $j(\zeta_n, \varphi_n, \varphi_n) \to j(\zeta, \varphi, \varphi)$ as $n \to \infty$.

Theorem 5.3. *(An existence result)[Theorem 3.1 in [70]] If Assumptions* 5.11-5.16 *hold true, then Problem* 5.5 *has at least one solution,* $(\boldsymbol{u}, \varphi, \lambda) \in X \times Y \times \Lambda$.

The proof of Theorem 5.3 was given in [70]. The key of the proof is the following fixed point result.

Lemma 5.2. *Let E be a metrizable locally convex topological vector space and let E_C be a weakly compact convex subset of E. Then, any weakly sequentially continuous map $f : E_C \to E_C$ has a fix point.*

For the proof of Lemma 5.2, we refer to [5].

To obtain uniqueness of a solution we have to make one more assumption on the conductivity.

Assumption 5.17. *(Conductivity operator II)*

$$\|K(\mu_1) - K(\mu_2)\|_{L^{d-1+\epsilon}(\Gamma_3)} \leq L_K \|\mu_1 - \mu_2\|_Z, \quad \mu_1, \mu_2 \in Z \text{ and } L_K < \infty \text{ fixed.}$$

Taking into consideration Assumption 5.17, in addition to (j_1)-(j_3), the functional j has the following property: for each pair $(\varphi, \psi) \in Y \times Y$, there exists $L > 0$ such that

$$|j(\zeta_1, \varphi, \psi) - j(\zeta_2, \varphi, \psi)| \leq L \|\zeta_1 - \zeta_2\|_Z \|\varphi\|_Y \|\psi\|_Y, \quad \zeta_1, \zeta_2 \in Z, \tag{5.29}$$

where $L = c_0^2 L_K$ and $c_0 > 0$ is the continuity constant of the trace operator γ associated with Lemma 5.1 and $p = 2$, $q = 2(d-1+\epsilon)/(d-2+\epsilon)$.

Theorem 5.4. *(A uniqueness result)[Theorem 3.6 in [70]] Let Assumptions* 5.11-5.16 *and Assumption 5.17 be true. Additionally, we assume that*

$$m_{\mathcal{C}} - \frac{c_0^4 L_K^2 \mathcal{B} M_a^2}{m_\beta \alpha^2} > 0 \text{ and } \frac{m_\beta}{2} - \frac{c_0^4 L_K^2 \mathcal{B} M_e^2}{m_\beta \alpha^2} > 0. \tag{5.30}$$

Then Problem 5.5 has a unique solution.

The third result is the following one.

Theorem 5.5. *(A stability result)[Theorem 3.7 in [70]] If Assumptions* 5.11-5.17 *and the hypothesis* (5.30) *hold true, then there exists $\mathcal{S} > 0$ such that*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_X + \|\varphi_1 - \varphi_2\|_Y + \|\lambda_1 - \lambda_2\|_Z \leq \mathcal{S}(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_X + \|q_1 - q_2\|_Y),$$

where $(\boldsymbol{u}_1, \varphi_1, \lambda_1)$ and $(\boldsymbol{u}_2, \varphi_2, \lambda_2)$ are the solutions of Problem 5.5 corresponding to the data $(\boldsymbol{f}_1, q_1) \in X \times Y$ and $(\boldsymbol{f}_2, q_2) \in X \times Y$, respectively.

The proofs of Theorems 5.4 and 5.5 can be found in [70]. See also [70] for a numerical example.

5.2 The case of viscoplastic materials

This section is based on the paper [11]. Here, we consider a frictionless contact problem with normal compliance and unilateral constraint and we investigate the behavior of the weak solution with respect to the normal compliance function and the penetration bound. After the description of the contact problem, we derive a new variational formulation which involves a dual Lagrange multiplier. Then we provide the unique weak solvability of the problem, which represents the first trait of novelty. The second trait of novelty consists in the fact that we prove the continuous dependence of the weak solution with respect to the normal compliance function and the penetration bound.

5.2.1 The model and its weak solvability

We consider a viscoplastic body that occupies the bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, with the boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. We assume that the boundary Γ is Lipschitz continuous and we denote by $\boldsymbol{\nu}$ its unit outward normal, defined almost everywhere. Let $T > 0$ and let $[0, T]$ be the time interval. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$, surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$ and, finally, we assume that the body is in contact with a deformable foundation on $\Gamma_3 \times (0, T)$. The contact is frictionless and we model it with a normal compliance and unilateral constraint condition.

Then, the classical formulation of the contact problem is the following.

Problem 5.6. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \times (0, T), \tag{5.31}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0, T), \tag{5.32}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{5.33}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{5.34}$$

$$\left.\begin{array}{l} u_\nu \le g, \ \ \sigma_\nu + p(u_\nu) \le 0, \\ (u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3 \times (0, T), \tag{5.35}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_3 \times (0, T), \tag{5.36}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{5.37}$$

Equation (5.31) represents the viscoplastic constitutive law of the material. Equation (5.32) is the equilibrium equation and we use it here since the process is assumed to be quasistatic. Conditions (5.33) and (5.34) are the displacement and traction boundary conditions, respectively, and condition (5.35) represents the normal compliance condition with unilateral constraint, introduced in [78]. Recall that here $g \geq 0$ is a given bound for the penetration and p represents a given normal compliance function. Condition (5.36) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (5.37) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

In the study of the mechanical problem (5.31)–(5.37) we made the following assumptions.

Assumption 5.18. $\mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$;
$\mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \leq i, j, k, l \leq d$;
There exists $m_\mathcal{E} > 0$ such that $\mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_\mathcal{E}\|\boldsymbol{\tau}\|^2$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. in Ω.

Assumption 5.19. $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$;
There exists $L_\mathcal{G} > 0$ such that $\|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \leq L_\mathcal{G}\left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right)$ *for all* $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, *a.e.* $\boldsymbol{x} \in \Omega$.
The mapping $\boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ is measurable on Ω, *for all* $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$.
The mapping $\boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0})$ belongs to Q.

Assumption 5.20. $p : \mathbb{R} \to \mathbb{R}_+$ *such that:*
there exists $L_p > 0$ $|p(r_1) - p(r_2)| \leq L_p|r_1 - r_2|$ *for all* $r_1,\ r_2 \in \mathbb{R}$;
$(p(r_1) - p(r_2))(r_1 - r_2) \geq 0$ for all $r_1,\ r_2 \in \mathbb{R}$;
$p(r) = 0$ *for all* $r < 0$.

Assumption 5.21. $\boldsymbol{f}_0 \in C([0,T]; L^2(\Omega)^d),\ \boldsymbol{f}_2 \in C([0,T]; L^2(\Gamma_2)^d)$.

Assumption 5.22. $\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q$.

Assumption 5.23. *There exists* $\widetilde{\boldsymbol{\theta}} \in V$ *such that* $\widetilde{\boldsymbol{\theta}} \cdot \boldsymbol{\nu} = 1$ *almost everywhere on* Γ_3.

We consider the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \}.$$

We also consider the Hilbert space

$$S = \{\, \boldsymbol{w} = \boldsymbol{v}_{|\Gamma_3} \qquad \boldsymbol{v} \in V\, \},$$

where $\boldsymbol{v}_{|\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{v} \in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3; \mathbb{R}^d)$ where $H^{1/2}(\Gamma_3; \mathbb{R}^d)$ denotes the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^d$. The dual of the space S will be denoted by D and the duality paring between D and

S will be denoted by $\langle \cdot, \cdot \rangle_{\Gamma_3}$. For more details on trace operators and trace spaces we refer to [1, 92], for instance.

We define the operators $L : V \to V$, $P : V \to V$ and the function $\boldsymbol{f} : [0,T] \to V$ by equalities

$$(L\boldsymbol{u},\, \boldsymbol{v})_V = \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \tag{5.38}$$

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da, \tag{5.39}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \tag{5.40}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$ and $t \in [0,T]$. Also, let $b : V \times D \to \mathbb{R}$ denote the bilinear form defined by

$$b(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \tag{5.41}$$

for all $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in D$ and consider the sets

$$K = \{\, \boldsymbol{v} \in V \;:\; v_\nu \leq 0 \quad \text{a.e. on } \Gamma_3 \,\}, \tag{5.42}$$

$$\Lambda = \{\, \boldsymbol{\mu} \in D \;:\; \langle \boldsymbol{\mu},\, \boldsymbol{v} \rangle_{\Gamma_3} \leq 0 \quad \text{for all } \boldsymbol{v} \in K \,\}. \tag{5.43}$$

Notice that

$$\boldsymbol{f} \in C([0,T]; V). \tag{5.44}$$

Also, it is worth to mention that the bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the "inf-sup" condition, i.e. there exists $\alpha > 0$ such that

$$\inf_{\boldsymbol{\mu} \in D,\ \boldsymbol{\mu} \neq \boldsymbol{0}_D} \ \sup_{\boldsymbol{v} \in V,\ \boldsymbol{v} \neq \boldsymbol{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V \|\boldsymbol{\mu}\|_D} \geq \alpha. \tag{5.45}$$

Denote by $\boldsymbol{\beta}(t)$ and $\boldsymbol{\lambda}(t)$ the viscoplastic stress and the Lagrange multiplier given by

$$\boldsymbol{\beta}(t) = \boldsymbol{\sigma}(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{5.46}$$

$$\langle \boldsymbol{\lambda}(t), \boldsymbol{v} \rangle_{\Gamma_3} = -\int_{\Gamma_3} (\sigma_\nu(t) + p(u_\nu(t))) v_\nu \, da \qquad \text{for all } \boldsymbol{v} \in V. \tag{5.47}$$

The weak formulation of the model is the following one.

Problem 5.7. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$, a viscoplastic stress field $\boldsymbol{\beta} : [0,T] \to Q$ and a Lagrange multiplier $\boldsymbol{\lambda} : [0,T] \to \Lambda$ such that, for all $t \in [0,T]$,*

$$\begin{aligned} &(L\boldsymbol{u}(t), \boldsymbol{v})_V + (\boldsymbol{\beta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}(t), \boldsymbol{v})_V \\ &\qquad + b(\boldsymbol{v}, \boldsymbol{\lambda}(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V, \end{aligned} \tag{5.48}$$

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \text{for all } \boldsymbol{\mu} \in \Lambda, \tag{5.49}$$

$$\boldsymbol{\beta}(t) = \int_0^t \mathcal{G}(\mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \boldsymbol{\beta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{5.50}$$

Let $\boldsymbol{\eta}$ be an arbitrary element of the space $C([0,T];V)$ and consider the following auxiliary problem.

Problem $\mathcal{P}_\eta^1$. *Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ and a Lagrange multiplier $\boldsymbol{\lambda}_\eta : [0,T] \to \Lambda$ such that, for all $t \in [0,T]$,*

$$\begin{aligned}(L\boldsymbol{u}_\eta(t), \boldsymbol{v})_V + (P\boldsymbol{u}_\eta(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}_\eta(t))& \\ = (\boldsymbol{f}(t) - \boldsymbol{\eta}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,&\end{aligned} \tag{5.51}$$

$$b(\boldsymbol{u}_\eta(t), \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t)) \le b(g\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}_\eta(t)) \quad \text{for all } \boldsymbol{\mu} \in \Lambda. \tag{5.52}$$

In the study of Problem $\mathcal{P}_\eta^1$ we have the following result.

Lemma 5.3. *[Lemma 4.1 in [11]] There exists a unique solution $(\boldsymbol{u}_\eta, \boldsymbol{\lambda}_\eta)$ of Problem $\mathcal{P}_\eta^1$ which satisfies*

$$\boldsymbol{u}_\eta \in C([0,T];V), \quad \boldsymbol{\lambda}_\eta \in C([0,T];\Lambda). \tag{5.53}$$

Moreover, if $(\boldsymbol{u}_i, \boldsymbol{\lambda}_i)$ represents the solution of Problem $\mathcal{P}_\eta^1$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_i \in C([0,T];V)$, $i = 1,2$, then there exists $c > 0$ such that

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \|\boldsymbol{\lambda}_1(t) - \boldsymbol{\lambda}_2(t)\|_D \le c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_V \quad \text{for all } t \in [0,T]. \tag{5.54}$$

In the next step we construct the following auxiliary problem for the viscoplastic stress field.

Problem $\mathcal{P}_\eta^2$. *Find a viscoplastic stress field $\boldsymbol{\beta}_\eta : [0,T] \to Q$ such that*

$$\boldsymbol{\beta}_\eta(t) = \int_0^t \mathcal{G}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)) + \boldsymbol{\beta}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{5.55}$$

for all $t \in [0,T]$.

In the study of this problem we have the following result.

Lemma 5.4. *[Lemma 4.2 in [11]] There exists a unique solution of Problem $\mathcal{P}_\eta^2$ which satisfies*

$$\boldsymbol{\beta}_\eta \in C([0,T];Q). \tag{5.56}$$

Moreover, if $\boldsymbol{\beta}_i$ represents the solution of Problem $\mathcal{P}_{\eta i}^2$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_i \in C([0,T];V)$, $i = 1,2$, then there exists $c > 0$ such that

$$\|\boldsymbol{\beta}_1(t) - \boldsymbol{\beta}_2(t)\|_Q \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V\,ds \qquad \text{for all } t \in [0,T]. \tag{5.57}$$

We now introduce the operator $\Theta : C([0,T];V) \to C([0,T];V)$ which maps every element $\boldsymbol{\eta} \in C([0,T];V)$ to the element $\Theta\boldsymbol{\eta} \in C([0,T];V)$ defined as follows: for each $\boldsymbol{\eta} \in C([0,T];V)$ and for each moment $t \in [0,T]$, $\Theta\boldsymbol{\eta}(t)$ is the unique element in V which satisfies the equality

$$(\Theta\boldsymbol{\eta}(t), \boldsymbol{v})_V = (\boldsymbol{\beta}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \text{for all } \boldsymbol{v} \in V; \tag{5.58}$$

here $\boldsymbol{\beta}_\eta$ represents the viscoplastic stress obtained in Lemma 5.4.

We proceed with the following property of the operator Θ.

Lemma 5.5. *The operator Θ has a unique fixed point $\boldsymbol{\eta}^* \in C([0,T];V)$.*

The unique solvability of Problem 5.7 is given by the following result.

Theorem 5.6. *[Theorem 3.1 in [11]] If Assumptions* 5.18-5.23 *hold true, then Problem* 5.7 *has a unique solution $(\boldsymbol{u}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ which satisfies*

$$\boldsymbol{u} \in C([0,T];V), \quad \boldsymbol{\beta} \in C([0,T];Q), \quad \boldsymbol{\lambda} \in C([0,T];\Lambda). \tag{5.59}$$

The proof of Theorem 5.6 was given in [11].

A triple of functions $(\boldsymbol{u}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ which satisfies (5.48)–(5.50) is called a *weak solution* of Problem 5.6. We conclude that, under Assumptions 5.18-5.23, Problem 5.6 has a unique weak solution with regularity (5.59). Moreover, we note that, once the weak solution is know, then the stress field $\boldsymbol{\sigma}$ can be easily computed by using equality (5.46). And, using standard arguments, it can be shown that $\boldsymbol{\sigma} \in C([0,T];Q_1)$.

5.2.2 A convergence result

In this subsection we discuss the behavior of the solution with respect to a perturbation of the normal compliance function p and the bound g. To this end, we assume in what follows that Assumptions 5.18–5.23 hold and we denote by $(\boldsymbol{u}, \boldsymbol{\beta}, \boldsymbol{\lambda})$ the solution of Problem 5.7. Also, for each $\rho > 0$ let $g^\rho \geq 0$ and consider a function p^ρ which satisfies

Assumption 5.24. $p^\rho : \mathbb{R} \to \mathbb{R}_+$ *such that*
There exists $L_p^\rho > 0 : \ |p^\rho(r_1) - p^\rho(r_2)| \leq L_p^\rho |r_1 - r_2| \quad$ for all $r_1, r_2 \in \mathbb{R}$;
$(p^\rho(r_1) - p^\rho(r_2))(r_1 - r_2) \geq 0 \quad$ for all $r_1, r_2 \in \mathbb{R}$.
$p^\rho(r) = 0$ *for all* $r < 0$.

We define the operator $P^\rho : V \to V$ by equality

$$(P^\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p^\rho(u_\nu) v_\nu \, d\Gamma \quad \text{for all } \boldsymbol{u}, \ \boldsymbol{v} \in V. \tag{5.60}$$

Then, we consider the following perturbation of the variational problem $\mathcal{P}_V$.

Problem 5.8. *Find a displacement field* $\boldsymbol{u}^\rho : [0,T] \to V$, *a viscoplastic stress field* $\boldsymbol{\beta}^\rho : [0,T] \to Q$ *and a Lagrange multiplier* $\boldsymbol{\lambda}^\rho : [0,T] \to \Lambda$ *such that, for all* $t \in [0,T]$,

$$\begin{aligned}&(L\boldsymbol{u}^\rho(t), \boldsymbol{v})_V + (\boldsymbol{\beta}^\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P^\rho \boldsymbol{u}^\rho(t), \boldsymbol{v})_V \\ &\qquad + b(\boldsymbol{v}, \boldsymbol{\lambda}^\rho(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,\end{aligned} \tag{5.61}$$

$$b(\boldsymbol{u}^\rho(t), \boldsymbol{\mu} - \boldsymbol{\lambda}^\rho(t)) \le b(g^\rho \widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}^\rho(t)) \quad \text{for all } \boldsymbol{\mu} \in \Lambda, \tag{5.62}$$

$$\boldsymbol{\beta}^\rho(t) = \int_0^t \mathcal{G}(\mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}^\rho(s)) + \boldsymbol{\beta}^\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\rho(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{5.63}$$

It follows from Theorem 5.6 that Problem 5.8 has a unique solution $(\boldsymbol{u}^\rho, \boldsymbol{\beta}^\rho, \boldsymbol{\lambda}^\rho)$ with the regularity expressed in (5.59). Consider now the following assumption on the normal compliance functions p^ρ and p.

Assumption 5.25. *There exists* $G : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $|p^\rho(r) - p(r)| \le G(\rho)(|r| + 1)$ *for all* $r \in \mathbb{R}$ *and* $\rho > 0$.

Then, we have the following estimate, which represents the main result in this subsection.

Theorem 5.7. *[Theorem 5.1 in [11]] If Assumptions* 5.18–5.25 *hold true, then there exists* $c > 0$ *which depends on* Ω, Γ_1, Γ_3, $\mathcal{E}$, $\mathcal{G}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, g, p, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ *and* T, *but does not depend on* ρ, *such that*

$$\begin{aligned}&\|\boldsymbol{u}^\rho - \boldsymbol{u}\|_{C([0,T];V)} + \|\boldsymbol{\beta}^\rho - \boldsymbol{\beta}\|_{C([0,T];Q)} + \|\boldsymbol{\lambda}^\rho - \boldsymbol{\lambda}\|_{C([0,T];D)} \\ &\qquad \le c\,(G(\rho)+1)\big[(G(\rho)+1)|g^\rho - g| + G(\rho)\big].\end{aligned} \tag{5.64}$$

Corollary 5.1. *[Corollary 5.2 in [11]] If Assumptions* 5.18–5.25 *hold true, and moreover, assume that*

$$g^\rho \to g, \quad G(\rho) \to 0 \quad \text{as} \quad \rho \to 0, \tag{5.65}$$

then the solution $(\boldsymbol{u}^\rho, \boldsymbol{\lambda}^\rho, \boldsymbol{\beta}^\rho)$ *of Problem 5.8 converges to the solution* $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\beta})$ *of Problem 5.7, i.e.*

$$\boldsymbol{u}^\rho \to \boldsymbol{u} \text{ in } C([0,T];V), \quad \boldsymbol{\beta}^\rho \to \boldsymbol{\beta} \text{ in } C([0,T];Q), \quad \boldsymbol{\lambda}^\rho \to \boldsymbol{\lambda} \text{ in } C([0,T];D),$$

as $\rho \to 0$.

The proofs of Theorem 5.7 and Corollary 5.1 were given in [11]. In addition to the mathematical interest, the convergence result in Corollary 5.1 is important from the mechanical point of view, since it shows that the weak solution of the viscoplastic contact problem $\mathcal{P}$ depends continuously on both the normal compliance function and the penetration bound.

Remark 5.1. *In [11] it was provided a numerical validation of this convergence result.*

Chapter 6

Contact problems involving multi-contact zones

This chapter is based on the papers [110, 113]. We are interested on the weak solvability of a class of contact models for elastic materials. Every model we propose is mathematically described by a boundary value problem which consists of a system of partial differential equations associated with four boundary conditions (the boundary being partitioned in four parts): a displacement condition, a traction condition and two contact conditions. The weak solvability of the boundary value problems we propose herein relies on new abstract results in the study of some generalized saddle point problems.

6.1 The case of linear elastic operators

This section presents the results we have got in the paper [110]. In this section we firstly prove abstract existence, uniqueness and boundedness results as well as abstract convergence results. Next, we discuss the existence, the uniqueness, the boundedness and the approximation of the weak solutions based on the abstract results.

6.1.1 Abstract results

Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two real Hilbert spaces and $\Lambda \subset Y$. We consider the following problem.

Problem 6.1. *Given $f, h \in X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda$ and*

$$a(u, v-u) + j(v) - j(u) + b(v-u, \lambda) \geq (f,\, v-u)_X \quad \text{for all } v \in X, \tag{6.1}$$

$$b(u, \mu-\lambda) \leq b(h, \mu-\lambda) \quad \text{for all } \mu \in \Lambda. \tag{6.2}$$

Assumption 6.1. *$a(\cdot,\cdot) : X \times X \to \mathbb{R}$ is a symmetric bilinear form such that*

- (i_1) *there exists* $M_a > 0 : \; |a(u,v)| \leq M_a \|u\|_X \|v\|_X$ *for all* $u, v \in X$;
- (i_2) *there exists* $m_a > 0 : \; a(v,v) \geq m_a \, \|v\|_X^2$ *for all* $v \in X$.

Assumption 6.2. *The functional* $j : X \to \mathbb{R}$ *is convex. In addition, there exists* $L_j > 0$ *such that*

$$|j(v) - j(u)| \leq L_j \|v - u\|_X \quad \textit{for all } u, v \in X.$$

Assumption 6.3. $b(\cdot, \cdot) : X \times Y \to \mathbb{R}$ *is a bilinear form such that*

- (j_1) *there exists* $M_b > 0 : \; |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ *for all* $v \in X$, $\mu \in Y$;
- (j_2) *there exists* $\alpha > 0 : \; \inf\limits_{\mu \in Y, \mu \neq 0_Y} \sup\limits_{v \in X, v \neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Assumption 6.4. Λ *is a closed convex subset of* Y *such that* $0_Y \in \Lambda$.

We can associate to Problem 6.1 the following functional:

$$\mathcal{L} : X \times \Lambda \to \mathbb{R}, \;\; \mathcal{L}(v,\mu) = \frac{1}{2} a(v,v) + j(v) + b(v - h, \mu) - (f, v)_X. \tag{6.3}$$

According to the saddle point theory in [50], this functional $\mathcal{L}$ admits at least one saddle point $(u, \lambda) \in X \times \Lambda$.

Theorem 6.1 (An existence and uniqueness result)**.** *[Theorem 2 in [110]] If Assumptions* 6.1-6.4 *hold true, then Problem* 6.1 *has at least one solution, unique in the first argument.*

Proposition 6.1. *[A boundedness result][Proposition 2 in [110]] Assumptions* 6.1-6.4 *hold true. If* $(u, \lambda) \in X \times \Lambda$ *is a solution of Problem 6.1, then there exist* K_1, $K_2 > 0$ *such that*

$$\|u\|_X \leq K_1; \quad \|\lambda\|_Y \leq K_2. \tag{6.4}$$

Setting $h = 0_X$, then Problem 6.1 leads us to the following semi-homogeneous problem.

Problem 6.2. *Given* $f \in X$*, find* $(u, \lambda) \in X \times Y$ *such that* $\lambda \in \Lambda$ *and*

$$\begin{array}{rcll} a(u, v-u) + j(v) - j(u) + b(v-u, \lambda) & \geq & (f, \, v-u)_X & \textit{for all } v \in X, \\ b(u, \mu - \lambda) & \leq & 0 & \textit{for all } \mu \in \Lambda. \end{array}$$

Corollary 6.1. *[Corollary 1 in [110]] If Assumptions* 6.1-6.4 *hold true, then Problem* 6.2 *has at least one solution, unique in the first argument. In addition,*

$$\begin{array}{rcl} \|u\|_X & \leq & \dfrac{1}{m_a}(\|f\|_X + L_j); \\ \|\lambda\|_Y & \leq & \dfrac{1}{\alpha}\left(1 + \dfrac{M_a}{m_a}\right)(\|f\|_X + L_j). \end{array}$$

The proofs of Theorem 6.1, Proposition 6.1 and Corollary 6.1 were delivered in [110].

Let ρ be a real positive number and $j_\rho : X \to \mathbb{R}$ be a functional which fulfills the following assumption.

Assumption 6.5. *The functional $j_\rho : X \to \mathbb{R}$ is convex. In addition,*

- *there exists a positive real number L, which is independent of ρ, such that*

$$|j_\rho(v) - j_\rho(u)| \leq L\|v-u\|_X \text{ for all } u, v \in X.$$

- *j_ρ is a Gâteaux differentiable functional.*

We denote by ∇j_ρ the Gâteaux differential of j_ρ.

Assumption 6.6.

- *There exists $L_{\nabla j_\rho} > 0$ such that*

$$\|\nabla j_\rho(v) - \nabla j_\rho(w)\|_X \leq L_{\nabla j_\rho}\|v-w\|_X \text{ for all } v, w \in X.$$

- *There exists $m_{\nabla j_\rho} > 0$ such that*

$$(\nabla j_\rho(v) - \nabla j_\rho(w), v-w)_X \geq m_{\nabla j_\rho}\|v-w\|_X^2 \text{ for all } v, w \in X.$$

Let us state the following regularized problem.

Problem 6.3. *Given $\rho > 0$ and $f, h \in X$, find $(u_\rho, \lambda_\rho) \in X \times Y$ such that $\lambda_\rho \in \Lambda \subset Y$ and, for all $v \in X$, $\mu \in \Lambda$,*

$$\begin{aligned} a(u_\rho, v-u_\rho) + j_\rho(v) - j_\rho(u_\rho) + b(v-u_\rho, \lambda_\rho) &\geq (f, v-u_\rho)_X && (6.5)\\ b(u_\rho, \mu-\lambda_\rho) &\leq b(h, \mu-\lambda_\rho). && (6.6) \end{aligned}$$

Lemma 6.1. *[Lemma 2 in [110]] A pair $(u_\rho, \lambda_\rho) \in X \times \Lambda$ verifies* (6.5) *if and only if it verifies*

$$a(u_\rho, v) + (\nabla j_\rho(u_\rho), v)_X + b(v, \lambda_\rho) = (f, v)_X \quad \text{for all } v \in X. \tag{6.7}$$

Let us introduce the following notation:

$$\mathcal{M} = \|f\|_X^2 + L^2 + \frac{m_a M_b\|h\|_X(\|f\|_X + L)}{\alpha} + \frac{M_b^2\|h\|_X^2 M_a^2}{\alpha^2}.$$

Proposition 6.2. *[Proposition 3 in [110]] Let $\rho > 0$. If Assumption* 6.1, *Assumptions* 6.3-6.6 *hold true, then Problem* 6.3 *has a unique solution (u_ρ, λ_ρ). Moreover,*

$$\begin{aligned} \|u_\rho\|_X &\leq \frac{2\,\mathcal{M}^{1/2}}{m_a}; \\ \|\lambda_\rho\|_Y &\leq \frac{\|f\|_X + L + \frac{2\,M_a\,\mathcal{M}^{1/2}}{m_a}}{\alpha}. \end{aligned} \tag{6.8}$$

Corollary 6.2. *[Corollary 2 in [110]] Let $(u_\rho, \lambda_\rho)_\rho$ be a sequence of solutions corresponding to a sequence of regularized problems. There exists $(u_{\rho'}, \lambda_{\rho'})_{\rho'}$ a subsequence of the sequence $(u_\rho, \lambda_\rho)_\rho$, and $\tilde{u} \in X$, $\tilde{\lambda} \in \Lambda$ such that,*

$$u_{\rho'} \rightharpoonup \tilde{u} \text{ in } X \text{ as } \rho' \to 0$$

and

$$\lambda_{\rho'} \rightharpoonup \tilde{\lambda} \text{ in } Y \text{ as } \rho' \to 0.$$

Assumption 6.7. *There exists $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

- *$\mathcal{F}(\rho) \to 0$ as $\rho \to 0$;*
- *for each $\rho > 0$, $|j_\rho(v) - j(v)| \leq \mathcal{F}(\rho)$ for all $v \in X$.*

Lemma 6.2. *[Lemma 3 in [110]] Let u_ρ be the first component of the unique pair solution of Problem 6.3. Then*

$$u_\rho \to u \text{ in } X \text{ as } \rho \to 0,$$

where u is the unique first component of a pair solution of Problem 6.1.

Corollary 6.3 (Corollary 3 in [110])**.** *The whole sequence $(u_\rho)_\rho$ converges strongly to $\tilde{u} = u$.*

Lemma 6.3. *[Lemma 4 in [110]] Let $(u_\rho, \lambda_\rho)_\rho$ be a sequence of solutions of a sequence of regularized problems. Then*

$$j(u_\rho) \to j(u) \text{ as } \rho \to 0; \tag{6.9}$$

$$j_\rho(u_\rho) \to j(u) \text{ as } \rho \to 0. \tag{6.10}$$

Proposition 6.3. *[Proposition 4 in [110]] The pair $(\tilde{u}, \tilde{\lambda})$ is a solution of Problem 6.1.*

For the proofs we send to [110].

Remark 6.1. *We can compute the unique first component of a pair solution of Problem 6.1 by computing the strong limit of the sequence $(u_\rho)_\rho$.*

6.1.2 3D contact models

In this subsection we discuss two contact models. The first model is mathematically described as follows.

Problem 6.4. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ *such that*

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{6.11}$$

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{6.12}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{6.13}$$

$$\boldsymbol{\sigma}\,\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{6.14}$$

$$-\sigma_\nu = F,\ \|\boldsymbol{\sigma}_\tau\| \le k|\sigma_\nu|, \boldsymbol{\sigma}_\tau = -k|\sigma_\nu|\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \qquad \text{on } \Gamma_3, \tag{6.15}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0},\ \sigma_\nu \le 0,\ u_\nu - g \le 0,\ \sigma_\nu\,(u_\nu - g) = 0 \qquad \text{on } \Gamma_4, \tag{6.16}$$

where $\mathcal{E}$ is the elastic tensor, $F : \Gamma_3 \to \mathbb{R}_+$ denotes the prescribed normal stress, $k : \Gamma_3 \to \mathbb{R}_+$ denotes the coefficient of friction and $g : \Gamma_4 \to \mathbb{R}_+$ denotes the gap.

Let us make the following assumptions.

Assumption 6.8. $\mathcal{E} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$ *is a fourth order tensor such that:*

- (i_1) $\mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d$;
- (i_2) *there exists* $m_{\mathcal{E}} > 0 : \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2$ *for all* $\boldsymbol{\tau} \in \mathbb{S}^3$, *a.e. in* Ω.

Assumption 6.9. *The density of the volume forces verifies* $\boldsymbol{f}_0 \in L^2(\Omega)^3$ *and the density of traction verifies* $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.

Assumption 6.10. *There exists* $g_{ext} : \Omega \to \mathbb{R}$ *such that* $g_{ext} \in H^1(\Omega)$, $\gamma g_{ext} = 0$ *almost everywhere on* Γ_1, $\gamma g_{ext} \ge 0$ *almost everywhere on* $\Gamma \setminus \Gamma_1$ $g = \gamma g_{ext}$ *almost everywhere on* Γ_4.

Assumption 6.11. *The prescribed normal stress verifies* $F \in L^\infty(\Gamma_3)$ *and* $F(\boldsymbol{x}) \ge 0$ *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Assumption 6.12. *The coefficient of friction verifies* $k \in L^\infty(\Gamma_3)$ *and* $k(\boldsymbol{x}) \ge 0$ *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Assumption 6.13. *The unit outward normal to* Γ_4 *is a constant vector.*

Let us introduce the space

$$V_1 = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \ :\ \boldsymbol{\gamma}\,\boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}. \tag{6.17}$$

We define a bilinear form $a_1 : V_1 \times V_1 \to \mathbb{R}$ such that

$$a_1(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\, dx \qquad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V_1. \tag{6.18}$$

Next, we define $\boldsymbol{f}_1 \in V_1$ such that, for all $\boldsymbol{v} \in V_1$,

$$(\boldsymbol{f}_1, \boldsymbol{v})_{V_1} = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x}) v_\nu(\boldsymbol{x})\, d\Gamma.$$

Besides, we introduce a functional j_1 as follows:

$$j_1 : V_1 \to \mathbb{R}_+ \quad j_1(\boldsymbol{v}) = \int_{\Gamma_3} F(\boldsymbol{x})\, k(\boldsymbol{x})\, \|\boldsymbol{v}_\tau(\boldsymbol{x})\|\, d\Gamma. \tag{6.19}$$

Let D_1 be the dual of the space

$$M_1 = \{\widetilde{v} = v_{\nu|_{\Gamma_4}} \quad \boldsymbol{v} \in V_1\}.$$

We define $\lambda \in D_1$ such that

$$\langle \lambda, w \rangle = - \int_{\Gamma_4} \sigma_\nu(\boldsymbol{x})\, w(\boldsymbol{x})\, d\Gamma \qquad \text{for all } w \in M_1,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D_1 and M_1. Furthermore, we define a bilinear form as follows,

$$b_1 : V_1 \times D_1 \to \mathbb{R}, \quad b_1(\boldsymbol{v}, \mu) = \langle \mu, v_{\nu|_{\Gamma_4}} \rangle \qquad \text{for all } \boldsymbol{v} \in V_1,\ \mu \in D_1. \tag{6.20}$$

Let us introduce the following subset of D_1,

$$\Lambda_1 = \Big\{ \mu \in D_1 \ :\ \langle \mu, v_{\nu|_{\Gamma_4}} \rangle \le 0 \qquad \text{for all } \boldsymbol{v} \in \mathcal{K}_1 \Big\}, \tag{6.21}$$

where

$$\mathcal{K}_1 = \{\boldsymbol{v} \in V_1 \ :\ v_\nu \le 0 \ \text{ almost everywhere on } \ \Gamma_4\}.$$

We are led to the following weak formulation of Problem 6.4.

Problem 6.5. *Find $\boldsymbol{u} \in V_1$ and $\lambda \in \Lambda_1$ such that, for all $\boldsymbol{v} \in V_1$, $\mu \in \Lambda_1$,*

$$\begin{aligned} a_1(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j_1(\boldsymbol{v}) - j_1(\boldsymbol{u}) + b_1(\boldsymbol{v} - \boldsymbol{u}, \lambda) \ &\ge\ (\boldsymbol{f}_1, \boldsymbol{v} - \boldsymbol{u})_{V_1}, \\ b_1(\boldsymbol{u}, \mu - \lambda) \ &\le\ b_1(g_{ext}\boldsymbol{\nu}_4, \mu - \lambda). \end{aligned}$$

A solution of Problem 6.5 is called a *weak solution* to Problem 6.4. The well-posedness of Problem 6.5 is given by the following theorem.

Theorem 6.2 (Theorem 3 in [110])**.** *If Assumptions* 6.8-6.13 *hold true, then Problem* 6.5 *has a bounded solution* $(\boldsymbol{u}, \lambda) \in V_1 \times \Lambda_1$, *unique in its first argument.*

Let $\rho > 0$. We consider the following regularized problem.

Problem 6.6. *Find $\boldsymbol{u}_\rho \in V_1$ and $\lambda_\rho \in \Lambda_1$ such that, for all $\boldsymbol{v} \in V_1$, $\mu \in \Lambda_1$,*

$$\begin{aligned} a_1(\boldsymbol{u}_\rho, \boldsymbol{v} - \boldsymbol{u}_\rho) + j_{1\rho}(\boldsymbol{v}) - j_{1\rho}(\boldsymbol{u}_\rho) + b_1(\boldsymbol{v} - \boldsymbol{u}_\rho, \lambda_\rho) \;&\geq\; (\boldsymbol{f}_1, \boldsymbol{v} - \boldsymbol{u}_\rho)_{V_1}, \\ b_1(\boldsymbol{u}_\rho, \mu - \lambda_\rho) \;&\leq\; b_1(g_{ext}\boldsymbol{\nu}_4, \mu - \lambda_\rho) \end{aligned}$$

where

$$j_{1\rho} : V_1 \to \mathbb{R} \quad j_{1\rho}(\boldsymbol{v}) = \int_{\Gamma_3} F(\boldsymbol{x}) k(\boldsymbol{x}) (\sqrt{\|\boldsymbol{v}_\tau(\boldsymbol{x})\|^2 + \rho^2} - \rho) \, d\Gamma.$$

Notice that, according to [147], the functional $j_{1\rho}$ is Gâteaux differentiable and denoting by $\nabla j_{1\rho}$ its Gâteaux differential we have:

- $\nabla j_{1\rho} : V_1 \to V_1 \quad (\nabla j_{1\rho}(\boldsymbol{w}), \boldsymbol{v})_{V_1} = \displaystyle\int_{\Gamma_3} F(\boldsymbol{x}) k(\boldsymbol{x}) \frac{\boldsymbol{w}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})}{\sqrt{\|\boldsymbol{w}_\tau(\boldsymbol{x})\|^2 + \rho^2}} \, d\Gamma;$
- $\|\nabla j_{1\rho}(\boldsymbol{v}) - \nabla j_{1\rho}(\boldsymbol{w})\|_{V_1} = \displaystyle\sup_{\boldsymbol{z} \in V_1, \boldsymbol{z} \neq 0_{V_1}} \frac{(\nabla j_{1\rho}(\boldsymbol{v}) - \nabla j_{1\rho}(\boldsymbol{w}), \boldsymbol{z})_{V_1}}{\|\boldsymbol{z}\|_{V_1}}$

$$\leq \frac{2\, c_{tr}^2 \|k\, F\|_{L^\infty(\Gamma_3)}}{\rho} \|\boldsymbol{v} - \boldsymbol{w}\|_{V_1} \quad \text{for all } \boldsymbol{w}, \boldsymbol{v} \in V_1,$$

where c_{tr} is a positive constant which fulfills the following inequality

$$\|\boldsymbol{z}_\tau\|_{L^2(\Gamma_3)} \leq c_{tr} \|\boldsymbol{z}\|_{V_1} \quad \text{for all } \boldsymbol{z} \in V_1. \tag{6.22}$$

In addition, for all $\boldsymbol{v} \in V_1$,

$$|j_{1\rho}(\boldsymbol{v}) - j_1(\boldsymbol{v})| \leq \mathcal{F}(\rho), \quad \text{where } \mathcal{F}(\rho) = \rho \int_{\Gamma_3} F(\boldsymbol{x})\, k(\boldsymbol{x}) \, d\Gamma.$$

On the other hand, for all $\boldsymbol{v}$, $\boldsymbol{w} \in V_1$ we have

$$|j_{1\rho}(\boldsymbol{v}) - j_{1\rho}(\boldsymbol{w})| \leq c_{tr} \|F\, k\|_{L^2(\Gamma_3)} \|\boldsymbol{v} - \boldsymbol{w}\|_{V_1}. \tag{6.23}$$

It is worth to emphasize that $(\boldsymbol{u}_\rho, \lambda_\rho) \in V_1 \times \Lambda_1$ is a solution of Problem 6.6 if and only if it verifies

$$(A_{1\rho}\boldsymbol{u}_\rho, \boldsymbol{v})_{V_1} + b_1(\boldsymbol{v}, \lambda_\rho) = (\boldsymbol{f}_1, \boldsymbol{v})_{V_1} \quad \text{for all } \boldsymbol{v} \in V_1, \tag{6.24}$$

$$b_1(\boldsymbol{u}_\rho, \mu - \lambda_\rho) \leq b_1(g_{ext}\boldsymbol{\nu}_4, \mu - \lambda_\rho) \quad \text{for all } \mu \in \Lambda_1, \tag{6.25}$$

where $A_{1\rho} : V_1 \to V_1$, $(A_{1\rho}\boldsymbol{v}, \boldsymbol{w})_{V_1} = a_1(\boldsymbol{v}, \boldsymbol{w}) + (\nabla j_{1\rho}(\boldsymbol{v}), \boldsymbol{w})_{V_1}$.

Remark 6.2. *The first component of a solution of Problem 6.5 (which is unique in the first argument), is the strong limit of the sequence $(\boldsymbol{u}_\rho)_\rho$, $\boldsymbol{u}_\rho$ being the first component of the solution of the problem* (6.24)-(6.25).

For details see [110].

Let us proceed with the second model we are interested on.

Problem 6.7. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ *such that*

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{6.26}$$

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{6.27}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{6.28}$$

$$\boldsymbol{\sigma}\,\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{6.29}$$

$$u_\nu = 0,\ \|\boldsymbol{\sigma}_\tau\| \le \zeta,\ \boldsymbol{\sigma}_\tau = -\zeta\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \qquad \text{if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \qquad \text{on } \Gamma_3, \tag{6.30}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0},\ \sigma_\nu \le 0,\ u_\nu - g \le 0,\ \sigma_\nu\,(u_\nu - g) = 0 \qquad \text{on } \Gamma_4, \tag{6.31}$$

where $\mathcal{E}$ is the elastic tensor, $\zeta : \Gamma_3 \to \mathbb{R}_+$ denotes the friction bound and $g : \Gamma_4 \to \mathbb{R}_+$ denotes the gap. We keep Assumptions 6.8-6.10 and Assumption 6.13. In addition, we made the following assumption.

Assumption 6.14. *The friction bound verifies* $\zeta \in L^\infty(\Gamma_3)$ *and* $\zeta(\boldsymbol{x}) \ge 0$ *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Let us introduce the space

$$V_2 = \Big\{\boldsymbol{v} \in V_1 \mid v_\nu = 0 \text{ a.e. on } \Gamma_3\Big\}$$

which is a closed subspace of the space V_1 defined in (6.17).

We define a bilinear form $a_2 : V_2 \times V_2 \to \mathbb{R}$ such that

$$a_2(\boldsymbol{u},\,\boldsymbol{v}) = \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\,dx \qquad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V_2. \tag{6.32}$$

Next, we define $\boldsymbol{f}_2 \in V_2$ such that,

$$(\boldsymbol{f}_2,\,\boldsymbol{v})_{V_2} = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x})\cdot\boldsymbol{\gamma v}(\boldsymbol{x})\,d\Gamma \qquad \text{for all } \boldsymbol{v} \in V_2.$$

Besides, we introduce a functional j_2 as follows:

$$j_2 : V_2 \to \mathbb{R}_+ \quad j_2(\boldsymbol{v}) = \int_{\Gamma_3} \zeta(\boldsymbol{x})\,\|\boldsymbol{v}_\tau(\boldsymbol{x})\|\,d\Gamma. \tag{6.33}$$

Let D_2 be the dual of the space

$$M_2 = \{\widetilde{v} = v_{\nu|_{\Gamma_4}} \quad \boldsymbol{v} \in V_2\}.$$

We define $\lambda \in D_2$ such that

$$\langle \lambda, w\rangle = -\int_{\Gamma_4} \sigma_\nu(\boldsymbol{x})w(\boldsymbol{x})\,d\Gamma \qquad \text{for all } w \in M_2,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between D_2 and M_2. Furthermore, we define a bilinear form as follows,

$$b_2 : V_2 \times D_2 \to \mathbb{R}, \quad b_2(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, v_{\nu|_{\Gamma_4}}\rangle \quad \text{for all } \boldsymbol{v} \in V_2,\, \mu \in D_2.$$

Let us introduce the following subset of D_2,

$$\Lambda_2 = \Big\{\mu \in D_2 \,:\, \langle \mu,\, w\rangle \leq 0 \quad \text{for all } w \in \mathcal{K}\Big\},$$

where

$$\mathcal{K} = \{w \in M_2 \,:\, w \leq 0 \;\; \text{almost everywhere on} \;\; \Gamma_4\}.$$

Notice that $\lambda \in \Lambda_2$. Moreover,

$$\begin{aligned} b_2(\boldsymbol{u}, \lambda) &= b_2(g_{ext}\boldsymbol{\nu}_4, \lambda) \\ b_2(\boldsymbol{u}, \mu) &\leq b_2(g_{ext}\boldsymbol{\nu}_4, \mu) \text{ for all } \mu \in \Lambda_2. \end{aligned}$$

We have the following weak formulation of Problem 6.7.

Problem 6.8. *Find $\boldsymbol{u} \in V_2$ and $\lambda \in \Lambda_2$ such that, for all $\boldsymbol{v} \in V_2$, $\mu \in \Lambda_2$,*

$$\begin{aligned} a_2(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j_2(\boldsymbol{v}) - j_2(\boldsymbol{u}) + b_2(\boldsymbol{v} - \boldsymbol{u}, \lambda) &\geq (\boldsymbol{f}_2,\, \boldsymbol{v} - \boldsymbol{u})_{V_2}, \\ b_2(\boldsymbol{u}, \mu - \lambda) &\leq b_2(g_{ext}\boldsymbol{\nu}_4, \mu - \lambda). \end{aligned}$$

A solution of Problem 6.8 is called a *weak solution* of Problem 6.7.

The well-posedness of Problem 6.8 is given by the following theorem.

Theorem 6.3. *[Theorem 4 in [110]] Assumptions* 6.8-6.10*, and Assumptions* 6.13-6.14 *hold true. Then, Problem* 6.8 *has a bounded solution $(\boldsymbol{u}, \lambda) \in V_2 \times \Lambda_2$, unique in its first argument.*

Let $\rho > 0$. We consider the following regularized problem.

Problem 6.9. *Let $\rho > 0$. Find $\boldsymbol{u}_\rho \in V_2$ and $\lambda_\rho \in \Lambda_2$ such that for all $\boldsymbol{v} \in V_2$, $\mu \in \Lambda_2$,*

$$\begin{aligned} a_2(\boldsymbol{u}_\rho, \boldsymbol{v} - \boldsymbol{u}_\rho) + j_{2\rho}(\boldsymbol{v}) - j_{2\rho}(\boldsymbol{u}_\rho) + b_2(\boldsymbol{v} - \boldsymbol{u}_\rho, \lambda_\rho) &\geq (\boldsymbol{f}_2,\, \boldsymbol{v} - \boldsymbol{u}_\rho)_{V_2}, \\ b_2(\boldsymbol{u}_\rho, \mu - \lambda_\rho) &\leq b_2(g_{ext}\boldsymbol{\nu}_4, \mu - \lambda_\rho) \end{aligned}$$

where $j_{2\rho} : V_2 \to \mathbb{R}$, $j_{2\rho}(\boldsymbol{v}) = \int_{\Gamma_3} \zeta(\boldsymbol{x})(\sqrt{\|\boldsymbol{v}_\tau(\boldsymbol{x})\|^2 + \rho^2} - \rho)\, d\Gamma$.

Notice that the functional $j_{2\rho}$ is Gâteaux differentiable and denoting by $\nabla j_{2\rho}$ its Gâteaux differential, we have

- $\nabla j_{2\rho} : V_2 \to V_2 \quad (\nabla j_{2\rho}(\boldsymbol{w}), \boldsymbol{v})_{V_2} = \displaystyle\int_{\Gamma_3} \zeta(\boldsymbol{x}) \frac{\boldsymbol{w}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})}{\sqrt{\|\boldsymbol{w}_\tau(\boldsymbol{x})\|^2 + \rho^2}}\, d\Gamma;$

- $\|\nabla j_{2\rho}(\boldsymbol{v}) - \nabla j_{2\rho}(\boldsymbol{w})\|_{V_2} = \sup\limits_{\boldsymbol{z}\in V_2, \boldsymbol{z}\neq O_{V_2}} \frac{(\nabla j_{2\rho}(\boldsymbol{v}) - \nabla j_{2\rho}(\boldsymbol{w}), \boldsymbol{z})_{V_2}}{\|\boldsymbol{z}\|_{V_2}}$

$$\leq \frac{2\, c_{tr}^2 \|\zeta\|_{L^\infty(\Gamma_3)}}{\rho} \|\boldsymbol{v} - \boldsymbol{w}\|_{V_2} \quad \text{for all } \boldsymbol{w}, \boldsymbol{v} \in V_2.$$

Furthermore, for all $\boldsymbol{v} \in V_2$,

$$|j_{2\rho}(\boldsymbol{v}) - j_2(\boldsymbol{v})| \leq \mathcal{F}(\rho) \text{ where } \mathcal{F}(\rho) = \rho \int_{\Gamma_3} \zeta(\boldsymbol{x})\, d\Gamma$$

and, for all $\boldsymbol{v}$, $\boldsymbol{w} \in V_2$,

$$j_{2\rho}(\boldsymbol{v}) - j_{2\rho}(\boldsymbol{w}) = \int_{\Gamma_3} \zeta(\boldsymbol{x}) \frac{(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{w}_\tau\|)(\|\boldsymbol{v}_\tau\| + \|\boldsymbol{w}_\tau\|)}{\sqrt{\|\boldsymbol{v}_\tau\|^2 + \rho^2} + \sqrt{\|\boldsymbol{w}_\tau\|^2 + \rho^2}}\, d\Gamma.$$

Moreover,

$$|j_{2\rho}(\boldsymbol{v}) - j_{2\rho}(\boldsymbol{w})| \leq L_{j_{2\rho}} \|\boldsymbol{v} - \boldsymbol{w}\|_{V_2};\ L_{j_{2\rho}} = c_{tr} \|\zeta\|_{L^2(\Gamma_3)}.$$

It is worth to emphasize that $(\boldsymbol{u}_\rho, \lambda_\rho) \in V_2 \times \Lambda_2$ is a solution of Problem 6.8 if and only if it verifies

$$\begin{aligned} (A_{2\rho}\boldsymbol{u}_\rho, \boldsymbol{v})_{V_2} + b_2(\boldsymbol{v}, \lambda_\rho) &= (\boldsymbol{f}_2, \boldsymbol{v})_{V_2} && \text{for all } \boldsymbol{v} \in V_2, && (6.34) \\ b_2(\boldsymbol{u}_\rho, \mu - \lambda_\rho) &\leq b_2(g_{ext}\boldsymbol{\nu}_4, \mu - \lambda_\rho) && \text{for all } \mu \in \Lambda_2, && (6.35) \end{aligned}$$

where $A_{2\rho} : V_2 \to V_2$, $(A_{2\rho}\boldsymbol{v}, \boldsymbol{w})_{V_2} = a_2(\boldsymbol{v}, \boldsymbol{w}) + (\nabla j_{2\rho}(\boldsymbol{v}), \boldsymbol{w})_{V_2}$.

Remark 6.3. *The unique first component of a pair solution of Problem 6.8 is the strong limit of the sequence $(\boldsymbol{u}_\rho)_\rho$, $\boldsymbol{u}_\rho$ being the first component of the solution of the problem* (6.34)-(6.35).

For details see [110].

6.2 The case of nonlinear elastic operators

This section is based on Section 2 and on a part of Section 4 of the paper [113]. In this section we firstly focus on an abstract problem governed by two convex functionals. Based on a saddle point technique, we deliver existence and uniqueness results. To illustrate the applicability of the abstract results we have got, two contact models are solved.

6.2.1 Abstract results

In this subsection we consider the following mixed variational problem.

Problem 6.10. *Given $f \in X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda \subset Y$ and*

$$\begin{aligned} J(v) - J(u) + b(v - u, \lambda) + \varphi(v) - \varphi(u) &\geq (f, v - u)_X && \textit{for all } v \in X, \\ b(u, \mu - \lambda) &\leq 0 && \textit{for all } \mu \in \Lambda. \end{aligned}$$

We made the following assumptions.

Assumption 6.15. *$(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 6.16. *$J : X \to [0,\infty)$ is a convex lower semicontinuous functional. In addition, there exist m_1, $m_2 > 0$ such that $m_1\|v\|_X^2 \geq J(v) \geq m_2\|v\|_X^2$ for all $v \in X$.*

Assumption 6.17. *$b : X \times Y \to \mathbb{R}$ is a bilinear form such that*

- *there exists $M_b > 0 : |b(v,\mu)| \leq M_b\|v\|_X\|\mu\|_Y$ for all $v \in X$, $\mu \in Y$,*
- *there exists $\alpha > 0 : \inf\limits_{\mu\in Y,\mu\neq 0_Y} \sup\limits_{v\in X, v\neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X\|\mu\|_Y} \geq \alpha$.*

Assumption 6.18. *Λ is a closed convex subset of Y that contains 0_Y.*

Assumption 6.19. *$\varphi : X \to [0,\infty)$ is a convex lower semicontinuous functional. In addition, there exists $q_1 > 0$ such that, for all $v \in X$, $\varphi(v) \leq q_1\|v\|_X$.*

Theorem 6.4. *[An existence result][Theorem 3 in [113]] If Assumptions* 6.15–6.19 *hold true, then Problem* 6.10 *has at least one solution.*

The proof of Theorem 6.4 can be found in [113].

In order to establish the uniqueness of the solution, additional assumptions are necessary.

Assumption 6.20. *$J : X \to [0,\infty)$ is a Gâteaux differentiable functional. In addition:*

- *there exists $m > 0$ such that*
$$(\nabla J(u) - \nabla J(v), u - v)_X \geq m\|u - v\|_X^2 \quad \text{for all } u, v \in X.$$
- *there exists $L > 0$ such that*
$$\|\nabla J(u) - \nabla J(v)\|_X \leq L\|u - v\|_X \quad \text{for all } u, v \in X.$$

Assumption 6.21. *$\varphi : X \to [0,\infty)$ is a Gâteaux differentiable functional.*

Let us define
$$\tilde{J} : X \to [0,\infty) \quad \tilde{J} = J + \varphi. \tag{6.36}$$

We consider the following auxiliary problem.

Problem $\tilde{1}$. *Find $u \in X$ and $\lambda \in \Lambda$ such that*
$$\begin{aligned} (\nabla\tilde{J}(u), v)_X + b(v,\lambda) &= (f,v)_X && \text{for all } v \in X,\\ b(u,\mu-\lambda) &\leq 0 && \text{for all } \mu \in \Lambda. \end{aligned}$$

According to Lemma 2 in [113], the set of the solutions of Problem 6.10 coincides with the set of the solutions of Problem $\tilde{1}$.

Theorem 6.5. *[An uniqueness result][Theorem 4 in [113]] If Assumptions* 6.15–6.21 *hold true, then Problem* 6.10 *has a unique solution.*

For the proof of Theorem 6.5 we send the reader to [113].

6.2.2 3D contact models

To illustrate the applicability of the previous abstract results, two contact models are discussed in this subsection. Each model involves a deformable body which occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary Γ partitioned in four parts. In order to describe the behavior of the material, we use a nonlinear constitutive law expressed by the subdifferential of a proper, convex, lower semicontinuous functional.

Problem 6.11. *[The First Model] Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, such that*

$$Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \qquad in\ \Omega, \tag{6.37}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \qquad in\ \Omega, \tag{6.38}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \qquad on\ \Gamma_1, \tag{6.39}$$

$$\boldsymbol{\sigma\nu}(\boldsymbol{x}) = \boldsymbol{f}_1(\boldsymbol{x}) \qquad on\ \Gamma_2, \tag{6.40}$$

$$\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = \boldsymbol{0},\ u_\nu(\boldsymbol{x}) \leq 0, \sigma_\nu(\boldsymbol{x}) \leq 0, \sigma_\nu(\boldsymbol{x}) u_\nu(\boldsymbol{x}) = 0 \qquad on\ \Gamma_3, \tag{6.41}$$

$$\begin{gathered} -\sigma_\nu(\boldsymbol{x}) = F(\boldsymbol{x}), \\ \|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \leq K(\boldsymbol{x})|\sigma_\nu(\boldsymbol{x})|, \boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -K(\boldsymbol{x})|\sigma_\nu(\boldsymbol{x})| \frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|}\ if\ \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} \qquad on\ \Gamma_4, \end{gathered} \tag{6.42}$$

where Γ_1, Γ_2, Γ_3 and Γ_4 is a partition of Γ such that the Lebesgue measure of Γ_1 is positive. Note that (6.37) is the equilibrium equation, (6.38) is the constitutive law, (6.39) is the displacement boundary condition, and (6.40) is the traction boundary condition. Finally, (6.41) is a frictionless unilateral contact condition with zero gap and (6.42) is a frictional contact condition with prescribed normal stress. The coefficient of friction K as well as the prescribed normal stress F are given functions. Details on the boundary contact conditions we use here can be found for instance in [59, 147].

In order to give a weak formulation we make the following assumptions.

Assumption 6.22. $\boldsymbol{f}_0 \in L^2(\Omega)^3$; $\quad \boldsymbol{f}_1 \in L^2(\Gamma_2)^3$.

Assumption 6.23. $\omega : \mathbb{S}^3 \to [0, \infty)$ *is a convex lower semicontinuous functional. In addition, there exist* α_1, $\alpha_2 > 0$ *such that*

$$\alpha_1 \|\boldsymbol{\varepsilon}\|^2 \geq \omega(\boldsymbol{\varepsilon}) \geq \alpha_2 \|\boldsymbol{\varepsilon}\|^2 \quad for\ all\ \boldsymbol{\varepsilon} \in \mathbb{S}^3.$$

Assumption 6.24. $F \in L^3(\Gamma_4)$, $F(\boldsymbol{x}) \geq 0$ *a.e. on* Γ_4; $K \in L^3(\Gamma_4)$, $K(\boldsymbol{x}) \geq 0$ *a.e. on* Γ_4.

The functional setting is as follows.

$$X = \{\boldsymbol{v} \in H^1(\Omega)^3 \ :\ \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}. \tag{6.43}$$

$$S = \{w = v_{\nu|_{\Gamma_3}} \quad \boldsymbol{v} \in X\}, \tag{6.44}$$

$$Y = S'. \tag{6.45}$$

Next, we define the functional

$$J : X \to [0, \infty), \quad J(\boldsymbol{v}) = \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))dx. \tag{6.46}$$

In addition, using Riesz's representation theorem we define $\boldsymbol{f} \in X$ as follows,

$$(\boldsymbol{f}, \boldsymbol{v})_X = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_1(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x}) v_\nu(\boldsymbol{x})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in X. \tag{6.47}$$

Furthermore, we can introduce the following convex and continuous functional.

$$\varphi : X \to [0, \infty) \quad \varphi(\boldsymbol{v}) = \int_{\Gamma_4} F(\boldsymbol{x})\, K(\boldsymbol{x})\, \|\boldsymbol{v}_\tau(\boldsymbol{x})\|\, d\Gamma. \tag{6.48}$$

We define $\lambda \in Y$ such that

$$\langle \lambda, w \rangle = -\int_{\Gamma_3} \sigma_\nu(\boldsymbol{x})\, w(\boldsymbol{x})\, d\Gamma \quad \text{for all } w \in S, \tag{6.49}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between Y and S.

Moreover, we define a bilinear form as follows,

$$b : X \times Y \to \mathbb{R}, \quad b(\boldsymbol{v}, \mu) = \langle \mu, v_{\nu|\Gamma_3} \rangle \quad \text{for all } \boldsymbol{v} \in X,\ \mu \in Y. \tag{6.50}$$

Let us introduce the following subset of Y,

$$\Lambda = \Big\{ \mu \in Y \ :\ \langle \mu, v_{\nu|\Gamma_3} \rangle \leq 0 \quad \text{for all } \boldsymbol{v} \in \mathcal{K} \Big\}, \tag{6.51}$$

where

$$\mathcal{K} = \{ \boldsymbol{v} \in X \ :\ v_\nu \leq 0 \ \text{ almost everywhere on } \ \Gamma_3 \}.$$

This first contact model is related to Problem 6.10 for *unbounded subset* Λ.

According to Theorem 6.4, Problem 6.11 has at least one weak solution.

Problem 6.12. *[The Second Model] Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, *such that*

$$\begin{aligned}
&Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} && \textit{in } \Omega, && (6.52)\\
&\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) && \textit{in } \Omega, && (6.53)\\
&\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} && \textit{on } \Gamma_1, && (6.54)\\
&\boldsymbol{\sigma\nu}(\boldsymbol{x}) = \boldsymbol{f}_1(\boldsymbol{x}) && \textit{on } \Gamma_2, && (6.55)\\
&u_\nu(\boldsymbol{x}) = 0,\ \|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \leq \zeta(\boldsymbol{x}),\ \boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -\zeta(\boldsymbol{x})\, \tfrac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|}\ \textit{if } \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} && \textit{on } \Gamma_3, && (6.56)\\
&-\sigma_\nu(\boldsymbol{x}) = F(\boldsymbol{x}), && && \\
&\|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \leq K(\boldsymbol{x})|\sigma_\nu(\boldsymbol{x})|, \boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -K(\boldsymbol{x})|\sigma_\nu(\boldsymbol{x})|\tfrac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|}\ \textit{if } \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} && \textit{on } \Gamma_4, && (6.57)
\end{aligned}$$

where Γ_1, Γ_2, Γ_3 and Γ_4 is a partition of Γ such that the Lebesgue measure of Γ_1 is positive, as in the previous example. Recall that (6.52) is the equilibrium equation, (6.53) is the constitutive law, (6.54) is the displacement boundary condition and (6.55) is the traction boundary condition. Herein (6.56) is a bilateral frictional contact condition with friction bound ζ. Finally, (6.57) is a frictional contact condition with prescribed normal stress. The functions ζ, K and F are given functions. For details on the boundary contact conditions written here see for instance [147] and the references therein.

In order to analyze this second example we adopt Assumptions 6.22-6.24. In addition, we make the following assumption.

Assumption 6.25. $\zeta \in L^2(\Gamma_3)$, $\zeta(\boldsymbol{x}) \geq 0$ *a.e. on* Γ_3.

Let us introduce the spaces

$$\begin{aligned}
X &= \Big\{\boldsymbol{v} \in H^1(\Omega)^3 \;:\; \boldsymbol{\gamma}\,\boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1,\; v_\nu = 0 \text{ a.e. on } \Gamma_3\Big\};\\
S &= \{\boldsymbol{z} = \boldsymbol{\gamma w}|_{\Gamma_3} \,|\, \boldsymbol{w} \in X\};\\
Y &= S'.
\end{aligned}$$

We define J, $\boldsymbol{f}$ and φ as in the previous subsection, see (6.46)-(6.48).

Next we introduce the following subset:

$$\Lambda = \Big\{\boldsymbol{\mu} \in Y \;:\; \langle\boldsymbol{\mu},\, \boldsymbol{z}\rangle \leq \int_{\Gamma_3} \zeta(\boldsymbol{x})\|\boldsymbol{z}(\boldsymbol{x})\|\, d\Gamma \quad \text{for all } \boldsymbol{z} \in S\Big\}, \tag{6.58}$$

$\langle\cdot,\cdot\rangle$ being the duality pairing between Y and S.

Let us define $\boldsymbol{\lambda} \in Y$,

$$\langle\boldsymbol{\lambda}, \boldsymbol{z}\rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{z}(\boldsymbol{x})\, d\Gamma \quad \text{for all } \boldsymbol{z} \in S. \tag{6.59}$$

We also define a bilinear form $b(\cdot,\cdot)$,

$$b : X \times Y \to \mathbb{R} \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle\boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3}\rangle. \tag{6.60}$$

This second model is related to Problem 6.10 for *bounded subset* Λ.

According to Theorem 6.4, Problem 6.12 has at least one weak solution. More details can be found in [113].

Chapter 7

Unilateral frictional contact problems

This chapter, based on some results we have got in the paper [113], draws the attention to an abstract mixed variational problem governed by a convex functional and a bifunctional which depends on a Lagrange multiplier in the first argument and is convex in the second argument. After we discuss the existence and the uniqueness of the solution of the abstract problem, we illustrate the applicability of the abstract result to the weakly solvability of a unilateral frictional contact problem.

7.1 Abstract results

In this section, based on Section 3 of the paper [113], we present the results in the study of the following mixed variational problem.

Problem 7.1. *Given $f \in X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda \subset Y$ and*

$$\begin{aligned} J(v) - J(u) + b(v-u,\lambda) + j(\lambda, v) - j(\lambda, u) &\geq (f,\, v-u)_X \qquad \text{for all } v \in X,\\ b(u, \mu - \lambda) &\leq 0 \qquad \text{for all } \mu \in \Lambda. \end{aligned}$$

In order to study Problem 7.1 we adopt Assumptions 6.15-6.18 and 6.20 from the previous chapter. In addition we made the following assumptions.

Assumption 7.1. $j : \Lambda \times X \to [0, \infty)$ *is a bifunctional such that:*

- j_1) *for all $\eta \in \Lambda$, $j(\eta, \cdot) : X \to [0, \infty)$ is a convex Gâteaux differentiable functional;*
- j_2) *for all $\eta \in \Lambda$, there exists $q_1 > 0$ (q_1 independent of η) such that*

$$j(\eta, v) \leq q_1 \|v\|_X \qquad \text{for all } v \in X;$$

- j_3) *for all $\eta \in \Lambda$, $(\nabla_2 j(\eta, u), u)_X \geq 0 \qquad$ for all $u \in X$;*

- j_4) *for all* $\eta \in \Lambda$, $\nabla_2 j(\eta, 0_X) = 0_X$;
- j_5) *for all* $\eta \in \Lambda$, *there exists* $L_j > 0$ *(*L_j *independent of* η*) such that*

$$\|\nabla_2 j(\eta, u) - \nabla_2 j(\eta, v)\|_X \leq L_j \|u - v\|_X \quad \text{for all } u, v \in X.$$

- j_6) *if* $(u_n)_n \subset X$ *and* $(\eta_n)_n \subset Y$ *are two sequences such that* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$ *and* $\eta_n \rightharpoonup \eta$ *in* Y *as* $n \to \infty$, *then* $\limsup_{n\to\infty} j(\eta_n, v) - j(\eta_n, u_n) \leq j(\eta, v) - j(\eta, u)$.

Notice that for each $(u, \eta) \in X \times \Lambda$, $\nabla_2 j(\eta, u)$ denotes the Gâteaux differential of j in u.

Assumption 7.2. *If* $(u_n)_n \subset X$ *and* $(\tau_n)_n \subset Y$ *are two sequences such that* $u_n \rightharpoonup \tilde{u}$ *in* X *as* $n \to \infty$ *and* $\tau_n \rightharpoonup \tilde{\tau}$ *in* Y *as* $n \to \infty$, *then* $\limsup_{n\to\infty} b(u_n, \tau_n) = b(\tilde{u}, \tilde{\tau})$.

Theorem 7.1. *[An existence result][Theorem 5 in [113]] If Assumptions* 6.15-6.18, 6.20, 7.1-7.2 *hold true, then Problem* 7.1 *has at least one solution.*

The proof of Theorem 7.1 can be found in [113]. The key of the proof was the construction of the following operator.

$$T : \Lambda \to \Lambda \quad T(\eta) = \lambda_\eta, \tag{7.1}$$

which is a weakly sequentially continuous map. In addition, it is worth to mention that $T_{|\pounds}$ has a fixed point, where

$$\pounds = \{\mu \in \Lambda \,|\, \|\mu\|_Y \leq \frac{\|f\|_X}{\alpha} + \frac{(L + L_j)\|f\|_X}{m\,\alpha}\}.$$

Thus, there exists $\eta^* \in \pounds \subset \Lambda$ such that $T(\eta^*) = \lambda_{\eta^*} = \eta^*$. The pair $(u_{\eta^*}, \lambda_{\eta^*}) \in X \times \Lambda$ is a solution of Problem 7.1.

In order to investigate the uniqueness of the solution, we made a new assumption.

Assumption 7.3. *For all* μ_1, $\mu_2 \in \Lambda$, v_1, $v_2 \in X$ *there exists* $G > 0$ *such that*

$$j(\mu_1, v_2) - j(\mu_1, v_1) + j(\mu_2, v_1) - j(\mu_2, v_2) \leq G\|v_1 - v_2\|_X.$$

Theorem 7.2. *[An uniqueness result][Theorem 6 in [113]] If Assumptions* 6.15-6.18, 6.20, 7.1-7.3 *hold true, then Problem* 7.1 *has a solution, unique in its first argument.*

The proof of Theorem 7.2 was given in [113].

7.2 A 3D contact model

To illustrate the applicability of the abstract results presented in Section 7.1, we consider the following 3D model (the third example in Section 4 of the paper [113]).

Problem 7.2. *Given $\rho > 0$, find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, such that*

$$
\begin{aligned}
&Div\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} && in\ \Omega, && (7.2)\\
&\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) && in\ \Omega, && (7.3)\\
&\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} && on\ \Gamma_1, && (7.4)\\
&\boldsymbol{\sigma\nu}(\boldsymbol{x}) = \boldsymbol{f}_1(\boldsymbol{x}) && on\ \Gamma_2, && (7.5)\\
&u_\nu(\boldsymbol{x}) \le 0, \sigma_\nu(\boldsymbol{x}) \le 0, \sigma_\nu(\boldsymbol{x}) u_\nu(\boldsymbol{x}) = 0 && && \\
&\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -k(\sigma_\nu)(\boldsymbol{x}) \frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\sqrt{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|^2 + \rho^2}} && on\ \Gamma_3, && (7.6)
\end{aligned}
$$

where Γ_1, Γ_2 and Γ_3 is a partition of Γ such that the Lebesgue measure of Γ_1 is positive. Recall that (7.2) is the equilibrium equation, (7.3) is the constitutive law, (7.4) is the displacement boundary condition and (7.5) is the traction boundary condition. Herein (7.6) is a unilateral contact condition with regularized Coulomb-type friction law. The friction law we use describes a situation when slip appears even for small tangential shears, which is the case when the surfaces are lubricated by a thin layer of non-Newtonian fluid, see [147] and the references therein.

In order to analyze the model, we adopt Assumptions 6.22 and 6.23 made in the previous chapter. Moreover, herein we consider

$$
\omega : \mathbb{S}^3 \to [0, \infty), \quad \omega(\boldsymbol{\varepsilon}) = \frac{1}{2} \mathcal{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \frac{\beta}{2} \|\boldsymbol{\varepsilon} - P_K \boldsymbol{\varepsilon}\|^2 \tag{7.7}
$$

where $\mathcal{A}$ is a fourth order symmetric tensor satisfying the ellipticity condition, β is a strictly positive constant, $K \subset \mathbb{S}^3$ denotes a closed, convex set containing the element $0_{\mathbb{S}^3}$ and $P_K : \mathbb{S}^3 \to K$ is the projection operator. Notice that the function ω fulfills the following property.

Property 7.1. *ω is a Gâteaux differentiable functional. In addition:*

- *ω_1) there exists $L_\omega > 0$ such that*

$$
\|\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})\| \le L_\omega \|\boldsymbol{\varepsilon} - \boldsymbol{\tau}\| \quad \textit{for all } \boldsymbol{\varepsilon}, \boldsymbol{\tau} \in \mathbb{S}^3;
$$

- *ω_2) there exists $m_\omega > 0$ such that*

$$
(\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})) \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\tau}) \ge m_\omega \|\boldsymbol{\varepsilon} - \boldsymbol{\tau}\|^2 \quad \textit{for all } \boldsymbol{\varepsilon}, \boldsymbol{\tau} \in \mathbb{S}^3,
$$

where

$$
\nabla\omega(\boldsymbol{\varepsilon}) \cdot \boldsymbol{\tau} = \mathcal{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\tau} + \beta(\boldsymbol{\varepsilon} - P_K \boldsymbol{\varepsilon}) \cdot \boldsymbol{\tau}.
$$

We keep the definitions for X in (6.43), J in (6.46), S in (6.44), Y in (6.45), λ in (6.49), $b(\cdot,\cdot)$ in (6.50) and Λ in (6.51). Notice that $\lambda = -\sigma_{\nu|_{\Gamma_3}}$.

Herein we consider a coefficient of friction as follows:

$$
k(\sigma_\nu) : \Gamma_3 \to [0, \infty) \quad k(\sigma_\nu)(\boldsymbol{x}) = \vartheta \frac{|(\mathcal{R}\sigma_{\nu|_{\Gamma_3}})(\boldsymbol{x})|}{1 + |(\mathcal{R}\sigma_{\nu|_{\Gamma_3}})(\boldsymbol{x})|}, \tag{7.8}
$$

where $\vartheta > 0$ and $\mathcal{R} : Y \to L^2(\Gamma_3)$ is a linear and compact operator.

In addition, we define $\boldsymbol{f} \in X$ as follows:

$$(\boldsymbol{f},\, \boldsymbol{v})_X = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_1(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in X.$$

Finally, we define

$$j : \Lambda \times X \to \mathbb{R}_+ \quad j(\eta, \boldsymbol{v}) = \int_{\Gamma_3} K(\eta)(\sqrt{\|\boldsymbol{v}_\tau(\boldsymbol{x})\|^2 + \rho^2} - \rho)\, d\Gamma, \tag{7.9}$$

where $K : \Lambda \to L^\infty(\Gamma_3)$,

$$K(\eta)(\boldsymbol{x}) = \frac{\vartheta |(\mathcal{R}\eta)(\boldsymbol{x})|}{1 + |(\mathcal{R}\eta)(\boldsymbol{x})|}. \tag{7.10}$$

Notice that $K(\mu) = -k(-\mu)$ for all $\mu \in \Lambda$.

According to Theorem 7.2, Problem 7.2 has a solution $(\boldsymbol{u}, \lambda)$, unique in its first argument. For details see [113].

Part II

A variational approach via bipotentials

Chapter 8

Unilateral frictionless contact problems

This chapter is based on the paper [106]. We consider a unilateral contact model for nonlinearly elastic materials, under the small deformations hypothesis, for static processes. The contact is modeled with Signorini's condition with zero gap and the friction is neglected on the potential contact zone. The behavior of the material is modeled by a subdifferential inclusion, the constitutive map being proper, convex, and lower semicontinuous. After describing the model, we give a weak formulation using a bipotential which depends on the constitutive map and its Fenchel conjugate. We look for the unknown into a Cartesian product of two nonempty, convex, closed, unbounded subsets of two Hilbert spaces. We prove the existence and the uniqueness of the weak solution based on minimization arguments for appropriate functionals associated with the variational system. How the proposed variational approach is related to previous variational approaches, is discussed too.

8.1 The model and its weak solvability via bipotentials

In this section we discuss the weak solvability via bipotentials theory for a unilateral frictionless contact model in the following physical setting. A body occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary, partitioned in three measurable parts, Γ_1, Γ_2 and Γ_3, such that the Lebesgue measure of Γ_1 is positive. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body can be in contact with a rigid foundation. According to this physical setting we formulate the following boundary value problem.

Problem 8.1. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$, *such that*

$$\operatorname{Div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \quad \text{in } \Omega, \tag{8.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \quad \text{in } \Omega, \tag{8.2}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \qquad \textit{on } \Gamma_1, \tag{8.3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \qquad \textit{on } \Gamma_2, \tag{8.4}$$

$$\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = \boldsymbol{0},\ u_\nu(\boldsymbol{x}) \leq 0,\ \sigma_\nu(\boldsymbol{x}) \leq 0,\ \sigma_\nu(\boldsymbol{x})u_\nu(\boldsymbol{x}) = 0 \qquad \textit{on } \Gamma_3. \tag{8.5}$$

Problem 8.1 has the following structure: (8.1) represents the equilibrium equation, (8.2) represents the constitutive law, (8.3) represents the displacements boundary condition, (8.4) represents the traction boundary condition and (8.5) represents the frictionless unilateral contact condition.

Assumption 8.1. *The constitutive function $\omega : \mathbb{S}^3 \to \mathbb{R}$ is convex and lower semicontinuous. In addition, there exist α, β such that $1 > \beta \geq \alpha > 0$ and $\beta\|\boldsymbol{\varepsilon}\|^2 \geq \omega(\boldsymbol{\varepsilon}) \geq \alpha\|\boldsymbol{\varepsilon}\|^2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^3$.*

Assumption 8.2. *The densities of the volume forces and traction verify*

$$\boldsymbol{f}_0 \in L^2(\Omega)^3 \textit{ and } \boldsymbol{f}_2 \in L^2(\Gamma_2)^3.$$

Let us introduce the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \ : \ \boldsymbol{\gamma v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}.$$

Let $\boldsymbol{f} \in V$ be such that

$$(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega)^3} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x}) d\Gamma \qquad \text{for all } \boldsymbol{v} \in V.$$

We introduce a subset of V as follows,

$$U_0 = \{\boldsymbol{v} \in V \ : \ v_\nu \leq 0 \text{ a.e. on } \Gamma_3\}.$$

Lemma 8.1. *[Lemma 1 in [106]] Let α, β be the constants in Assumption 8.1. Then*

$$(1-\beta)\|\boldsymbol{\tau}\|^2 \leq \omega^*(\boldsymbol{\tau}) \leq \frac{1}{4\alpha}\|\boldsymbol{\tau}\|^2 \qquad \textit{for all } \boldsymbol{\tau} \in \mathbb{S}^3. \tag{8.6}$$

The proof of this lemma can be found in [106]. We associate with the constitutive map ω the bipotential $B : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{R}$,

$$B(\boldsymbol{\tau}, \boldsymbol{\mu}) = \omega(\boldsymbol{\tau}) + \omega^*(\boldsymbol{\mu}) \qquad \text{for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}^3, \tag{8.7}$$

where ω^* is Fenchel's conjugate of the function ω.

Notice that there exists $C = C(\alpha, \beta) > 0$ such that

$$B(\boldsymbol{\tau}, \boldsymbol{\mu}) \geq C(\|\boldsymbol{\tau}\|^2 + \|\boldsymbol{\mu}\|^2) \qquad \text{for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}^3. \tag{8.8}$$

We introduce the Hilbert space

$$L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\mu} = (\mu_{ij}) \ : \ \mu_{ij} \in L^2(\Omega),\ \mu_{ij} = \mu_{ji} \text{ for all } i, j \in \{1, 2, 3\}\}.$$

It is worth to note that

$$B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot)), \boldsymbol{\tau}(\cdot)) \in L^1(\Omega) \quad \text{for all } \boldsymbol{v} \in V,\ \boldsymbol{\tau} \in L^2_s(\Omega)^{3\times 3}.$$

We define now the form

$$b : V \times L^2_s(\Omega)^{3\times 3} \to \mathbb{R} \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \int_\Omega B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})), \boldsymbol{\mu}(\boldsymbol{x}))\, dx. \tag{8.9}$$

Consider the following subset of $L^2_s(\Omega)^{3\times 3}$,

$$\Lambda = \{\boldsymbol{\mu} \in L^2_s(\Omega)^{3\times 3} \ : \ (\boldsymbol{\mu}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}} \geq (\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in U_0\}. \tag{8.10}$$

We have the following weak formulation of Problem 8.1.

Problem 8.2. *Find $\boldsymbol{u} \in U_0 \subset V$ and $\boldsymbol{\sigma} \in \Lambda \subset L^2_s(\Omega)^{3\times 3}$ such that*

$$\begin{aligned} &b(\boldsymbol{v}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad && \textit{for all } \boldsymbol{v} \in U_0 \\ &b(\boldsymbol{u}, \boldsymbol{\mu}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) \geq 0 && \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

Each solution $(\boldsymbol{u}, \boldsymbol{\sigma}) \in U_0 \times \Lambda$ of Problem 8.2 is called a *weak solution* of Problem 8.1.

Theorem 8.1. *(An existence result)[Theorem 3 in [106]] If Assumptions 8.1-8.2 hold true, then Problem* 8.2 *has at least one solution $(\boldsymbol{u}, \boldsymbol{\sigma}) \in U_0 \times \Lambda$.*

In order to get the uniqueness, additional assumptions were needed.

Assumption 8.3. *The constitutive function ω and its Fenchel's conjugate ω^* are strictly convex.*

Theorem 8.2. *(An uniqueness result)[Theorem 4 in [106]] If Assumptions 8.1-8.3 hold true, then Problem 8.2 has a unique solution.*

The proofs of Theorems 8.1 and 8.2 can be found in [106].

8.2 New approach versus previous approaches

In this section we discuss Problem 8.1 for a special class of nonlinear materials, such that the following additional assumption holds true.

Assumption 8.4. *The constitutive function ω is Gâteaux differentiable and its gradient $\nabla\omega$ verifies:*
there exists $L > 0$ such that $\|\nabla\omega(\boldsymbol{\varepsilon}_1) - \nabla\omega(\boldsymbol{\varepsilon}_2)\| \leq L\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3$;
there exists $m > 0$ such that $(\nabla\omega(\boldsymbol{\varepsilon}_1) - \nabla\omega(\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$ for all $\boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3$.

In this special case, the constitutive law (8.2) becomes $\boldsymbol{\sigma}(\boldsymbol{x}) = \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})))$ in Ω, the literature offering us two variational approaches: the primal variational formulation and the dual variational formulation. More precisely, Problem 8.1 has the following variational formulation in displacements.

Primal variational formulation. *Find $\boldsymbol{u} \in U_0$ such that*

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in U_0.$$

Herein the operator $A : V \to V$ is defined as follows: for each $\boldsymbol{u} \in V$, $A\boldsymbol{u}$ is the element of V that satisfies

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \nabla\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))\, dx$$

for all $\boldsymbol{v} \in V$. The primal variational formulation has a unique solution $\boldsymbol{u} \in U_0$, see e.g. Theorem 5.10 in [147].

On the other hand, Problem 8.1 has the following weak formulation in terms of stress.

The dual variational formulation. *Find $\boldsymbol{\sigma} \in \Lambda$ such that*

$$((\nabla\omega)^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma})_{L^2(\Omega)^{3\times 3}} \geq 0 \quad \text{for all } \boldsymbol{\tau} \in \Lambda.$$

The dual variational formulation has a unique solution $\boldsymbol{\sigma} \in \Lambda$, see e.g. Theorem 5.12 in [147].

Let us state the following auxiliary result.

Theorem 8.3. *[Theorem 5 in [106]] Assumptions 8.1-8.2 and Assumption 8.4 hold true.*

1) If $\tilde{\boldsymbol{u}}$ is the solution of the primal variational formulation and $\tilde{\boldsymbol{\sigma}}$ is the function given by $\tilde{\boldsymbol{\sigma}} = \nabla\omega(\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}))$ then $\tilde{\boldsymbol{\sigma}}$ is the unique solution of the dual variational formulation.

2) If $\tilde{\boldsymbol{\sigma}}$ is the solution of the dual variational formulation then $\tilde{\boldsymbol{\sigma}} = \nabla\omega(\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}))$ where $\tilde{\boldsymbol{u}}$ is the solution of the primal variational formulation.

The proof of this theorem is a straightforward consequence of Theorem 5.13 in [147].

The main result of this section is the following theorem.

Theorem 8.4. *[Theorem 6 in [106]] Assumptions 8.1-8.2 and Assumption 8.4 hold true.*

i) If $\tilde{\boldsymbol{u}}$ is the unique solution of the primal variational formulation and $\tilde{\boldsymbol{\sigma}}$ is the unique solution of the dual variational formulation, then the pair $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\sigma}})$ is a solution of Problem 8.2.

ii) If $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of Problem 8.2, *then the first component $\boldsymbol{u}$ is the unique solution of the primal variational formulation.*

iii) If, in addition, ω is strictly convex then the unique solution of Problem 8.2, $(\boldsymbol{u}, \boldsymbol{\sigma})$, coincides with the pair $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\sigma}})$ consisting of the unique solution of the primal variational formulation and the unique solution of the dual variational formulation.

We underline that, if the constitutive function ω fulfills Assumptions 8.1, Assumption 8.4 and, in addition is strictly convex, then the unique solution of Problem 8.2 coincides with the pair consisting of the unique solution of the primal variational formulation and the unique solution of the dual variational formulation. Let us give an example of such a constitutive function ω:

$$\omega : \mathbb{S}^3 \to \mathbb{R},\ \omega(\boldsymbol{\tau}) = \frac{1}{2}\mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} + \frac{\zeta}{2}\|\boldsymbol{\tau} - P_{\mathcal{K}}\boldsymbol{\tau}\|^2, \tag{8.11}$$

where $\mathcal{E} : \mathbb{S}^3 \to \mathbb{S}^3$, $\mathcal{E} = (\mathcal{E}_{ijkl})$, $\mathcal{E}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $1 \le i, j, k, l \le 3$, λ, μ and ζ being positive coefficients of the material, small enough (e.g. $\frac{3}{2}\lambda + \mu + \frac{\zeta}{2} < 1$), $\mathcal{K} \subset \mathbb{S}^3$ is a nonempty, closed and convex set and $P_{\mathcal{K}} : \mathbb{S}^3 \to \mathcal{K}$ represents the projection operator on $\mathcal{K}$.

In order to study the properties of the functional ω a very helpful reference was [139].

Chapter 9

Frictional contact problems

This chapter is based on the paper [108]. The frictional contact model we investigate in the present paper is a 3D nonlinearly elastostatic model, under the small deformation hypothesis. Mathematically, we describe it as a boundary value problem consisting of a system of a partial differential vectorial equation (equilibrium equation) and a subdifferential inclusion (constitutive law), associated with a homogeneous displacement boundary condition, a traction boundary condition and a frictional contact condition. The constitutive law indicates us that the stress belongs to the subdifferential of a proper, convex, lower semicontinuous functional. In order to model the frictional contact we use a static version of Coulomb's law of dry friction with prescribed normal stress. Based on minimization arguments for appropriate functionals associated with the variational system, the existence and the uniqueness of the weak solution of this model it was proved. In addition to prove the existence and the uniqueness of the weak solution as a "global" solution, allowing to compute simultaneously the displacement field and the Cauchy stress tensor, another relevant aspect of this approach it was discussed for a particular class of constitutive functions: the weak solution in the new approach coincides with the pair consisting of the unique solution of the primal variational formulation and the unique solution of the dual variational formulation. Due to the particular feature of the mechanical model we treat in this chapter, the weak formulation herein is more complex than those presented in the previous chapter; it involves not only a bipotential function but also a potential which depends on the prescribed normal stress and on the coefficient of friction.

9.1 The model and its weak solvability via bipotentials

We consider a body that occupies a bounded domain $\Omega \subset \mathbb{R}^3$, with smooth boundary, partitioned in three measurable parts, Γ_1, Γ_2 and Γ_3, such that the Lebesgue measure of Γ_1 is positive. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body is in frictional contact with a foundation, the normal stress being prescribed. According to this physical setting we formulate the following boundary value problem.

Problem 9.1. *Find* $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$, *such that*

$$\operatorname{Div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{0} \quad \text{in } \Omega, \tag{9.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \quad \text{in } \Omega, \tag{9.2}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{9.3}$$

$$\boldsymbol{\sigma\nu}(\boldsymbol{x}) = \boldsymbol{f}_2(\boldsymbol{x}) \quad \text{on } \Gamma_2, \tag{9.4}$$

$$-\sigma_\nu(\boldsymbol{x}) = F(\boldsymbol{x}) \quad \text{on } \Gamma_3 \tag{9.5}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \leq k(\boldsymbol{x})\,|\sigma_\nu(\boldsymbol{x})|, \\ &\boldsymbol{\sigma}_\tau(\boldsymbol{x}) = -\,k(\boldsymbol{x})\,|\sigma_\nu(\boldsymbol{x})|\,\frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|} \quad \text{if} \quad \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{9.6}$$

Problem 9.1 has the following structure: (9.1) represents the equilibrium equation, (9.2) represents the constitutive law, (9.3) represents the homogeneous displacements boundary condition, (9.4) represents the traction boundary condition and (9.5)-(9.6) model the frictional contact with prescribed normal stress.

In order to study Problem 9.1 we keep Assumption 8.1 on the constitutive function ω and Assumption 8.2 for the density of the volume forces $\boldsymbol{f}_0$ and the density of the traction $\boldsymbol{f}_2$.

In addition we made the following assumptions.

Assumption 9.1. *The prescribed normal stress verifies* $F \in L^2(\Gamma_3)$ *and* $F(\boldsymbol{x}) \geq 0$ *a.e.* $\boldsymbol{x} \in \Gamma_3$.

Assumption 9.2. *The coefficient of friction verifies* $k \in L^\infty(\Gamma_3)$ *and* $k(\boldsymbol{x}) \geq 0$ *a.e.* $\boldsymbol{x} \in \Gamma_3$.

On the other hand, we introduce the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \;:\; \boldsymbol{\gamma}\,\boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}$$

and define $\boldsymbol{f} \in V$,

$$(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega)^3} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\,\boldsymbol{v}(\boldsymbol{x}) d\Gamma \quad \text{for all } \boldsymbol{v} \in V.$$

Notice that $k(\cdot)\, F(\cdot)\, \|\boldsymbol{v}_\tau(\cdot)\| \in L^1(\Gamma_3)$. This allows us to consider the following functional

$$j : V \to \mathbb{R}_+ \quad j(\boldsymbol{v}) = \int_{\Gamma_3} k(\boldsymbol{x})\, F(\boldsymbol{x})\, \|\boldsymbol{v}_\tau(\boldsymbol{x})\|\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V. \tag{9.7}$$

As in the previous chapter, we associate with the constitutive map ω the bipotential B : $\mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{R}$,

$$B(\boldsymbol{\tau}, \boldsymbol{\mu}) = \omega(\boldsymbol{\tau}) + \omega^*(\boldsymbol{\mu}) \quad \text{for all } \boldsymbol{\tau}, \boldsymbol{\mu} \in \mathbb{S}^3, \tag{9.8}$$

and we introduce a form $b(\cdot,\cdot)$ as follows,

$$b : V \times L^2_s(\Omega)^{3\times 3} \to \mathbb{R} \quad b(\boldsymbol{v},\boldsymbol{\mu}) = \int_\Omega B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})),\boldsymbol{\mu}(\boldsymbol{x}))\,dx. \tag{9.9}$$

Consider now the following subset of $L^2_s(\Omega)^{3\times 3}$,

$$\Lambda = \{\boldsymbol{\mu} \in L^2_s(\Omega)^{3\times 3} \,:\, (\boldsymbol{\mu},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}} + j(\boldsymbol{v}) \geq (\boldsymbol{f},\boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V\}. \tag{9.10}$$

Lemma 9.1. *[Lemma 2 in [108]] The subset Λ is an unbounded, closed, convex subset of $L^2_s(\Omega)^{3\times 3}$.*

The proof of Lemma 9.1 was given in [108].

Problem 9.1 has the following weak formulation.

Problem 9.2. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\sigma} \in \Lambda \subset L^2_s(\Omega)^{3\times 3}$ such that*

$$\begin{aligned} b(\boldsymbol{v},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V \quad & \textit{for all } \boldsymbol{v} \in V \\ b(\boldsymbol{u},\boldsymbol{\mu}) - b(\boldsymbol{u},\boldsymbol{\sigma}) \geq 0 \quad & \textit{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

This is a new variational system governed by the functional j.

Each solution $(\boldsymbol{u},\boldsymbol{\sigma})$ of Problem 9.2 is called a *weak solution* of Problem 9.1.

We define a functional $\mathcal{L}$ as follows,

$$\mathcal{L} : V \times \Lambda \to \mathbb{R} \quad \mathcal{L}(\boldsymbol{v},\boldsymbol{\mu}) = b(\boldsymbol{v},\boldsymbol{\mu}) + j(\boldsymbol{v}) - (\boldsymbol{f},\boldsymbol{v})_V.$$

Let us consider the following minimization problem.

$$\mathcal{L}(\boldsymbol{u},\boldsymbol{\sigma}) = \min_{(\boldsymbol{v},\boldsymbol{\mu})\in V\times\Lambda} \mathcal{L}(\boldsymbol{v},\boldsymbol{\mu}). \tag{9.11}$$

Theorem 9.1. *(An existence result)[Theorem 3 in [108]] If Assumptions* 8.1, 8.2, 9.1 *and* 9.2 *hold true, then Problem* 9.2 *has at least one solution $(\boldsymbol{u},\boldsymbol{\sigma}) \in V \times \Lambda$ which is a solution of the minimization problem (9.11).*

The study of the uniqueness of the solution was made under the following additional assumption.

Assumption 9.3. *The constitutive function ω and its Fenchel's conjugate ω^* are strictly convex.*

Theorem 9.2. *(An uniqueness result)[Theorem 4 in [106]] If Assumptions* 8.1, 8.2, 9.1, 9.2 *and* 9.3 *hold true, then Problem* 9.2 *has a unique solution.*

The proofs of Theorems 9.1 and 9.2 were given in [108].

9.2 New approach versus previous approaches

In this section we discuss Problem 9.1 adopting Assumption 8.4 for the constitutive function ω. In this special case, Problem 9.1 has the following variational formulation in displacements, see [147].

Primal variational formulation. *Find* $\boldsymbol{u} \in V$ *such that*

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V.$$

The primal variational formulation has a unique solution $\boldsymbol{u} \in V$, see for example Theorem 5.21 in [147].

Also, in the special case we treat in this section, Problem 9.1 has the following variational formulation in terms of stress.

The dual variational formulation. *Find* $\boldsymbol{\sigma} \in \Lambda$ *such that*

$$((\nabla\omega)^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma})_{L^2(\Omega)^{3\times 3}} \geq 0 \quad \text{for all } \boldsymbol{\tau} \in \Lambda.$$

The dual variational formulation has a unique solution $\boldsymbol{\sigma} \in \Lambda$, see Theorem 5.32 in [147].

A straightforward consequence of Theorem 5.34 in [147] is the following theorem.

Theorem 9.3. *[Theorem 5 in [108]] Assumptions* 8.1, 8.2, 9.1, 9.2 *and Assumption* 8.4 *hold true.*

1) If $\widetilde{\boldsymbol{u}}$ *is the solution of the primal variational formulation and* $\widetilde{\boldsymbol{\sigma}}$ *is the function given by* $\widetilde{\boldsymbol{\sigma}} = \nabla\omega(\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}))$ *then* $\widetilde{\boldsymbol{\sigma}}$ *is the unique solution of the dual variational formulation.*

2) If $\widetilde{\boldsymbol{\sigma}}$ *is the solution of the dual variational formulation then* $\widetilde{\boldsymbol{\sigma}} = \nabla\omega(\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}))$ *where* $\widetilde{\boldsymbol{u}}$ *is the solution of the primal variational formulation.*

The main result of this section is the following theorem.

Theorem 9.4. *[Theorem 6 in [108]] Assumptions* 8.1, 8.2, 9.1, 9.2 *and Assumption* 8.4 *hold true.*

i) If $\widetilde{\boldsymbol{u}}$ *is the unique solution of the primal variational formulation and* $\widetilde{\boldsymbol{\sigma}}$ *is the unique solution of the dual variational formulation, then the pair* $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\sigma}})$ *is a solution of Problem* 9.2.

ii) If $(\boldsymbol{u}, \boldsymbol{\sigma})$ *is a solution of Problem* 9.2, *then the first component* $\boldsymbol{u}$ *is the unique solution of the primal variational formulation.*

iii) If, in addition, ω *is strictly convex then the unique solution of Problem* 9.2, $(\boldsymbol{u}, \boldsymbol{\sigma})$, *coincides with the pair* $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\sigma}})$ *consisting of the unique solution of the primal variational formulation and the unique solution of the dual variational formulation.*

The proof of Theorem 9.4 was given in [108].

Part III

A variational approach via history-dependent quasivariational inequalities on unbounded time interval

Chapter 10

Preliminaries

In this chapter we present some preliminaries. Firstly we recall an abstract fixed point result for operators defined on the Fréchet space of continuous functions on $\mathbb{R}_+ = [0, \infty)$ with values on a real Banach space $(X, \|\cdot\|_X)$, denoted $C(\mathbb{R}_+, X)$. Then, we recall an existence and uniqueness result of the solution for an abstract history-dependent quasivariational inequality formulated on the unbounded time interval $[0, \infty)$.

10.1 A fixed point result

Let $(X, \|\cdot\|_X)$ be a real Banach space, $\mathbb{N}^*$ represents the set of positive integers and $\mathbb{R}_+ = [0, \infty)$. We consider the functional space of continuous functions defined on $\mathbb{R}_+$ with values on X, that is

$$C(\mathbb{R}_+; X) = \{\ x : \mathbb{R}_+ \to X \ |\ \ x \text{ is continuous } \}.$$

Let us present some preliminaries on the space $C(\mathbb{R}_+; X)$; details on the Fréchet space $C(\mathbb{R}_+, X)$ including some basic properties can be found in [39, 96].

For all $n \in \mathbb{N}^*$, we denote by $C([0, n]; X)$ the space of continuous functions defined on $[0, n]$ with values on X, that is

$$C([0, n]; X) = \{\ x : [0, n] \to X \ |\ \ x \text{ is continuous } \}.$$

The space $C([0, n]; X)$ is a real Banach space with the norm

$$\|x\|_n = \max_{t \in [0,n]} \|x(t)\|_X \tag{10.1}$$

and, moreover, for any $\lambda > 0$ the norm (10.1) is equivalent with Bielecki's norm,

$$\|x\|_{\lambda,n} = \max_{t \in [0,n]} \left\{ e^{-\lambda t} \|x(t)\|_X \right\}. \tag{10.2}$$

Consider now two sequences of real numbers $(\lambda_n)_{n\in\mathbb{N}^*}$ and $(\beta_n)_{n\in\mathbb{N}^*}$ such that

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...., \tag{10.3}$$

$$\beta_n > 0 \quad \forall\, n \in \mathbb{N}^*, \quad \sum_{n=1}^{\infty} \beta_n \, < \infty. \tag{10.4}$$

For any $x\,, y \in C(\mathbb{R}_+; X)$ define

$$d(x, y) = \sum_{n=1}^{\infty} \beta_n \, \frac{|x-y|_n}{1+|x-y|_n}\,, \tag{10.5}$$

where, for all $n \in \mathbb{N}^*$,

$$|x|_n = \|x\|_{\lambda_n, n} = \max_{t\in[0,n]} \big\{e^{-\lambda_n t}\|x(t)\|_X\big\}. \tag{10.6}$$

It is well known that d is a distance on $C(\mathbb{R}_+; X)$ and the metric space $(C(\mathbb{R}_+; X), d)$ is complete, i.e. is a Fréchet space.

We note that, for all $n \in \mathbb{N}^*$, $|\cdot|_n$ and $\|\cdot\|_n$ are equivalent norms on the space $C([0, n]; X)$. Also, we recall that the convergence of a sequence $(x_p)_{p\in\mathbb{N}^*} \subset C(\mathbb{R}_+; X)$ to the element $x \in C(\mathbb{R}_+; X)$, is characterized by the following equivalences:

$$\begin{aligned} d(x_p, x) \to 0 \ \text{ as } \ p \to \infty \ &\Leftrightarrow \ \lim_{p\to\infty} |x_p - x|_n = 0 \quad \forall\, n \in \mathbb{N}^* \\ &\Leftrightarrow \ \lim_{p\to\infty} \|x_p - x\|_n = 0 \quad \forall\, n \in \mathbb{N}^*. \end{aligned} \tag{10.7}$$

According to (10.7), the convergence in the metric space $(C(\mathbb{R}_+; X), d)$ does not depend on the choice of sequences $(\lambda_n)_{n\in\mathbb{N}^*}$ and $(\beta_n)_{n\in\mathbb{N}^*}$ which satisfy (10.3) and (10.4). For this reason, we write $C(\mathbb{R}_+; X)$ instead of $(C(\mathbb{R}_+; X), d)$ and we refer to $C(\mathbb{R}_+; X)$ as to a Fréchet space. Also, note that:

$$\begin{aligned} &(x_p)_{p\in\mathbb{N}^*} \subset C(\mathbb{R}_+; X) \ \text{ is a Cauchy sequence if and only if} \\ &\forall\, \varepsilon > 0, \forall\, n \in \mathbb{N}^*, \exists\, N = N(\varepsilon, n) \ \text{ such that } \ |x_p - x_q|_n < \varepsilon \quad \forall\, p,\, q \ge N. \end{aligned} \tag{10.8}$$

Theorem 10.1. *[Theorem 2.1 in [144]] Let $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator. Assume that there exists $m \in \mathbb{N}^*$ with the following property: for all $n \in \mathbb{N}^*$ there exist two constants $c_n \ge 0$ and $k_n \in [0, 1)$ such that*

$$\|\Lambda x(t) - \Lambda y(t)\|_X^m \le k_n \|x(t) - y(t)\|_X^m + c_n \int_0^t \|x(s) - y(s)\|_X^m \, ds \tag{10.9}$$

for all $x,\, y \in C(\mathbb{R}_+; X)$ and for any $t \in [0, n]$. Then the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

Corollary 10.1. *[Corollary 2.5 in [144]] Let* $\Lambda : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$ *be a nonlinear operator. Assume that there exist* $m \in \mathbb{N}^*$, $\alpha \in [0,1)$ *and a continuous function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, *such that*

$$\|\Lambda\, x(t) - \Lambda\, y(t)\|_X^m \leq \alpha\, \|x(t) - y(t)\|_X^m + \gamma(t) \int_0^t \|x(s) - y(s)\|_X^m \, ds$$

for all $x,\, y \in C(\mathbb{R}_+;X)$ *and for any* $t \in \mathbb{R}_+$. *Then the operator* Λ *has a unique fixed point* $\eta^* \in C(\mathbb{R}_+;X)$.

The proofs of Theorem 10.1 and Corollary 10.1 can be found in [144].

10.2 An abstract history-dependent quasivariational inequality

Let X be a real Hilbert space with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$. Let also Y be a normed space with the norm denoted $\|\cdot\|_Y$ and let $\mathcal{L}(X,Y)$ denote the space of linear continuous operators from X to Y with the usual norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. Finally, for $n \in \mathbb{N}^*$ we denote by $C([0,n];\mathcal{L}(X,Y))$ the space of continuous functions defined on the bounded interval $[0,n]$ with values in $\mathcal{L}(X,Y)$.

Let K be a subset of X and consider the operators $A : K \to X$, $\mathcal{S} : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;Y)$ and the functionals $\varphi : Y \times X \to \mathbb{R}$, $j : X \times X \to \mathbb{R}$. Moreover, let $f : \mathbb{R}_+ \to X$. Then we consider the problem of finding a function $u \in C(\mathbb{R}_+;X)$, such that for all $t \in \mathbb{R}_+$, the inequality below holds:

$$\begin{aligned} u(t) \in K, \quad & (Au(t), v - u(t))_X + \varphi(\mathcal{S}u(t), v) - \varphi(\mathcal{S}u(t), u(t)) \\ & + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t))_X \qquad \text{for all } v \in K. \end{aligned} \tag{10.10}$$

Note that (10.10) represents a time-dependent variational inequality governed by two functionals φ and j which depend on the solution and, therefore, we refer to (10.10) as a quasivariational inequality. Also, to avoid any confusion, we note that here and below the notation $Au(t)$ and $\mathcal{S}u(t)$ are short hand notation for $A(u(t))$ and $(\mathcal{S}u)(t)$, i.e. $Au(t) = A(u(t))$ and $\mathcal{S}u(t) = (\mathcal{S}u)(t)$, for all $t \in \mathbb{R}_+$.

In the study of (10.10) were used the following assumptions.

Assumption 10.1. K is a closed, convex, nonempty subset of X.

Assumption 10.2.

There exists $m > 0$ such that $(Au_1 - Au_2, u_1 - u_2)_X \geq m\, \|u_1 - u_2\|_X^2$ *for all* $u_1, u_2 \in K$.

There exists $L > 0$ *such that* $\|Au_1 - Au_2\|_X \leq L\, \|u_1 - u_2\|_X$ *for all* $u_1, u_2 \in K$.

Assumption 10.3. For all $y \in Y$, $\varphi(y, \cdot)$ is convex and lower semicontinuous on X. There exists $\alpha > 0$ such that $\varphi(y_1, u_2) - \varphi(y_1, u_1) + \varphi(y_2, u_1) - \varphi(y_2, u_2) \leq \alpha \, \|y_1 - y_2\|_Y \, \|u_1 - u_2\|_X$ *for all* $y_1,\, y_2 \in Y,\, u_1,\, u_2 \in X$.

Assumption 10.4. For all $x \in X$, $j(u, \cdot)$ is convex and lower semicontinuous on X. There exists $\beta > 0$ such that $j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \leq \beta \, \|u_1 - u_2\|_X \, \|v_1 - v_2\|_X$ *for all* $u_1,\, u_2,\, v_1,\, v_2 \in X$.

Assumption 10.5. $\beta < m$.

Assumption 10.6. $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$; for all $n \in \mathbb{N}^*$ there exists $r_n > 0$ such that
$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Y \leq r_n \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \ \textit{for all}\ u_1,\, u_2 \in C(\mathbb{R}_+; X),\ t \in [0, n].$$

Assumption 10.7. $f \in C(\mathbb{R}_+; X)$.

Assumption 10.6 is satisfied for the operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ given by

$$\mathcal{S}v(t) = R\left(\int_0^t v(s)\, ds + v_0\right) \quad \text{for all } v \in C(\mathbb{R}_+; X),\ t \in \mathbb{R}_+, \tag{10.11}$$

where $R : X \to Y$ is a Lipschitz continuous operator and $v_0 \in X$. It is also satisfied for the *Volterra operator* $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ given by

$$\mathcal{S}v(t) = \int_0^t R(t - s)\, v(s)\, ds \quad \text{for all } v \in C(\mathbb{R}_+; X),\ t \in \mathbb{R}_+, \tag{10.12}$$

where now $R \in C(\mathbb{R}_+; \mathcal{L}(X, Y))$. In the case of the operator (10.11), inequality (10.6) holds with c_n being the Lipschitz constant of the operator R, for all $n \in \mathbb{N}^*$, and in the case of the operator (10.12) it holds with

$$r_n = \|R\|_{C([0,n]; \mathcal{L}(X,Y))} = \max_{t \in [0,n]} \|R(t)\|_{\mathcal{L}(X,Y)} \qquad \text{for all } n \in \mathbb{N}^*.$$

Clearly, in the case of the operators (10.11) and (10.12) the current value $\mathcal{S}v(t)$ at the moment t depends on the history of the values of v at the moments $0 \leq s \leq t$ and, therefore, we refer the operators of the form (10.11) or (10.12) as history-dependent operators. We extend this definition to all operators $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ satisfying condition (10.6) and, for this reason, we say that the quasivariational inequalities of the form (10.10) are history-dependent quasivariational inequalities. Their main feature consists in the fact that, at any moment $t \in \mathbb{R}_+$ the functional φ depends on the history of the solution up to the moment t, $\mathcal{S}u(t)$. This feature makes the difference with respect to the quasivariational inequalities studied in literature in which, usually, φ was assumed to depend on the current value of the solution, $u(t)$.

Based on arguments of monotonicity and convexity, combined with the fixed point result, Corollary 10.1, we have the following result.

Theorem 10.2. *[Theorem 2 in [146]] If Assumptions* 10.1–10.7 *hold true, then the variational inequality* (10.10) *has a unique solution* $u \in C(\mathbb{R}_+; X)$.

The proof of Theorem 10.2 can be found in [146].

Chapter 11

Viscoplastic problems

This chapter is based on the paper [10]. We consider two quasistatic problems which describe the contact between a viscoplastic body and an obstacle, the so-called foundation. The contact is frictionless and is modelled with normal compliance of such a type that the penetration is not restricted in the first problem, but is restricted with unilateral constraint, in the second one. For each problem we derive a variational formulation, then we prove its unique solvability. Next, we prove the convergence of the weak solution of the first problem to the weak solution of the second problem, as the stiffness coefficient of the foundation converges to infinity.

11.1 Mechanical models and their weak solvability

In this section we discuss the weak solvability of two viscoplastic contact models in the following physical setting. A viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $(d \in \{2,3\})$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\mathrm{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the process is quasistatic, and we study the contact process in the interval of time $\mathbb{R}_+ = [0, \infty)$.

In the first problem the contact is modeled with normal compliance in such a way that the penetration is not limited. Under these conditions, the classical formulation of the problem is the following.

Problem 11.1. *Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad in \quad \Omega \times (0, \infty), \tag{11.1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad in \quad \Omega \times (0, \infty), \tag{11.2}$$

$$\tag{11.3}$$

$$\boldsymbol{u} = \mathbf{0} \qquad on \quad \Gamma_1 \times (0, \infty), \tag{11.4}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad on \quad \Gamma_2 \times (0, \infty), \tag{11.5}$$

$$-\sigma_\nu = p(u_\nu) \qquad on \quad \Gamma_3 \times (0, \infty), \tag{11.6}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \qquad on \quad \Gamma_3 \times (0, \infty), \tag{11.7}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad in \quad \Omega. \tag{11.8}$$

In order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x}$ or t. Equation (11.1) represents the viscoplastic constitutive law of the material; equation (11.2) is the equilibrium equation; conditions (11.4) and (11.5) are the displacement and traction boundary conditions, respectively, and condition (11.7) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (11.8) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively. The function p involved in the contact condition (11.6) verifies the following assumption.

Assumption 11.1. $p : \mathbb{R} \to \mathbb{R}_+$ *such that*
There exists $L_p > 0$ *such that* $|p(r_1) - p(r_2)| \leq L_p |r_1 - r_2|$ *for all* $r_1, r_2 \in \mathbb{R}$.
$(p(r_1) - p(r_2))(r_1 - r_2) \geq 0$ *for all* $r_1, r_2 \in \mathbb{R}$.
$p(r) = 0$ *for all* $r < 0$.

Condition (11.6) combined with Assumption 11.1 shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); also, when there is penetration (i.e. when $u_\nu \geq 0$), then the reaction of the foundation is towards the body (since $\sigma_\nu \leq 0$) and it is increasing with the penetration (since p is an increasing function). Finally, we note that in this condition the penetration is not restricted and the normal stress is uniquely determined by the normal displacement.

Condition (11.6) was first introduced by Oden and Martin, see [95, 127], in the study of dynamic contact problems with elastic and viscoelastic materials. The term *normal compliance* for this condition was first used by Klarbring, Mikelič and Shillor, see [84, 85]. A first example of normal compliance function p which satisfies condition (11.6) is

$$p(r) = c_\nu r_+ \tag{11.9}$$

where c_ν is a positive constant. In this case condition (11.6) shows that the reaction of the foundation is proportional to the penetration. A second example of normal compliance function p which satisfies condition (11.6) is given by

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if} \quad r \leq \alpha, \\ c_\nu \alpha & \text{if} \quad r > \alpha, \end{cases}$$

where α is a positive coefficient related to the wear and hardness of the surface and, again, $c_\nu > 0$. In this case the contact condition (11.6) means that when the penetration is too large, i.e. when it exceeds α, the obstacle backs off and offers no additional resistance to the penetration. We conclude that in this case the foundation has an elastic-perfectly plastic behavior.

In the second problem the contact is again modeled with normal compliance but in such a way that the penetration is limited and associated to a unilateral constraint. The classical formulation of the problem is the following.

Problem 11.2. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega \times (0, \infty), \tag{11.10}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega \times (0, \infty), \tag{11.11}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{11.12}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2 \times (0, \infty), \tag{11.13}$$

$$\left.\begin{array}{l} u_\nu \le g, \ \ \sigma_\nu + p(u_\nu) \le 0, \\ (u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3 \times (0, \infty), \tag{11.14}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_3 \times (0, \infty), \tag{11.15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{11.16}$$

Here $g \ge 0$ is given and p is a function which satisfies the following assumption.

Assumption 11.2. $p :]-\infty, g] \to \mathbb{R}_+$ *is a given function such that:*

There exists $L_p > 0 : \ |p(r_1) - p(r_2)| \le L_p |r_1 - r_2|$ *for all* $r_1, r_2 \le g$.

$(p(r_1) - p(r_2))(r_1 - r_2) \ge 0$ *for all* $r_1, r_2 \le g$.

$p(r) = 0$ *for all* $r < 0$.

Recall that condition (11.14) was first introduced in [78]. Combined with Assumption 11.2 it shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); moreover, the penetration is limited (since $u_\nu \le g$) and g represents its maximum value. When $0 \le u_\nu < g$ then the reaction of the foundation is uniquely determined by the normal displacement (since $-\sigma_\nu = p(u_\nu)$) and, when $u_\nu = g$, the normal stress is not uniquely determined but is submitted to the restriction $-\sigma_\nu \ge p(g)$. Such a condition shows that the contact follows a normal compliance condition of the form (11.6) but up to the limit g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g. For this reason we refer to the contact condition (11.14) as a normal compliance contact condition with finite penetration and unilateral constraint or, for simplicity, a normal compliance condition with finite penetration. We conclude from above that this case models an elastic-rigid behavior of the foundation. Also, note that when $g = 0$ condition (11.14) becomes the classical Signorini contact condition in a form with a zero gap function,

$$u_\nu \le 0, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu u_\nu = 0 \qquad \text{on } \Gamma_3 \times (0, \infty).$$

Moreover, when $g > 0$ and $p = 0$, condition (11.6) becomes the Signorini contact condition in a form with a gap function,

$$u_\nu \le g, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu (u_\nu - g) = 0 \qquad \text{on } \Gamma_3 \times (0, \infty).$$

The last two conditions model the contact with a perfectly rigid foundation.

We made the following assumptions.

Assumption 11.3. $\mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d;\ \ \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i, j, k, l \le d.$
There exists $m_\mathcal{E} > 0$ such that $\mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_\mathcal{E} \|\boldsymbol{\tau}\|^2$ *for all* $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. in Ω.

Assumption 11.4. $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$.
There exists $L_\mathcal{G} > 0$ such that $\|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \le L_\mathcal{G} \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right)$ *for all* $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$.
The mapping $\boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ is measurable on Ω, for any $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$.
The mapping $\boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0})$ belongs to Q.

Assumption 11.5. $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d)$.

Assumption 11.6. $\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q$.

Assumption 11.7. $\boldsymbol{u}_0 \in U, \quad \boldsymbol{\sigma}_0 \in Q$,

In Assumption 11.7, U denotes the set of admissible displacements defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}. \tag{11.17}$$

We define the operator $P : V \to V$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, d\Gamma \quad \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{11.18}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, d\Gamma \quad \text{ for all } \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{11.19}$$

We have the following variational formulation of Problem 11.1.

Problem 11.3. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to V$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{11.20}$$

and

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V \tag{11.21}$$

for all $t \in \mathbb{R}_+$.

We have the following variational formulation for Problem 11.2.

Problem 11.4. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{11.22}$$

and

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in U \tag{11.23}$$

hold, for all $t \in \mathbb{R}_+$.

In the study of the Problem 11.3 we obtained the following results.

Lemma 11.1. *[Lemma 4.3 in [10]] Assumptions* 11.3, 11.4 *and* 11.6 *hold. Then, for each function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ there exists a unique function $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ such that*

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+. \tag{11.24}$$

Moreover, the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ satisfies the following property: for every $n \in \mathbb{N}$ there exists $r_n > 0$ such that

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q \leq r_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \tag{11.25}$$
$$\forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].$$

Next, using the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ defined in Lemma 11.1 we obtained the following equivalence result.

Lemma 11.2. *[Lemma 4.4 in [10]] Assumptions* 11.1 *and* 11.3–11.6 *hold and let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be a couple of functions such that $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Then, $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of Problem* 11.3 *if and only if for all $t \in \mathbb{R}_+$, the following equalities hold:*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{u}(t), \tag{11.26}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \ \ \forall\, \boldsymbol{v} \in V. \tag{11.27}$$

Let us define the operator $A : V \to V$ and the functional $\varphi : Q \times V \to \mathbb{R}$ by equalities

$$(A\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (P\boldsymbol{v}, \boldsymbol{w})_V \quad \forall\, \boldsymbol{v},\ \boldsymbol{w} \in V, \tag{11.28}$$

$$\varphi(\boldsymbol{\tau}, \boldsymbol{v}) = (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{\tau} \in Q,\ \ \boldsymbol{v} \in V. \tag{11.29}$$

With this notation we consider the problem of finding a function $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:

$$\begin{aligned}(A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \varphi(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t))& \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in V.&\end{aligned} \tag{11.30}$$

Applying Theorem 10.2, we have got the following result.

Theorem 11.1. *[Theorem 4.1 in [10]] If Assumptions* 11.1 *and* 11.3–11.6 *hold true, then Problem 11.3 has a unique solution, which satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; V), \qquad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q). \tag{11.31}$$

The proof of Theorem 11.1 can be found in [10].

In the study of the Problem 11.4 we obtained the following results.

Lemma 11.3. *[Lemma 4.5 in [10]] Assumptions* 11.2–11.5 *and* 11.7 *hold and let* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *be a couple of functions such that* $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ *and* $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$*. Then,* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *is a solution of Problem* 11.4 *if and only if for all* $t \in \mathbb{R}_+$*, the equality and the inequality below hold:*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{u}(t), \tag{11.32}$$

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q& \\ +(P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.&\end{aligned} \tag{11.33}$$

Using similar arguments with those used to prove the previous theorem we have got the following result.

Theorem 11.2. *[Theorem 4.2 in [10]] If Assumptions* 11.2–11.5 *and* 11.7 *hold true, then Problem 11.4 has a unique solution, which satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; U), \qquad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q). \tag{11.34}$$

The proof of Theorem 11.2 was given in [10].

11.2 A convergence result

Everywhere in this section we assume that the function p satisfies Assumption 11.2 and let q be a function which satisfies the following assumption.

Assumption 11.8. $q : [g, +\infty) \to \mathbb{R}_+$ *is a given function such that:*
There exists $L_q > 0 : |q(r_1) - q(r_2)| \le L_q |r_1 - r_2|$ *for all* $r_1, r_2 \ge g$.
$(q(r_1) - q(r_2))(r_1 - r_2) > 0$ *for all* $r_1, r_2 \ge g$, $r_1 \neq r_2$.
$q(g) = 0$.

Let $\mu > 0$ and consider the function p_μ defined by

$$p_\mu(r) = \begin{cases} p(r) & \text{if} \quad r \le g, \\ \frac{1}{\mu}\, q(r) + p(g) & \text{if} \quad r > g. \end{cases} \tag{11.35}$$

Using Assumptions 11.2 and 11.8 it follows that the function p_μ satisfies:

$$\begin{cases} p_\mu : \mathbb{R} \to \mathbb{R}_+. \\ \text{There exists } L_{p_\mu} > 0 \text{ such that} \\ \qquad |p_\mu(r_1) - p_\mu(r_2)| \le L_{p_\mu} |r_1 - r_2| \quad \text{for all } r_1, r_2 \in \mathbb{R}. \\ (p_\mu(r_1) - p_\mu(r_2))(r_1 - r_2) \ge 0 \quad \text{for all } r_1, r_2 \in \mathbb{R}. \\ p_\mu(r) = 0 \ \text{ for all } r < 0. \end{cases}$$

This allows us to consider the operator $P_\mu : V \to V$ defined by

$$(P_\mu \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\mu(u_\nu) v_\nu \, da \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V. \tag{11.36}$$

We note that P_μ is a monotone, Lipschitz continuous operator.

We also consider the contact problem with normal compliance and infinite penetration when the contact condition (11.6) is replaced with

$$-\sigma_\nu = p_\mu(u_\nu) \qquad \text{on } \Gamma_3 \times (0, \infty). \tag{11.37}$$

In this condition μ represents a penalization parameter which may be interpreted as a deformability of the foundation, and then $\frac{1}{\mu}$ is the surface stiffness coefficient. Indeed, when μ is smaller the reaction force of the foundation to penetration is larger and so the same force will result in a smaller penetration, which means that the foundation is less deformable. When μ is larger the reaction force of the foundation to penetration is smaller, and so the foundation is less stiff and more deformable.

The variational formulation of the problem with normal compliance and finite penetration associated to function p_μ is as follows.

Problem 11.5. *Find a displacement field* $\boldsymbol{u}_\mu : \mathbb{R}_+ \to V$ *and a stress field* $\boldsymbol{\sigma}_\mu : \mathbb{R}_+ \to Q$ *such that, for all* $t \in \mathbb{R}_+$, *the following equalities hold:*

$$\boldsymbol{\sigma}_\mu(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\mu(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),$$

$$(\boldsymbol{\sigma}_\mu(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P_\mu \boldsymbol{u}_\mu(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \qquad \text{for all } \boldsymbol{v} \in V.$$

It follows from Theorem 11.1 that Problem 11.5 has a unique solution $(\boldsymbol{u}_\mu, \boldsymbol{\sigma}_\mu)$ which satisfies (11.31). In addition, according to Theorem 11.2, Problem 11.4 has a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (11.34). The behavior of the solution $(\boldsymbol{u}_\mu, \boldsymbol{\sigma}_\mu)$ as $\mu \to 0$ is given in the following result.

Theorem 11.3. *[Theorem 5.1 in [10]] If Assumptions* 11.1–11.8 *hold, then the solution* $(\boldsymbol{u}_\mu, \boldsymbol{\sigma}_\mu)$ *of Problem* 11.5 *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *of Problem* 11.4, *that is*

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V + \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \tag{11.38}$$

as $\mu \to 0$, *for all* $t \in \mathbb{R}_+$.

For the proof of Theorem 11.3 we refer to [10].

In addition to the mathematical interest in the result above, this result is important from the mechanical point of view, since it shows that the weak solution of the viscoplastic contact problem with normal compliance and finite penetration may be approached as closely as one wishes by the solution of the viscoplastic contact problem with normal compliance and infinite penetration, with a sufficiently small deformability coefficient.

Remark 11.1. *A numerical validation of this convergence result can be found in Section 6 of the paper [10]. Fully discrete schemes for the numerical approximation of the contact problems were introduced and implemented. Finally, numerical simulations in the study of a two-dimensional example were presented.*

Chapter 12

Electro-elasto-viscoplastic contact problems

This chapter is based on the paper [20]. We consider a mathematical model which describes the quasistatic frictionless contact between a piezoelectric body and a foundation. The novelty of the model consists in the fact that the foundation is assumed to be electrically conductive, the material's behavior is described with an electro-elastic-visco-plastic constitutive law, the contact is modelled with normal compliance and finite penetration and the problem is studied on unbounded time interval. We derive a variational formulation of the problem and prove existence, uniqueness and regularity results.

12.1 The mechanical model

In this section we describe an electro-elastic-visco-plastic model in the following physical setting. An electro-elasto-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ. The body is subject to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. The boundary of the body is subjected to mechanical and electrical constraints. To describe the mechanical constraints we consider a partition of Γ into three measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. We assume that the body is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in contact with an electrically conductive obstacle. The contact is frictionless and is modelled with a version of the normal compliance condition with finite penetration, which takes into account the conductivity of the foundation. To describe the electrical constraints we consider a partition of $\Gamma_1 \cup \Gamma_2$ into two measurable sets Γ_a and Γ_b such that $\text{meas}(\Gamma_a) > 0$. We assume that the electrical potential vanishes on Γ_a and the surface electric charges of density q_b are prescribed on Γ_b. Also, during the process, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when contact is lost. We assume that the problem is quasistatic, and we study the problem in the interval of time $\mathbb{R}_+ = [0, \infty)$.

The classical formulation of the contact problem defined above, is as follows.

Problem 12.1. *Find a displacement field* $\boldsymbol{u}:\Omega\times\mathbb{R}_+\to\mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma}:\Omega\times\mathbb{R}_+\to\mathbb{S}^d$*, an electric potential field* $\varphi:\Omega\times\mathbb{R}_+\to\mathbb{R}$*, and an electric displacement field* $\boldsymbol{D}:\Omega\times\mathbb{R}_+\to\mathbb{R}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \mathcal{E}^*\boldsymbol{E}(\dot{\varphi}) + \mathcal{G}(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{D},\boldsymbol{E}(\varphi)) \quad \text{in} \quad \Omega\times(0,\infty), \tag{12.1}$$

$$\dot{\boldsymbol{D}} = \boldsymbol{\beta}\boldsymbol{E}(\dot{\varphi}) + \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{D},\boldsymbol{E}(\varphi),\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega\times(0,\infty), \tag{12.2}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega\times(0,\infty), \tag{12.3}$$

$$\mathit{div}\,\boldsymbol{D} = q_0 \quad \text{in} \quad \Omega\times(0,\infty), \tag{12.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1\times(0,\infty), \tag{12.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2\times(0,\infty), \tag{12.6}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a\times(0,\infty), \tag{12.7}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = q_b \quad \text{on} \quad \Gamma_b\times(0,\infty), \tag{12.8}$$

$$\left.\begin{array}{l} u_\nu \le g,\ \sigma_\nu + h_\nu(\varphi-\varphi_F)p_\nu(u_\nu) \le 0, \\ (u_\nu - g)(\sigma_\nu + h_\nu(\varphi-\varphi_F)p_\nu(u_\nu)) = 0, \end{array}\right\} \quad \text{on} \quad \Gamma_3\times(0,\infty), \tag{12.9}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_3\times(0,\infty), \tag{12.10}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = p_e(u_\nu)h_e(\varphi-\varphi_F) \quad \text{on} \quad \Gamma_3\times(0,\infty), \tag{12.11}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\ \boldsymbol{\sigma}(0)=\boldsymbol{\sigma}_0,\ \varphi(0)=\varphi_0,\ \boldsymbol{D}(0)=\boldsymbol{D}_0 \quad \text{in} \quad \Omega. \tag{12.12}$$

In order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x}\in\Omega\cup\Gamma$ and $t\in\mathbb{R}_+$. Equations (12.1) and (12.2) represent the electro-elasto-viscoplastic constitutive law of the material. Equations (12.3) and (12.4) are the equilibrium equations for the stress and the electric displacement fields, respectively. Conditions (12.5) and (12.6) are the displacement and traction boundary conditions, and conditions (12.7)–(12.8) represent the electric boundary conditions. Also, (12.12) represents the initial conditions in which $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, φ_0, and $\boldsymbol{D}_0$ denote the initial displacement, the initial stress, the initial electric potential field and the initial electric displacement field, respectively.

We turn to the boundary conditions (12.9)–(12.11) which describe the mechanical and electrical conditions on the potential contact surface Γ_3; there, $g>0$ is a given bound for the normal displacement and φ_F denotes the electric potential of the foundation.

First, (12.9) represents the normal compliance contact condition with finite penetration in which p_ν is a prescribed nonnegative function which vanishes when its argument is negative and

h_ν is a positive function, the stiffness coefficient. Recall that this condition, first introduced in [78] in the study of a purely mechanic contact problem, contains as particular cases both the Signorini contact condition and the classical normal compliance contact condition described, see for instance [59, 138]. We note that (12.9) shows that when there is no contact (i.e. when $u_\nu < 0$) then $\sigma_\nu = 0$ and, therefore, the normal pressure vanishes; when there is contact (i.e. when $u_\nu \geq 0$) then $\sigma_\nu \leq 0$ and, therefore, the reaction of the foundation is towards the body. The function g represents the maximum interpenetration of body's and foundations's asperities. Note also that the stiffness coefficient is assumed to depend on the difference between the potential of the foundation and the body's surface which is one of the novelties of the model.

Next, condition (12.10) shows that the tangential stress on the contact surface vanishes. We use it here since we assume that the contact process is frictionless. An important extension of the results in this paper would take into consideration frictional conditions on the contact surface Γ_3.

Finally, (12.11) is a regularized electrical contact condition on Γ_3, similar to that used in [8, 9, 90]. Here p_e represents the electrical conductivity coefficient, which vanishes when its argument is negative, and h_e is a given function. Condition (12.11) shows that when there is no contact at a point on the surface (i.e. when $u_\nu < 0$), then the normal component of the electric displacement field vanishes, and when there is contact (i.e. when $u_\nu \geq 0$) then there may be electrical charges which depend on the difference between the potential of the foundation and the body's surface. Note also that if the foundation is assumed to be insulated then there are no charges on Γ_3 during the process and, therefore, $\boldsymbol{D} \cdot \boldsymbol{\nu} = 0$ on $\Gamma_3 \times (0, \infty)$. This condition can be recovered from (12.11) by taking $p_e \equiv 0$.

Note that in (12.1)–(12.12) the coupling between the mechanical unknowns $(\boldsymbol{u}, \boldsymbol{\sigma})$ and the electrical unknowns $(\varphi, \boldsymbol{D})$ arises both in the constitutive equations (12.1)–(12.2) and the contact conditions (12.9)–(12.11). This feature of the problem (12.1)–(12.12) is a consequence of the assumption that the foundation is conductive. It represents one of the differences with respect to the model treated in [60] and leads to additional mathematical difficulties.

12.2 Weak formulation and main results

In this section we discuss the weak solvability of the electro-elastic-visco-plastic model (12.1)–(12.12). We assume that the elasticity tensor, the piezoelectric tensor and the electric permittivity tensor satisfy the following conditions.

Assumption 12.1. $\mathcal{A} = (\mathcal{A}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$.

$\mathcal{A}_{ijkl} = \mathcal{A}_{klij} = \mathcal{A}_{jikl} \in L^\infty(\Omega),\ 1 \leq i, j, k, l \leq d.$

There exists $m_\mathcal{A} > 0$ such that $\mathcal{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_\mathcal{A} \|\boldsymbol{\tau}\|^2$ *for all* $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. in Ω.

Assumption 12.2. $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$.

$e_{ijk} = e_{ikj} \in L^\infty(\Omega),\ 1 \leq i, j, k \leq d.$

Assumption 12.3. $\boldsymbol{\beta} = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$.

$\beta_{ij} = \beta_{ji} \in L^\infty(\Omega),\ 1 \le i, j \le d.$

There exists $m_\beta > 0$ *such that* $\boldsymbol{\beta}\boldsymbol{E} \cdot \boldsymbol{E} \ge m_\beta \|\boldsymbol{E}\|^2$ *for all* $\boldsymbol{E} \in \mathbb{R}^d$, *a.e.* $\boldsymbol{x} \in \Omega$.

Assumption 12.4. $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{S}^d$.

There exists $L_{\mathcal{G}} > 0$ such that

$\|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \varepsilon_1, \boldsymbol{D}_1, \boldsymbol{E}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \boldsymbol{D}_2, \boldsymbol{E}_2)\| \le L_{\mathcal{G}}\,(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\varepsilon_1 - \varepsilon_2\| + \|\boldsymbol{D}_1 - \boldsymbol{D}_2\| + \|\boldsymbol{E}_1 - \boldsymbol{E}_2\|)$ *for all* $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, $\boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega$.

The mapping $\boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{E})$ is measurable on Ω, for any $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$ and $\boldsymbol{D}, \boldsymbol{E} \in \mathbb{R}^d$.

The mapping $\boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ belongs to Q.

Assumption 12.5. $G : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}^d$.

There exists $L_G > 0$ such that

$\|G(\boldsymbol{x}, \boldsymbol{D}_1, \boldsymbol{E}_1, \boldsymbol{\sigma}_1, \varepsilon_1) - G(\boldsymbol{x}, \boldsymbol{D}_2, \boldsymbol{E}_2, \boldsymbol{\sigma}_2, \varepsilon_2)\| \le L_G\,(\|\boldsymbol{D}_1 - \boldsymbol{D}_2\| + \|\boldsymbol{E}_1 - \boldsymbol{E}_2\| + \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|)$

for all $\boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathbb{R}^d$, $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$.

The mapping $\boldsymbol{x} \mapsto G(\boldsymbol{x}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ is measurable on Ω, for any $\boldsymbol{D}, \boldsymbol{E} \in \mathbb{R}^d$ and $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$.

The mapping $\boldsymbol{x} \mapsto G(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ belongs to $L^2(\Omega)^d$.

These assumptions are reasonable from physical point of view, see for instance [45, 59, 73, 138, 150]. In some applications, $\mathcal{G}$ and G are linear functions.

The functions p_r and h_r (for $r = \nu$, e) are such that the following assumptions hold true.

Assumption 12.6. $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$.

There exists $L_r > 0$ *such that*

$|p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \le L_r |u_1 - u_2|$ *for all* $u_1, u_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

There exists $\overline{p}_r > 0$ *such that* $0 \le p_r(\boldsymbol{x}, u) \le \overline{p}_r$ *for all* $u \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

The mapping $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u)$ *is measurable on* Γ_3, *for any* $u \in \mathbb{R}$.

$p_r(\boldsymbol{x}, u) = 0$ *for all* $u \le 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.

Assumption 12.7. $h_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$.

There exists $l_r > 0$ *such that*

$|h_r(\boldsymbol{x}, \varphi_1) - h_r(\boldsymbol{x}, \varphi_2)| \le l_r |\varphi_1 - \varphi_2|$ *for all* $\varphi_1, \varphi_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

There exists $\overline{h}_\nu > 0$ *such that* $0 \le h_\nu(\boldsymbol{x}, \varphi) \le \overline{h}_\nu$ *for all* $\varphi \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

There exists $\overline{h}_e > 0$ *such that* $|h_e(\boldsymbol{x}, \varphi)| \le \overline{h}_e$ *for all* $\varphi \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

The mapping $\boldsymbol{x} \mapsto h_r(\boldsymbol{x}, u)$ *is measurable on* Γ_3, *for any* $\varphi \in \mathbb{R}$.

We also assume that the bound of the normal displacement and the electrical potential of the foundation are as follows.

Assumption 12.8. $g \in L^2(\Gamma_3)$, $g \geq 0$ a.e. on Γ_3.

Assumption 12.9. $\varphi_F \in L^2(\Gamma_3)$.

Moreover, the density of the body forces and tractions, the volume and surface electric charge densities have the following regularity.

Assumption 12.10. $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)$, $\quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d)$,

Assumption 12.11. $q_0 \in C(\mathbb{R}_+; L^2(\Omega))$, $\quad q_b \in C(\mathbb{R}_+; L^2(\Gamma_b))$.

Finally, the initial data satisfy the following assumptions.

Assumption 12.12. $\boldsymbol{u}_0 \in U$, $\quad \boldsymbol{\sigma}_0 \in Q_1$,

Assumption 12.13. $\varphi_0 \in W$, $\quad \boldsymbol{D}_0 \in \mathcal{W}_1$.

Notice that U denotes the set of admissible displacements defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \leq g \text{ on } \Gamma_3 \,\}. \tag{12.13}$$

Alternatively, we assume that there exists $p \in [1, \infty]$ such that

Assumption 12.14. $\boldsymbol{f}_0 \in W^{1,p}_{loc}(\mathbb{R}_+; L^2(\Omega)^d)$, $\quad \boldsymbol{f}_2 \in W^{1,p}_{loc}(\mathbb{R}_+; L^2(\Gamma_2)^d)$,

Assumption 12.15. $q_0 \in W^{1,p}_{loc}(\mathbb{R}_+; L^2(\Omega))$, $\quad q_b \in W^{1,p}_{loc}(\mathbb{R}_+; L^2(\Gamma_b))$.

Besides Assumptions 12.12–12.13, the initial data satisfy the following compatibility conditions:

Assumption 12.16. $(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_0))_Q + J_\nu(\varphi_0, \boldsymbol{u}_0, \boldsymbol{v} - \boldsymbol{u}_0) \geq (\boldsymbol{f}(0), \boldsymbol{v} - \boldsymbol{u}_0)_V$ *for all* $\boldsymbol{v} \in U$,
$(\boldsymbol{D}_0, \nabla\psi)_{L^2(\Omega)^d} + (q(0), \psi)_W = J_e(\boldsymbol{u}_0, \varphi_0, \psi)$ *for all* $\psi \in W$.

Here and below $J_\nu : W \times V \times V \to \mathbb{R}$ and $J_e : V \times W \times W \to \mathbb{R}$ denote the functionals given by

$$J_\nu(\varphi, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_F) p_\nu(u_\nu) v_\nu \, da \tag{12.14}$$

$$J_e(\boldsymbol{u}, \varphi, \psi) = \int_{\Gamma_3} p_e(u_\nu) h_e(\varphi - \varphi_F) \psi \, da, \tag{12.15}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\varphi, \psi \in W$.

We consider the functions $\boldsymbol{f} : \mathbb{R}_+ \to V$ and $q : \mathbb{R}_+ \to W$ defined by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da \quad \text{for all } \boldsymbol{v} \in V,\ t \in \mathbb{R}_+, \tag{12.16}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_b} q_b(t) \psi \, da \quad \text{for all } \psi \in W,\ t \in \mathbb{R}_+. \tag{12.17}$$

We have the following variational formulation of Problem 12.1.

Problem 12.2. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$, a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q_1$, an electric potential field $\varphi : \mathbb{R}_+ \to W$ and an electric displacement field $\boldsymbol{D} : \mathbb{R}_+ \to \mathcal{W}_1$ such that*

$$\begin{aligned}\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{E}^*\boldsymbol{E}(\varphi(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{D}(s), \boldsymbol{E}(\varphi(s)))\, ds \qquad (12.18)\\ +\boldsymbol{\sigma}_0 - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \mathcal{E}^*\boldsymbol{E}(\varphi_0),\end{aligned}$$

$$\begin{aligned}\boldsymbol{D}(t) = \boldsymbol{\beta}\boldsymbol{E}(\varphi(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t G(\boldsymbol{D}(s), \boldsymbol{E}(\varphi(s)), \boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \qquad (12.19)\\ +\boldsymbol{D}_0 - \boldsymbol{\beta}\boldsymbol{E}(\varphi_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + J_\nu(\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - J_\nu(\varphi(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \qquad (12.20)\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in U,\end{aligned}$$

$$(\boldsymbol{D}(t), \nabla\psi)_{L^2(\Omega)^d} + (q(t), \psi)_W = J_e(\boldsymbol{u}(t), \varphi(t), \psi) \quad \text{for all } \psi \in W, \qquad (12.21)$$

for all $t \in \mathbb{R}_+$.

We consider the spaces $X = V \times W$, $Y = Q \times L^2(\Omega)^d$, together with the canonical inner products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$ and the associated norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. In addition, for the convenience of the reader we shall use the short hand notation

$$\begin{aligned}\widetilde{\mathcal{G}}(\boldsymbol{u}, \varphi, \boldsymbol{\sigma}, \boldsymbol{D}) = \mathcal{G}\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi) + \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{E}(\varphi), \boldsymbol{\beta}\boldsymbol{E}(\varphi) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{D}\right),\\ \widetilde{G}(\boldsymbol{u}, \varphi, \boldsymbol{\sigma}, \boldsymbol{D}) = G\left(\boldsymbol{\beta}\boldsymbol{E}(\varphi) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{D}, \boldsymbol{E}(\varphi), \boldsymbol{\varepsilon}(\boldsymbol{u}), \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi) + \boldsymbol{\sigma}\right).\end{aligned}$$

Lemma 12.1. *[Lemma 5.1 in [20]] For all $(\boldsymbol{u}, \varphi) \in C(\mathbb{R}_+; X)$ there exists a unique couple of functions $(\boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi), \boldsymbol{D}^I(\boldsymbol{u}, \varphi)) \in C^1(\mathbb{R}_+; Y)$ such that, for all $t \in \mathbb{R}_+$, the following equalities hold:*

$$\begin{aligned}\boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi)(t) = \int_0^t \widetilde{\mathcal{G}}(\boldsymbol{u}(s), \varphi(s), \boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi)(s), \boldsymbol{D}^I(\boldsymbol{u}, \varphi)(s))\, ds \qquad (12.22)\\ +\boldsymbol{\sigma}_0 - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \mathcal{E}^*\boldsymbol{E}(\varphi_0),\end{aligned}$$

$$\begin{aligned}\boldsymbol{D}^I(\boldsymbol{u}, \varphi)(t) = \int_0^t \widetilde{G}(\boldsymbol{u}(s), \varphi(s), \boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi)(s), \boldsymbol{D}^I(\boldsymbol{u}, \varphi)(s))\, ds \qquad (12.23)\\ +\boldsymbol{D}_0 - \boldsymbol{\beta}\boldsymbol{E}(\varphi_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0).\end{aligned}$$

Lemma 12.1 allows us to consider the operator $\mathcal{S} : C(\mathbb{R}_+, X) \to C^1(\mathbb{R}_+, Y)$ defined by

$$\mathcal{S}(x) = \left(\boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi), -\boldsymbol{D}^I(\boldsymbol{u}, \varphi)\right) \quad \forall\, x = (\boldsymbol{u}, \varphi) \in C(\mathbb{R}_+, X). \qquad (12.24)$$

Moreover, it leads to the following equivalence result.

Lemma 12.2. *[Lemma 5.2 in [20]] A quadruple of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{D}, \varphi)$ which satisfy $\boldsymbol{u} \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$, $\varphi \in C(\mathbb{R}_+; W)$, $\boldsymbol{D} \in C(\mathbb{R}_+; \mathcal{W}_1)$ is a solution of Problem* 12.2 *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t) + \boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi)(t), \tag{12.25}$$

$$\boldsymbol{D}(t) = -\boldsymbol{\beta}\nabla\varphi(t) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \boldsymbol{D}^I(\boldsymbol{u}, \varphi)(t), \tag{12.26}$$

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &\quad + (\boldsymbol{\sigma}^I(\boldsymbol{u}, \varphi)(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + J_\nu(\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - J_\nu(\varphi(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \\ &\qquad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U,\end{aligned} \tag{12.27}$$

$$\begin{aligned}&(\boldsymbol{\beta}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} - \big(\boldsymbol{D}^I(\boldsymbol{u}, \varphi)(t), \nabla\psi\big)_{L^2(\Omega)^d} \\ &\quad + J_e(\boldsymbol{u}(t), \varphi(t), \psi) = (q(t), \psi)_W \quad \forall\, \psi \in W,\end{aligned} \tag{12.28}$$

for all $t \in \mathbb{R}_+$.

To proceed, we consider the set $K = U \times W$, the operator $A : X \to X$, the functionals $\varphi : Y \times X \to \mathbb{R}$ and $j : X \times X \to \mathbb{R}$, and the function $f : \mathbb{R}_+ \to X$ defined by

$$\begin{aligned}(Ax, y)_X = {} & (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \\ & - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} + (\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d},\end{aligned} \tag{12.29}$$

$$\varphi(z, x) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + (\boldsymbol{D}, \nabla\varphi)_{L^2(\Omega)^d}, \tag{12.30}$$

$$j(x, y) = J_\nu(\varphi, \boldsymbol{u}, \boldsymbol{v}) + J_e(\boldsymbol{u}, \varphi, \psi), \tag{12.31}$$

$$f = (\boldsymbol{f}, q), \tag{12.32}$$

for all $x = (\boldsymbol{u}, \varphi)$, $y = (\boldsymbol{v}, \psi) \in X$ and $z = (\boldsymbol{\sigma}, \boldsymbol{D}) \in Y$. Note that the definition of the operator A follows by using Riesz's representation theorem.

The next step is provided by the following result.

Lemma 12.3. *[Lemma 5.3 in [20]] Let $t \in \mathbb{R}_+$, $\boldsymbol{u} \in C(\mathbb{R}_+, U)$, $\varphi \in C(\mathbb{R}_+, W)$ and denote $x = (\boldsymbol{u}, \varphi) \in C(\mathbb{R}_+, K)$. Then* (12.27)–(12.28) *hold if and only if $x(t)$ satisfies the inequality*

$$\begin{aligned}&(Ax(t), y - x(t))_X + \varphi(\mathcal{S}x(t), y) - \varphi(\mathcal{S}x(t), x(t)) \\ &\quad + j(x(t), y) - j(x(t), x(t)) \geq (f(t), x(t) - y)_X \quad \forall\, y \in K.\end{aligned} \tag{12.33}$$

We continue with the following existence and uniqueness result.

Lemma 12.4. *[Lemma 5.4 in [20]] There exists $L_0 > 0$ which depends on Ω, Γ_1, Γ_3, $\mathcal{A}$ and $\boldsymbol{\beta}$ such that there exists a unique function $x \in C(\mathbb{R}_+, K)$ which satisfies the inequality* (12.33) *for all $t \in \mathbb{R}_+$, if $\overline{h}_\nu L_\nu + \overline{h}_e L_e + \overline{p}_\nu l_\nu + \overline{p}_e l_e < L_0$.*

Based on these preliminaries steps, we have got the following results in the study of Problem 12.2 .

Theorem 12.1. *[Theorem 4.1 in [20]] If Assumptions* 12.1–12.13 *hold true, then there exists* $L_0 > 0$ *which depends on* Ω, Γ_1, Γ_3, $\mathcal{A}$ *and* $\boldsymbol{\beta}$ *such that Problem 12.2 has a unique solution, if*

$$\overline{h}_\nu L_\nu + \overline{h}_e L_e + \overline{p}_\nu l_\nu + \overline{p}_e l_e < L_0. \tag{12.34}$$

Moreover, the solution satisfies

$$\boldsymbol{u} \in C(\mathbb{R}_+; U), \quad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1), \quad \varphi \in C(\mathbb{R}_+; W), \quad \boldsymbol{D} \in C(\mathbb{R}_+; \mathcal{W}_1). \tag{12.35}$$

Theorem 12.2. *[Theorem 4.2 in [20]] If the inequality* (12.34) *and Assumptions* 12.1–12.13 *hold true, denoting by* $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ *the solution of Problem 12.2 obtained in Theorem* 12.1, *then:*

1) *Under Assumptions* 12.14–12.15, *the solution has the regularity*

$$\boldsymbol{u} \in W^{1,p}_{loc}(\mathbb{R}_+; U), \;\; \boldsymbol{\sigma} \in W^{1,p}_{loc}(\mathbb{R}_+; Q_1), \;\; \varphi \in W^{1,p}_{loc}(\mathbb{R}_+; W), \;\; \boldsymbol{D} \in W^{1,p}_{loc}(\mathbb{R}_+, \mathcal{W}_1) \tag{12.36}$$

and the following equalities hold, for almost any $t \in \mathbb{R}_+$*:*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \mathcal{E}^*\boldsymbol{E}(\dot{\varphi}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{D}(t), \boldsymbol{E}(\varphi(t))), \tag{12.37}$$

$$\dot{\boldsymbol{D}}(t) = \boldsymbol{\beta}\boldsymbol{E}(\dot{\varphi}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{D}(t), \boldsymbol{E}(\varphi(t)), \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))). \tag{12.38}$$

2) *Under Assumption 12.16, the solution satisfies the initial conditions*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \;\; \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \;\; \varphi(0) = \varphi_0, \;\; \boldsymbol{D}(0) = \boldsymbol{D}_0. \tag{12.39}$$

A quadruple of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ which satisfies (12.18)–(12.21) for all $t \in \mathbb{R}_+$ is called a weak solution to the piezoelectric contact Problem 12.1. We conclude that Theorem 12.1 provides the unique weak solvability of Problem 12.1 and Theorem 12.2 provides a regularity result of its weak solution.

Note that condition (12.34) represents a smallness assumption on the functions involved in the boundary conditions of Problem 12.1. It is satisfied if, for instance, either the quantities $\overline{p}_\nu$, $\overline{h}_\nu$, $\overline{p}_e$, $\overline{h}_e$, or the quantities L_ν, L_e, l_ν, l_e are small enough. And, this means that either the range of the functions p_ν, p_e, h_ν, h_e, or the range of their derivatives with respect the second variable (which exists, a.e.), is small enough. We conclude that the result in Theorem 12.1 works in the case when either the normal compliance function, the stiffness coefficient, the electrical conductivity coefficient and the electric charge function are small enough, or their variation is small enough.

The proofs of Theorems 12.1 and 12.2, given in [20], are based on the abstract result, Theorem 10.2.

CAREER EVOLUTION AND DEVELOPMENT PLANS

Chapter 13

Further research directions

The overall goal is to improve the understanding of real-world problems governed by Partial Differential Equations. The mathematical modeling via partial differential equations is foundational to the field of mechanical engineering; it provides necessary information for efficient design of technical systems. In particular, the contact models are used to analyze and test complex industrial systems. Since it is not possible to find strong solutions for complex problems, a good alternative is the weak solvability which allows to built efficient numerical approximations for the weak solutions. This is a motivation for the candidate to continue to do research on the direction of calculus of variations with applications in contact mechanics.

The most relevant further research directions envisaged from the candidate are the following:

- *Qualitative and numerical analysis in the study of mixed variational problems* The candidate intend to improve and extend the results in the papers [11, 68, 69, 70, 98, 99, 100, 101, 104, 105, 107, 111, 113]. Delivering uniqueness/multiplicity results, is one of the targets; the following references can be helpful, [33, 120]. How we can approximate the solutions is also of great interest. Regularization or perturbation techniques are envisaged. Also the candidate is interested to the solvability of a class of mixed variational problems via hemivariational inequalities theory. The notion of hemivariational inequality was introduced in [128] based on the properties of generalized gradient introduced and studied in [37, 38]. During the last two decades, a large number of works were devoted to the hemivariational inequalities theory related to contact models; for a contribution of the candidate in the field see the papers [4, 40, 41]. One target of the candidate is to extend and improve the results obtained in the paper [107], by replacing into the mixed variational system the variational inequality with a hemivariational inequality. Such a study allows to investigate contact models with non-convex potentials via calculus of variations with Lagrange multipliers. The analysis of contact problems with adhesion or damage, via mixed variational formulations, is also under attention; the following references can be helpful [140, 141, 142].

- *Qualitative and numerical analysis in the study of variational systems via bipotentials* The candidate intends to improve and extend the results in [102, 106, 108] to more complex variational systems via bipotentials (possibly non-separated). In particular time-dependent/evolutionary models are envisaged. Numerical algorithms are also of interest for the candidate.

- *Variational formulations/ weak solutions via weighted Sobolev spaces in contact mechanics* A first step was already done, see [19]; see also [56, 89] for some fundamental mathematical tools. The next steps will be related to the weak solvability of complex contact models for various kind of materials; e.g. piezoelectric problems with singularities and degeneracies (the following references can be helpful: [6, 9, 14, 17, 58, 60, 90, 123]).

- *Variational formulations/ weak solutions via Lebesgue spaces with variable exponent, in contact mechanics* A first step was made in [21] for regularized antiplane contact problems governed by nonlinearly elastic materials of Henky type. The candidate intends to improve and extends the previous study to the non-regularized case. Also, the study of a class of non-newtonian fluids is of interest for the candidate. The references [47, 51, 52, 53, 54, 86] can be useful.

- *Variational formulations/ weak solutions in contact mechanics, for materials with dry porosity.* The study of the behavior of non-classical materials (like materials with voids, porous materials with dry porosity) is a challenging topic. To start, the candidate intends to consult the following works [34, 35, 43, 71, 77, 126]. In the future the candidate envisages to investigate the behavior of poro-therm materials, such a study being motivated by the large applicability of such kind of materials.

- *Optimal control problems in contact mechanics* For the optimal control of variational inequalities we can refer for instance to [12, 18, 55, 91, 118, 119, 149]. Moreover, the recent book [125] is devoted to the optimal control of linear or nonlinear elliptic problems, including variational inequalities. Despite their mechanical relevance, optimal control problems for contact models are not so frequent in the literature, the contact problems being strongly nonlinear problems. The main aim of the candidate is to study optimal control problems which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. In the paper [103] a first step was already done. Moreover, the candidate intends to study the optimal control for abstract variational problems related to contact models, such as variational problems with Lagrange multipliers and variational systems via bipotentials.

- *Mathematical study for dissipative dynamic contact problems* At this item, the main interest lies in existence and uniqueness results for dynamic contact problems in elasticity, which are dissipative. To give an example, we can consider a rod which is connected to a dashpot at its left end and, at its right end it can come in contact with an obstacle; the obstacle can be deformable (in such a case we have to use a normal compliance contact condition) or rigid (in this case we have to use the unilateral Signorini's contact condition. The techniques in the paper [82] and the references therein can be helpful. A first step was already done, see the conference paper [46].

- *Asymptotic analysis in contact mechanics* We can found in the literature some stability results for displacement-traction problems, see for example [73]; but not for contact problems. Due to their nonlinearity, the contact problems are difficult to be investigated from the point of view of the asymptotic analysis. However there are some premises. For instance, the papers [10, 20, 144, 146] were devoted to the existence and the uniqueness of the solution of a class of contact problems on the unbounded time interval $[0, \infty)$. Using the weak solutions at all moments $t \in [0, \infty)$, one idea is to define a dynamical system, to seek for it the equilibrium points and to use a technique via Lyapunov functionals in order to study the asymptotic behavior.

- *Regularity results* There are very few regularity results for contact problems. The field is wide open and progress is likely to be slow. Let us give an example of a regularity result we focus on. It is known that in the mixed variational approach, the weak solutions of contact problems are pairs (u, λ), $\lambda \in D$ where D is a dual space, see e.g. $D = S'$, $S = \{\gamma v|_{\Gamma_3} \,|\, v \in H^1(\Omega), \gamma v = 0 \text{ a.e. on } \Gamma_1\}$ in Section 2.2 of the present manuscript. However, the numerical analysis requires $L^2(\Gamma_3)$-regularity for λ and, currently, this is an open problem. The techniques in the book [49] and the references therein can be helpful.

- *Convergence results* Using similar techniques with those used in [11], the candidate intends to investigate the convergence of the solutions of some regularized or perturbed problem to the solution of the originate problem.

- *Viscoelastic problems via fractional differential operators* Fractional order operators are suitable to model memory effects of various materials and systems of technical interest. In particular, they can help to model viscoelastic materials, see e.g. [7, 31]. We also refer to [2] for an efficient numerical method to integrate the constitutive law of fractional order viscoelasticity. The fractional order derivatives were used to conceive a new component *spring-pot* that interpolates between pure elastic and viscous behaviors. In [44] the authors modified a standard linear solid model replacing a dashpot with a spring-pot of order α; the fractional model was tested in human arterial segments. The candidate intends to explore the weak formulations/weak solvability for spring-pot models.

  More general, the candidate will focus on fractional calculus of variations, including weak solutions of fractional partial differential equations. This topic began to be developed starting with 1996 in order to better describe non-conservative mechanical systems. Currently, the list of applications includes material sciences and mechanics of fractal and complex media, see e.g. [32, 93, 94], just to mention a few.

Let us pick up *a few open problems*.

- In Section 7.1 it was discussed the following abstract problem: given $f \in X$, find $(u,\lambda) \in X \times Y$ such that $\lambda \in \Lambda \subset Y$ and

$$\begin{aligned} J(v) - J(u) + b(v-u,\lambda) + j(\lambda,v) - j(\lambda,u) &\geq (f,\, v-u)_X \quad &&\text{for all } v \in X, \\ b(u,\mu-\lambda) &\leq 0 &&\text{for all } \mu \in \Lambda. \end{aligned}$$

According to Theorem 7.2, this problem has a solution, unique in its first argument.

Let us draw the attention on a few points of interest.

1. Under the assumptions made in Section 7.1 the uniqueness/the multiplicity of the solution in the second argument is an open question.
2. Let us focus on Assumptions 6.20 and 7.1 in the present manuscript; herein $J : X \to [0,\infty)$ is a Gâteaux differentiable functional and for all $\eta \in \Lambda$, $j(\eta,\cdot) : X \to [0,\infty)$ is a convex Gâteaux differentiable functional. The approach adopted in previous work essentially relies on this two hypotheses. The proof of the existence of the solutions (u,λ) for a non-differential functional J and a non-differential, in the second argument, bifunctional j is of great interest from the mathematical point of view as well as from the applications point of view, such a mathematical problem being connected to more complex models with a better physical significance. The uniqueness/the multiplicity of the solution in the second argument in a "non-differential framework" is also interesting.
3. to write an efficient approximating algorithm is also an unsolved problem at this moment.

- In Subsection 5.1.2 it was formulated the following mixed problem: find $(\boldsymbol{u},\varphi,\lambda) \in X \times Y \times \Lambda$ such that

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + e(\boldsymbol{v},\varphi) + b(\boldsymbol{v},\lambda) &= (\boldsymbol{f},\boldsymbol{v})_X, \qquad &&\boldsymbol{v} \in X, \\ c(\varphi,\psi) - e(\boldsymbol{u},\psi) + j(\lambda,\varphi,\psi) &= (q,\psi)_Y, &&\psi \in Y, \\ b(\boldsymbol{u},\mu-\lambda) &\leq 0, &&\mu \in \Lambda. \end{aligned}$$

This variational formulation correspond to a frictionless unilateral contact model for electro-elastic material. Let us mention a few points of interest here:

1. a better regularity of λ (L^2-regularity)
2. to consider the frictional case; in this case existence, uniqueness, stability results are expected and a numerical approach is also envisaged.

- In Part II of the present manuscript we discussed the weak solvability for a class of contact problems via bipotentials theory. The following two variational problems were formulated.

  (1) Find $\boldsymbol{u} \in U_0 \subset V$ and $\boldsymbol{\sigma} \in \Lambda \subset L^2_s(\Omega)^{3\times 3}$ such that

$$\begin{aligned} b(\boldsymbol{v},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) &\geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V \quad && \text{for all } \boldsymbol{v} \in U_0 \\ b(\boldsymbol{u},\boldsymbol{\mu}) - b(\boldsymbol{u},\boldsymbol{\sigma}) &\geq 0 && \text{for all } \boldsymbol{\mu} \in \Lambda, \end{aligned}$$

  see Problem 8.2, and

  (2) Find $\boldsymbol{u} \in V$ and $\boldsymbol{\sigma} \in \Lambda \subset L^2_s(\Omega)^{3\times 3}$ such that

$$\begin{aligned} b(\boldsymbol{v},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) + j(\boldsymbol{v}) - j(\boldsymbol{u}) &\geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V \quad && \text{for all } \boldsymbol{v} \in V \\ b(\boldsymbol{u},\boldsymbol{\mu}) - b(\boldsymbol{u},\boldsymbol{\sigma}) &\geq 0 && \text{for all } \boldsymbol{\mu} \in \Lambda, \end{aligned}$$

  see Problem 9.2 in the present manuscript.

  To solve such kind of variational problems in an abstract framework for "non-separated" forms b is of great interest in the next period.

- In Section 12.1 it was discussed the following model: find a displacement field $\boldsymbol{u} : \Omega\times\mathbb{R}_+ \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, and an electric displacement field $\boldsymbol{D} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ such that

$$\dot{\boldsymbol{\sigma}} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \mathcal{E}^*\boldsymbol{E}(\dot{\varphi}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{D}, \boldsymbol{E}(\varphi)) \quad \text{in} \quad \Omega\times(0,\infty),$$

$$\dot{\boldsymbol{D}} = \boldsymbol{\beta}\boldsymbol{E}(\dot{\varphi}) + \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{D}, \boldsymbol{E}(\varphi), \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in} \quad \Omega\times(0,\infty),$$

$$\begin{aligned} \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 &= \boldsymbol{0} \quad && \text{in} \quad \Omega\times(0,\infty), \\ \operatorname{div}\boldsymbol{D} &= q_0 && \text{in} \quad \Omega\times(0,\infty), \end{aligned}$$

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0} \quad && \text{on} \quad \Gamma_1\times(0,\infty), \\ \boldsymbol{\sigma\nu} &= \boldsymbol{f}_2 && \text{on} \quad \Gamma_2\times(0,\infty), \end{aligned}$$

$$\begin{aligned} \varphi &= 0 \quad && \text{on} \quad \Gamma_a\times(0,\infty), \\ \boldsymbol{D}\cdot\boldsymbol{\nu} &= q_b && \text{on} \quad \Gamma_b\times(0,\infty), \\ &\left.\begin{aligned} &u_\nu \leq g,\ \sigma_\nu + h_\nu(\varphi-\varphi_F)p_\nu(u_\nu) \leq 0, \\ &(u_\nu - g)(\sigma_\nu + h_\nu(\varphi-\varphi_F)p_\nu(u_\nu)) = 0, \end{aligned}\right\} && \text{on} \quad \Gamma_3\times(0,\infty), \\ \boldsymbol{\sigma}_\tau &= \boldsymbol{0} && \text{on} \quad \Gamma_3\times(0,\infty), \\ \boldsymbol{D}\cdot\boldsymbol{\nu} &= p_e(u_\nu)h_e(\varphi-\varphi_F) && \text{on} \quad \Gamma_3\times(0,\infty), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0,\ \varphi(0) &= \varphi_0,\ \boldsymbol{D}(0) = \boldsymbol{D}_0 && \text{in} \quad \Omega. \end{aligned}$$

  We emphasize that this is a *frictionless* problem. To study *the frictional case* is an interesting continuation of the present work.

- In Chapter 1 of the present thesis (Subsections 1.1.2 and 1.2.3), *a priori error estimates* were presented for a class of piezoelectric contact problems, see [69, 68] for details; also, in [69, 68], *efficient algorithms* to approximate the weak solutions were described. Such kind of results are welcome in order to continue the study of the generalized saddle point problems described in Chapters 2-7 of the present manuscript, firstly for the stationary problems and nextly for the time-dependent or evolutionary problems.

- In the paper [4] it was studied the following mathematical model: find $u,\, \varphi : \bar{\Omega} \to \mathbf{R}$ such that

$$(\mathbf{P}) : \begin{cases} \mathrm{div}\,(\mu(\boldsymbol{x})\nabla u(\boldsymbol{x}) + e(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})) + f_0(\boldsymbol{x}) = 0 & \text{in } \Omega, \\ \mathrm{div}\,(e(\boldsymbol{x})\nabla u(\boldsymbol{x}) - \beta(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})) = q_0(\boldsymbol{x}) & \text{in } \Omega, \\ u(\boldsymbol{x}) = 0 & \text{on } \Gamma_1, \\ \varphi(\boldsymbol{x}) = 0 & \text{on } \Gamma_A, \\ \mu(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) + e(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) = f_2(\boldsymbol{x}) & \text{on } \Gamma_2, \\ e(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) - \beta(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) = q_B(\boldsymbol{x}) & \text{on } \Gamma_B, \\ -\mu(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) - e(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) \in h(\boldsymbol{x}, u(\boldsymbol{x}))\partial j(\boldsymbol{x}, u(\boldsymbol{x})) & \text{on } \Gamma_3, \\ e(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) - \beta(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) \in \bar{\partial}\varphi(\boldsymbol{x}, \varphi(\boldsymbol{x}) - \varphi_F(\boldsymbol{x})) & \text{on } \Gamma_3. \end{cases}$$

This model describes the antiplane shear deformation of a piezoelectric cylinder in frictional contact with a conductive foundation. The study was made under the following assumptions.

(**H1**): $\mu \in L^\infty(\Omega)$, $\beta \in L^\infty(\Omega)$, $e \in L^\infty(\Omega)$. There exist $\beta^*, \mu^* \in \mathbf{R}$ such that $\beta(\boldsymbol{x}) \geq \beta^* > 0$ and $\mu(\boldsymbol{x}) \geq \mu^* > 0$ almost everywhere in Ω.

(**H2**): $f_0 \in L^2(\Omega)$, $q_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)$, $q_B \in L^2(\Gamma_B)$, $\varphi_F \in L^\infty(\Gamma_3)$.

(**H3**): $h : \Gamma_3 \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function (i.e. $h(\cdot, t) : \Gamma_3 \to \mathbf{R}$ is measurable, for all $t \in \mathbf{R}$, and $h(\boldsymbol{x}, \cdot) : \mathbf{R} \to \mathbf{R}$ is continuous, a.e. on Γ_3). There exists a positive constant h_0 such that $0 \leq h(\boldsymbol{x}, t) \leq h_0$, for all $t \in \mathbf{R}$, a.e. on Γ_3.

(**H4**): $j : \Gamma_3 \times \mathbf{R} \to \mathbf{R}$ is a function which is measurable with respect to the first variable, and there exists $k \in L^2(\Gamma_3)$ such that a.e. on Γ_3 and for all $t_1, t_2 \in \mathbf{R}$ we have

$$|j(\boldsymbol{x}, t_1) - j(\boldsymbol{x}, t_2)| \leq k(\boldsymbol{x})|t_1 - t_2|.$$

(**H5**): $\varphi : \Gamma_3 \times \mathbf{R} \to \mathbf{R}$ is a functional such that $\varphi(\cdot, t) : \Gamma_3 \to \mathbf{R}$ is measurable for each $t \in \mathbf{R}$ and $\varphi(\boldsymbol{x}, \cdot) : \mathbf{R} \to \mathbf{R}$ is convex and lower semicontinuous a.e. on Γ_3.

Theorem 13.1 (Theorem 2 in [4]). *Assume conditions* (**H1**)-(**H5**) *are fulfilled. Then there exists at least one weak solution for problem (***P***).*

An interesting continuation of the previous work is related to the case of piezoelectric materials having some "perfect" insulators or "perfect" conductors points. Such anisotropic media lead to degenerate and singular mathematical problems. Notice that the presence of some "perfect" insulators or "perfect" conductors points imposes, from the mathematical point of view, some changes in the hypothesis (**H1**). In particular, we have to assume that $\inf_{\overline{\Omega}} \beta = 0$, $\sup_{\overline{\Omega}} \beta = \infty$. Solving such a problem is an open question.

- In [103] the optimal control for an antiplane model it was investigated. Let us sketch below the framework and the results. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set, with Lipschitz continuous boundary Γ partitioned in three measurable parts Γ_1, Γ_2, Γ_3 such that the Lebesgue measures of Γ_i is strictly positive, for each $i \in \{1, 2, 3\}$.

  We consider the following mechanical problem: *find a displacement field* $u : \bar{\Omega} \to \mathbb{R}$ *such that*

$$\operatorname{div}(\mu(x)\,\nabla u(x)) + f_0(x) = 0 \qquad \text{in } \Omega, \tag{13.1}$$

$$u(x) = 0 \qquad \text{on } \Gamma_1, \tag{13.2}$$

$$\mu(x)\,\partial_\nu u(x) = f_2(x) \qquad \text{on } \Gamma_2, \tag{13.3}$$

$$\left.\begin{array}{l} |\mu(x)\,\partial_\nu\, u(x)| \leq g(x, |u(x)|), \\ \mu(x)\,\partial_\nu u(x) = -g(x, |u(x)|)\,\frac{u(x)}{|u(x)|} \quad \text{if } u(x) \neq 0 \end{array}\right\} \quad \text{on } \Gamma_3. \tag{13.4}$$

Let us assume that

$$\mu \in L^\infty(\Omega), \quad \mu(x) \geq \mu^* > 0 \text{ a.e. in } \Omega, \mu^* \text{ big enough}, \tag{13.5}$$

$$f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2), \tag{13.6}$$

$$g : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that there exists } L_g > 0 : \tag{13.7}$$

$$|g(x, r_1) - g(x, r_2)| \leq L_g\,|r_1 - r_2|\ \forall\, r_1,\, r_2 \in \mathbb{R}_+, \ \text{ a.e. } x \in \Gamma_3; \tag{13.8}$$

$$\text{the mapping } x \mapsto g(x,\, r) \text{ is Lebesgue measurable on } \Gamma_3, \forall\, r \in \mathbb{R}_+; \tag{13.9}$$

$$\text{the mapping } x \mapsto g(x, 0) \text{ belongs to } L^2(\Gamma_3). \tag{13.10}$$

Furthermore, we consider the Hilbert space

$$V = \{v \in H^1(\Omega) \,|\, \gamma v = 0 \ a.e.\, on\, \Gamma_1\}.$$

We are led to the following weak formulation of the problem (13.1)-(13.4): *Given* $f_0 \in L^2(\Omega)$ *and* $f_2 \in L^2(\Gamma_2)$, *find* $u \in V$ *such that*

$$(Au, v-u)_V + j(u,v) - j(u,u) \geq (f, v-u)_V \quad \forall\, v \in V, \tag{13.11}$$

where

$$A : V \to V; \quad (Au, v)_V = \int_\Omega \mu(x) \nabla u(x) \cdot \nabla v(x)\, dx \quad \forall\, u,\, v \in V, \tag{13.12}$$

$$j : V \times V \to \mathbb{R}; \quad j(u,\, v) = \int_{\Gamma_3} g(x, |\gamma u(x)|)\, |\gamma v(x)|\, ds \qquad \forall\, u,\, v \in V, \tag{13.13}$$

$$(f, v)_V = \int_\Omega f_0(x)\, v(x)\, dx + \int_{\Gamma_2} f_2(x)\, \gamma v(x)\, d\Gamma \quad \forall v \in V. \tag{13.14}$$

Theorem 13.2. *[Theorem 3.6 in [103]] Assume* (13.5), (13.6) *and* (13.7). *Then, the problem* (13.11) *has a unique solution* $u \in V$ *which depends Lipschitz continuously on* f.

For a fixed function $f_0 \in L^2(\Omega)$, we consider the following *state problem.*

(PS1) *Let* $f_2 \in L^2(\Gamma_2)$ *(called control) be given. Find* $u \in V$ *such that*

$$\begin{aligned}(Au, v-u)_V + j(u,v) - j(u,u) \geq &\int_\Omega f_0(x)\, (v(x) - u(x))\, dx \\ &+ \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u(x))\, d\Gamma \quad \forall\, v \in V.\end{aligned} \tag{13.15}$$

For every control $f_2 \in L^2(\Gamma_2)$, the *state problem* (PS1) has a unique solution $u \in V$, $u = u(f_2)$.

Now, we would like to act a control on Γ_2 such that the resulting stress $\boldsymbol{\sigma}$ be as close as possible to a given target

$$\boldsymbol{\sigma}_d = \begin{pmatrix} 0 & 0 & \mu\,\dfrac{\partial u_d}{\partial x_1} \\ 0 & 0 & \mu\,\dfrac{\partial u_d}{\partial x_2} \\ \mu\,\dfrac{\partial u_d}{\partial x_1} & \mu\,\dfrac{\partial u_d}{\partial x_2} & 0 \end{pmatrix}$$

where u_d is a given function. Note that, since

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_d\|_{L^2(\Omega)^{3\times 3}} = \sqrt{2}\|\mu \nabla(u - u_d)\|_{L^2(\Omega)} \leq \sqrt{2}\|\mu\|_{L^\infty(\Omega)}\|u - u_d\|_V,$$

$\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_d$ will be close one from another if the difference between the functions u and u_d is small in the sense of $V-$norm. To give an example of a target of interest, u_d, we can consider $u_d = 0$. In this situation, by acting a control f_2 on Γ_2, the tension $\boldsymbol{\sigma}$ is small in the sense of L^2- norm, even if f_0 don't vanishes in Ω.

Let $\alpha, \beta > 0$ be two positive constants and let us define the following functional

$$L : L^2(\Gamma_2) \times V \to \mathbb{R}, \quad L(f_2, u) = \frac{\alpha}{2}\|u - u_d\|_V^2 + \frac{\beta}{2}\|f_2\|^2_{L^2(\Gamma_2)}. \tag{13.16}$$

Furthermore, we denote

$$\mathcal{V}_{ad} = \{[u, f_2] \mid [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that (13.15) is verified}\}$$

and we introduce the following *optimal control problem*,

(POC1) *Find* $[u^*, f_2^*] \in \mathcal{V}_{ad}$ *such that* $L(f_2^*, u^*) = \min_{[u,f_2]\in\mathcal{V}_{ad}} \Big\{L(f_2, u)\Big\}.$

Theorem 13.3. *[Theorem 3.7 in [103]] Assume* (13.5), (13.6), (13.7). *Then,* $(POC1)$ *has at least one solution* (u^*, f_2^*).

Let $\rho > 0$. We define a functional $j_\rho : V \times V \to \mathbb{R}$ as follows,

$$j_\rho(u, v) = \int_{\Gamma_3} g(x, \sqrt{(\gamma u(x))^2 + \rho^2} - \rho)(\sqrt{(\gamma v(x))^2 + \rho^2} - \rho)d\Gamma \quad \forall\, u, v \in V. \tag{13.17}$$

Let us state the following problem: *Given* $\rho > 0$, $f_0 \in L^2(\Omega)$ *and* $f_2 \in L^2(\Gamma_2)$, *find the displacement field* $u^\rho \in V$ *such that*

$$\begin{aligned}(Au^\rho, v - u^\rho)_V + j_\rho(u^\rho, v) - j_\rho(u^\rho, u^\rho) \geq& \int_\Omega f_0(x)\,(v(x) - u^\rho(x))\,dx \\ &+ \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u^\rho(x))\,d\Gamma \quad \forall\, v \in V.\end{aligned} \tag{13.18}$$

Theorem 13.4. *[Theorem in 4.11 [103]] Assume* (13.5), (13.6), (13.7). *Then, problem* (13.18) *has a unique solution* $u^\rho \in V$ *which depends Lipschitz continuously on* f.

Let us fix $\rho > 0$ and $f_0 \in L^2(\Omega)$. We introduce the following regularized problem

(PS2) *Let* $f_2 \in L^2(\Gamma_2)$ *(called control). Find* $u \in V$ *such that*

$$\begin{aligned}(Au, v - u)_V + j_\rho(u, v) - j_\rho(u, u) \geq& (f_0, v - u)_{L^2(\Omega)} \\ &+ (f_2, \gamma v - \gamma u)_{L^2(\Gamma_2)} \qquad \forall\, v \in V.\end{aligned} \tag{13.19}$$

Let us define the following admissible set,

$$\mathcal{V}^\rho_{ad} = \{[u, f_2] \mid [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that (13.19) is verified}\}.$$

(POC2) *Find* $[\bar{u}, \bar{f}_2] \in \mathcal{V}^\rho_{ad}$ *such that* $L(\bar{f}_2, \bar{u}) = \min_{[u,f_2]\in\mathcal{V}^\rho_{ad}} \Big\{L(f_2, u)\Big\}.$

Theorem 13.5 (Theorem 4.13 in [103]). *Assume* (13.5), (13.6), (13.7). *Then, (POC2) has at least one solution* $(\bar{u}, \bar{f}_2)$.

Let us replace the hypotheses (b) and (d) in (13.7), with the following stronger ones,

$$\left.\begin{array}{l} g(x,\cdot) \in C^1 \text{ a.e. } x \in \Gamma_3, \\ \text{there exists } L_g > 0 : \left|\partial_2 g(x,r)\right| \leq L_g \ \forall r \in \mathbb{R}_+, \ \text{ a.e } x \in \Gamma_3, \\ \text{there exists } M > 0 : |g(x,r)| \leq M \ \forall r \in \mathbb{R}_+, \ \text{ a.e. } x \in \Gamma_3. \end{array}\right\} \tag{13.20}$$

Theorem 13.6. *(Optimality condition)[Theorem 4.14 in [103]] Any optimal control $\bar{f}_2$ of the state problem (PS2) verifies*

$$\bar{f}_2 = -\frac{1}{\beta}\gamma(p(\bar{f}_2)), \tag{13.21}$$

where $p(\bar{f}_2)$ is the unique solution of the variational equation

$$\alpha(\bar{u} - u_d, w)_V = (p(\bar{f}_2), Aw + D_2^2 j_\rho(\bar{u}, \bar{u})w)_V \quad \forall w \in V, \tag{13.22}$$

and, for all $v \in V$,

$$\begin{aligned}(D_2^2 j_\rho(\bar{u}, \bar{u})w, v)_V = &\int_{\Gamma_3} \partial_2 g(x, \sqrt{(\gamma\bar{u}(x))^2 + \rho^2} - \rho)\frac{(\gamma\bar{u}(x))^2}{(\gamma\bar{u}(x))^2 + \rho^2}\gamma w(x)\gamma v(x) d\Gamma \\ &+ \int_{\Gamma_3} g(x, \sqrt{(\gamma\bar{u}(x))^2 + \rho^2} - \rho)\frac{\rho^2}{[(\gamma\bar{u}(x))^2 + \rho^2]^{3/2}}\gamma w(x)\gamma v(x) d\Gamma,\end{aligned}$$

$\bar{u} = u(\bar{f}_2)$ being the solution of (PS2) with $f_2 = \bar{f}_2$.

Under the hypotheses (13.5), (13.6), (a) and (c) of (13.7) and (13.20), we have two convergence results.

Theorem 13.7. *[Theorem 5.16 in [103]] Let $\rho > 0$, $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$ be given. If u^ρ, $u \in V$ are the solutions of problems $(PS2)$ and $(PS1)$, respectively, then,*

$$u^\rho \to u \ \text{in } V \ \text{as } \rho \to 0. \tag{13.23}$$

Theorem 13.8. *[Theorem 5.17 in [103]] Let $[\bar{u}^\rho, \bar{f}_2{}^\rho]$ be a solution of the problem (POC2). Then, there exists a solution of the problem (POC1), $[u^*, f_2^*]$, such that*

$$\begin{aligned} \bar{u}^\rho \to u^* \ &\text{in } V \ \text{as } \rho \to 0, \\ \bar{f}_2{}^\rho \rightharpoonup f_2^* \ &\text{in } L^2(\Gamma_2) \ \text{as } \rho \to 0. \end{aligned} \tag{13.24}$$

Following such a technique, the following questions are under attention in the future:

1. an extension of this study to the general case 3D;
2. to study the optimal control for 3D models taking into account various contact conditions; to start, a contact condition with normal compliance is envisaged.

Chapter 14

Further plans

14.1 On the scientific and professional career

After obtaining the Ph.D. degree in Mathematics, the candidate published in internationally recognized journals such as SIAM Journal on Scientific Computing, Zeitschrift für Angewandte Mathematik und Mechanik, Nonlinear Analysis -Theory, Methods and Applications, Nonlinear Analysis: Real World Applications, Journal of Mathematical Analysis and Applications, Mathematics and Mechanics of Solids, Communications on Pure and Applied Analysis, Journal of Global Optimization, Proceedings of The Royal Society of Edinburgh, Section: A Mathematics, The Quarterly Journal of Mechanics and Applied Mathematics, European Journal of Applied Mathematics, The Australian and New Zealand Industrial and Applied Mathematical Journal, Acta Applicandae Mathematicae, Quarterly of Applied Mathematics, Advanced Nonlinear Studies. *In the future* the candidate intends to do a research activity allowing to continue to publish in international journals of hight level.

The dissemination of the results is also under attention. During the years the candidate attended several international conferences. *In the future* the candidate intend to participate to prestigious international meetings in order to disseminate the best results. Also the candidate intends to be involved in the organization of scientific meetings.

The research activity of the candidate was realized mainly at the Department of Mathematics of the University of Craiova, where the author has a permanent position, but also at some Departments of Mathematics from other universities in Europe: Stuttgart University, Technische University of Munchen, University of Perpignan, where the candidate has had research collaborations concretized in the publication of some scientific papers with colleagues from abroad. *In the future* the candidate wishes to continue the collaborations started in the past and to establish new contacts.

In recent years the candidate was reviewer at several journals. *In the future* the candidate intends to extend the editorial activities for scientific journals.

The candidate intend to apply for national/international/interdisciplinary research projects as manager or member of teams. A few steps were already made: director of a GRANT PN-II-RU-TE CNCS-UEFISCDI; responsible of the Romanian side for a French-Romanian research project LEA Math Mode CNRS-IMAR; member of several teams for national, international or interdisciplinary research projects.

14.2 On the academical career

The teaching activities of the candidate were realized at the University of Craiova where along the years his activity has concretized in teaching seminars or courses on different topics: Theoretical Mechanics (seminar); Real Analysis (seminar); PDE's (seminar); Applied Nonlinear Analysis (course and seminar for MASTER); Control Theory (course and seminar for MASTER), Mathematical Modeling by Differential Equations (course and laboratory for MASTER), Mathematical Modeling in Contact Mechanics (course and seminar for MASTER); Singular Problems in Mathematical Physics (course and seminar for MASTER), Special Chapters of PDE's (course and seminar), Evolution Equations (course and seminar for MASTER), Numerical Analysis for PDE's (laborator for MASTER), etc. The candidate was co-author of two monographs published at Springer and Cambridge University Press. These monographs can be found in several libraries such as: Cornell University Library, McGill University Library, Stanford University, Mathematics and Statistics Library, The University of Arizona, Denver University Libraries, UCLA Library (University of California, Los Angeles Library), The University of Manchester, University of Colorado, Eastern Michigan University Library, to give a few examples. *In the future*, the candidate plans to publish Lecture Notes and new monographs addressed to students or researchers.

During the last 10 years the candidate advised several bachelor's degree or dissertation theses; also, in recent years the candidate has collaborated with PH.D. students who became co-authors and collaborators of the candidate (Ionică Andrei, Maria-Magdalena Boureanu, Raluca Ciurcea, Nicuşor Costea). By obtaining this habilitation the candidate plans to extend her advising activity to Ph.D. theses.

Bibliography

[1] R. A. Adams. *Sobolev spaces*, Academic Press, 1975.

[2] K. Adolfsson, M. Enelud, S. Larsson, Adaptive discretization of fractional order viscoelasticityusing sparse time history, *Comput. Methods Appl. Mech. Engrg.* **193** (2004) 4567-4590.

[3] S. Amdouni, P. Hild, V. Lleras, M. Moakher, and Y. Renard, A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies. *ESAIM: M2AN Mathematical Modelling and Numerical Analysis*, **46** (2012), 813–839.

[4] I. Andrei, N. Costea and A. Matei, Antiplane shear deformation of piezoelectric bodies in contact with a conductive support, *Journal of Global Optimization*; DOI 10.1007/s10898-011-9815-x; **56**(1) (2013), 103-119.

[5] O. Arino, S. Gautier and J.P. Penot. A fixed point theorem for sequantially continuous mappings with applications to ordinary differential equations, *Funkcial. Ekvac.* **27** (1984), 273-279.

[6] A. Arora, A. Arora, V.K. Dwivedi, P.J. George, K. Sreenivas, V. Gupta, Zinc oxide thin film-based MEMS acoustic sensor with tunnel for pressure compensation, *Sensors and Actuators A: Physical*, **141**(2), (2008), 256-261.

[7] R.L. Bagley, P.J. Torvik, Fractional calculus- a different approach to the analysis of viscoelastically damped structures, *AIAA J.* **21**, (1983) 741-748.

[8] M. Barboteu and M. Sofonea, Analysis and Numerical Approach of a Piezoelectric Contact Problem, *Annals of AOSR, Series on Mathematics and its Applications*, **1**(2009), 7-30.

[9] M. Barboteu, M. Sofonea, Modelling and Analysis of the Unilateral Contact of a Piezoelectric Body with a Conductive Support, *Journal of Mathematical Analysis and Applications*, **358**(1) (2009), 110-124.

[10] M. Barboteu, A. Matei and M. Sofonea, Analysis of Quasistatic Viscoplastic Contact Problems with Normal Compliance, *The Quarterly Journal of Mechanics and Applied Mathematics*, **65**(4) (2012), 555-579.

[11] M. Barboteu, A. Matei and M. Sofonea, On the behavior of the solution of a viscoplastic contact problem, *Quarterly of Applied Mathematics*, DOI: http://dx.doi.org/10.1090/S0033-569X-2014-01345-4, **72**(4) (2014), 625-647.

[12] V. Barbu, *Optimal Control of Variational Inequalities*, Pitman Advanced Publishing, Boston, 1984.

[13] A. Berga, Mathematical and numerical modeling of the non-associated plasticity of soils Part 1: The boundary value problem, *International Journal of Non-Linear Mechanics*, **47**(1) (2012), 26-35.

[14] P. Bisegna, F. Lebon and F. Maceri, The unilateral frictional contact of a piezoelectric body with a rigid support, in *Contact Mechanics*, J.A.C. Martins and Manuel D.P. Monteiro Marques (Eds.), (Kluwer, Dordrecht, 2002), 347-354.

[15] G. Bodovillé: On damage and implicit standard materials. *C. R. Acad. Sci.*, Paris, Sér. II, Fasc. b, Méc. Phys. Astron. **327** (8) (1999), 715–720.

[16] G. Bodovillé and G. de Saxcé: Plasticity with non-linear kinematic hardening: modelling and shakedown analysis by the bipotential approach, *Eur. J. Mech. A Solids* **20** (2001), 99–112.

[17] A. Borrelli, C. O. Horgan and M. C. Patria, Saint-Venant's principle for antiplane shear deformations of linear piezoelectic materials, *SIAM J. Appl. Math.* **62** (2002), 2027–2044.

[18] J.F. Bonnans, D. Tiba, Pontryagin's principle in the control of semiliniar elliptic variational inequalities, *Applied Mathematics and Optimization*, **23**(1) (1991), 299-312.

[19] M. Boureanu and A. Matei, Weak solutions for antiplane models involving elastic materials with degeneracies, *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, **61**(1) (2010), 73-85.

[20] M. Boureanu, A. Matei and M. Sofonea, Analysis of a Contact Problem for Electro- elastic-visco-plastic Materials, *Communications on Pure and Applied Analysis*, **11**(3) (2012), 1185-1203.

[21] M. Boureanu, A. Matei and M. Sofonea, Nonlinear problems with p(.)-growth conditions and applications to antiplane contact models, *Advanced Nonlinear Studies*, **14** (2014), 295-313.

[22] D. Braess, *Finite Elements*, Cambridge University Press, 1997.

[23] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[24] H. Brézis, *Analyse fonctionnelle–Théorie et applications*, Masson, Paris, 1987.

[25] M. Buliga and G. de Saxcé, C. Vallée: Bipotentials for Non-monotone Multivalued Operators: Fundamental Results and Applications, *Acta Appl Math.*, **110**(2) (2010), 955-972.

[26] M. Buliga, G. de Saxcé, C. Vallée: Non-maximal cyclically monotone graphs and construction of a bipotential for the Coulomb's dry friction law, *J. Convex Anal.* **17**(1) (2010), 81-94

[27] M. Buliga, G. de Saxcé, C. Vallée: Existence and construction of bipotentials for graphs of multivalued laws, *Journal of Convex Analysis*, **15**(1) (2008), 87-104.

[28] M. Buliga and G. de Saxcé, C. Vallée, *A variational formulation for constitutive laws described by bipotentials*, arXiv:1110.6598v1 [math.FA].

[29] M. Campillo, C. Dascălu and I.R. Ionescu. Instability of a periodic system of faults. *Geophys. Int. J.* **159** (2004), 212-222.

[30] M. Campillo and I.R. Ionescu. Initiation of antiplane shear instability under slip dependent friction. *J. Geophys. Res.* **102** B9 (1997), 363-371.

[31] M. Caputo, F. Mainardi, A new dissipation model based on memory mechanism, *Pure Appl. Geophys.* **91** (1971) 134-147.

[32] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, vol. 378 of CISM Courses and Lectures, Springer, Vienna, Austria, 1997.

[33] K.C. Chang, Variational methods for nondifferentiable functionals and applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981) 102-129.

[34] M. Ciarletta, D. Ieşan, *Non-classical elastic solids,* Pitman Research Notes in Mathematics, ser. 93, Longman Scientific & Technical, 1993.

[35] M. Ciarletta, D. Ieşan, Some results in the dynamical theory of porous elastic bodies, *J. Elast.*, **50**, 3-14, 1998.

[36] C. Ciulcu, D. Motreanu and M. Sofonea, Analysis of an elastic contact problem with slip dependent coefficient of friction, *Mathematical Inequalities and Applications*, pp. 465-479, 4/2001.

[37] F.H. Clarke, generalized gradients and applications, *Trans. Amer. Math. Soc.* **205**(1975), 247-262.

[38] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, Interscience, New York, 1983.

[39] C. Corduneanu, Problèmes globaux dans la théorie des équations intégrales de Volterra, *Ann. Math. Pure Appl.*, **67** (1965), 349–363.

[40] N. Costea and A. Matei, Weak solutions for nonlinear antiplane problems leading to hemivariational inequalities, *Nonlinear Analysis: Theory, Methods and Applications*, **72** (2010), 3669-3680.

[41] N. Costea and A. Matei, Contact models leading to variational-hemivariational in- equalities, *Journal of Mathematical Analysis and Applications*, **386**(2) (2012), 647-660.

[42] N. Costea and V. Radulescu, Existence results for hemivariational inequalities involving relaxed $\eta - \alpha$-monotone mappings, *Communications in Applied Analysis,* **13**(3) (2009), 293–304.

[43] S.C. Cowin, J.W. Nunziato, *Linear elastic materials with voids,* J. Elasticity, **13** (1983), 125-147.

[44] D.O. Craiem, F.J. Rojo, J.M. Atienza, G.V. Guinea, R.L. Armentano, Fractional calculus applied to model arterial viscoelasticity, *Latin American Applied Research*, **38** (2008), 141-145.

[45] N. Cristescu and I. Suliciu, *Viscoplasticity,* Martinus Nijhoff Publishers, Editura Tehnica, Bucharest, 1982.

[46] D. Danciu, A.C. Matei, S.D. Micu and I. Roventa, Nonlinear feedback control and Artificial Intelligence computational methods applied to a dissipative dynamic contact problem, conference paper, *Proceedings of the 11th International Conference on Informatics in Control, Automation and robotics*, 2014, (ICINCO 2014), Vienna, Austria.

[47] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Mathematical Inequalities and Applications*, **7**, (2004), 245–253.

[48] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[49] C. Eck, J. Jarušek and M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics 270, Chapman/CRC Press, New York, 2005.

[50] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*, Classics in Applied Mathematics **28** SIAM, Philadelphia, PA, 1999.

[51] K. Fan, Some properties of convex sets related to fixed point theorems, *Math. Ann.* **266** (1984), 519-537.

[52] X. Fan, Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.*, **339** (2008), 1395–1412.

[53] X. Fan, Solutions for $p(x)$–Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, **312** (2005), 464–477.

[54] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446.

[55] A. Friedman, Optimal Control for Variational Inequalities, *SIAM, Journal on Control and Optimization*, **24**(3) (1986), 439-451.

[56] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, *Rend. Sem. Mat. Univ. Padova*, **27** (1957), 284–305.

[57] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman Advanced Publishing Program, 1985.

[58] F. M. Guo, Z. Q. Zhu, Y. F. Long, W. M. Wang, S. Z. Zhu, Z. S. Lai, N. Li, G. Q. Yang, W. Lu, Study on low voltage actuated MEMS rf capacitive switches, *Sensors and Actuators A: Physical*, **108** (1-3) (2003), 128-133.

[59] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society–International Press, 2002.

[60] W. Han, Sofonea and K. Kazmi, *Analysis and numerical solution of a frictionless contact problem for electro-elastic-visco-plastic materials*, Computer Methods in Applied Mechanics and Engineering, **196** (2007), 3915–3926.

[61] J. Haslinger, I. Hlaváček and J. Nečas, Numerical methods for unilateral problems in solid mechanics, in *Handbook of Numerical Analysis*, P.G. Ciarlet and J.L. Lions ed., **IV**, North-Holland, Amsterdam, 313–485, 1996.

[62] P. Hild and P.Laborde, Quadratic finite element methods for unilateral contact problems, *Applied Numerical Mathematics*, **41** (2002), 410-421.

[63] P. Hild, Y. Renard, A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics. *Numer. Math.* **115** (2010), 101–129.

[64] M. Hjiaj, G. Bodovillé, G. de Saxcé: Matériaux viscoplastiques et loi de normalité implicites, *C. R. Acad. Sci.,* Paris, Sér. II, Fasc. b, Méc. Phys. Astron. **328** (2000), 519–524.

[65] T.-V. Hoarau-Mantel and A. Matei, Analysis of a viscoelastic antiplane contact problem with slip dependent friction, *International Journal of Applied Mathematics and Computer Science*, **12**(1) (2002), 51-59.

[66] S. Hüeber and B. Wohlmuth, An optimal a priori error estimate for non-linear multibody contact problems, *SIAM J. Numer. Anal.* **43** (2005), 157-173.

[67] S. Hüeber and B. Wohlmuth, A primal-dual active set strategy for non-linear multibody contact problems, *Comput. Methods Appl. Mech. Engrg.* **194** (2005), 3147-3155.

[68] S. Hüeber, A. Matei and B. Wohlmuth, A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity, *Bull. Math. Soc. Math. Roumanie*, **48**(96)(2) (2005), 209-232.

[69] S. Hüeber, A. Matei and B. Wohlmuth, Efficient algorithms for problems with friction, *SIAM Journal on Scientific Computing*, **29**(1) (2007), 70-92.

[70] S. Hüeber, A. Matei, B. Wohlmuth, A contact problem for electro-elastic materials, *Journal of Applied Mathematics and Mechanics(ZAMM)*, **93**(10-11) (2013), 789-800.

[71] D. Ieşan, A theory of thermoelastic materials with voids, *Acta Mech.*, **60**, 67-89, 1986.

[72] I.R. Ionescu, J.-C. Paumier, On the contact problem with slip displacements dependent friction in elastostatics, *Int. J. Eng. Sci.*, **34**(4) (1996), 471-491.

[73] I.R. Ionescu and M. Sofonea, *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1993.

[74] I.R. Ionescu, C. Dascalu and M. Campillo, Slip-weakening friction on a periodic System of faults: spectral analysis, *Z. Angew. Math. Phys. (ZAMP)* **53** (2002), 980–995.

[75] I.R. Ionescu and S. Wolf, Interaction of faults under slip dependent friction. Nonlinear eingenvalue analysis, *Mathematical Methods in Applied Sciences (M^2AS)* **28** (2005), 77–100.

[76] T. Ikeda, *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford, 1990.

[77] G. Iovane, A.V. Nasednik, *Finite element dynamic analysis of anisotropic elastic solids with voids*, Computers and Structures, **87** (2009), 981-989.

[78] J. Jarušek and M. Sofonea, On the solvability of dynamic elastic-visco-plastic contact problems, *Zeitschrift für Angewandte Matematik und Mechanik (ZAMM)*, **88** (2008), 3–22.

[79] K.L. Johnson, *Contact Mechanics*, Cambridge University Press, Cambridge, 1987.

[80] S. Kalinski and J. Petikiewicz, *Equations of motion coupled with the field of temperature in magnetic field involving mechanical and electromagnetic relaxations for anisotropic bodies*, in Proc. Vibr. Probl., 1960, pp. 17–28.

[81] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[82] J.U. Kim, A boundary thin obstacle for a wave equation, *Commun. in Partial Differential Equations*, **14**(8-9) (1989), 1011-1026.

[83] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, 2000.

[84] A. Klarbring, A. Mikelič and M. Shillor, Frictional contact problems with normal compliance, *Int. J. Engng. Sci.* **26** (1988), 811–832.

[85] A. Klarbring, A. Mikelič and M. Shillor, On friction problems with normal compliance, *Nonlinear Analysis* **13** (1989), 935–955.

[86] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.*, **41** (1991), 592–618.

[87] A. Kufner, O. John, and S. Fucík, *Function Spaces*, in: Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: 406 Analysis, Noordhoff International Publishing, Leyden, 1977.

[88] T.A. Laursen, *Computational Contact and Impact Mechanics*, Springer, Berlin, 2002.

[89] S. Leonardi, Solvability of degenerated quasilinear elliptic equations, *Nonlinear Anal.*, **26** (1996), 1053–1060.

[90] Z. Lerguet, M. Shillor and M. Sofonea, A frictional contact problem for an electro-viscoelastic body, *Electronic Journal of Differential Equations*, **170** (2007), 1-16.

[91] J.-L. Lions, *Contrôle optimale des systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Paris, 1968.

[92] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes*, Dunod, Paris, 1968.

[93] F. Mainardi, *Fractional calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.

[94] A. B. Malinowska, D. F. M. Torres, *Introduction to the Fractional Calculus of Variations*, Imperial College Press, 2012.

[95] J. A. C. Martins and J. T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis : Theory, Methods and Applications* **11** (1987), 407–428.

[96] J. J. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York-London, 1966.

[97] A. Matei, V.V. Motreanu and M. Sofonea, A quasistatic antiplane contact problem with slip dependent friction, *Advances in Nonlinear Variational Inequalities*, **4**(2) (2001), 1-21.

[98] A. Matei, A variational approach for an electro-elastic unilateral contact problem, *Mathematical Modelling and Analysis*, **14**(3) (2009), 323-334.

[99] A. Matei and R. Ciurcea, Contact problems for nonlinearly elastic materials: weak solvability involving dual Lagrange multipliers, *Australian and New Zealand Industrial and Applied Mathematics Journal(ANZIAM JOURNAL)*, **52** (2010), 160–178.

[100] A. Matei and R. Ciurcea, Weak solvability for a class of contact problems, *Annals of the Academy of Romanian Scientists Series on Mathematics and its Applications* **2**(1) (2010), 25-44.

[101] A. Matei and R. Ciurcea, Weak solutions for contact problems involving viscoelastic materials with long memory, *Mathematics and Mechanics of Solids*, **16**(4) (2011), 393 - 405.

[102] A. Matei and C. Niculescu, Weak solutions via bipotentials in mechanics of deformable solids, *J. Math. Anal. Appl.,* **379**(1) (2011), 15-25.

[103] A. Matei and S. Micu. Boundary optimal control for nonlinear antiplane problems. *Nonlinear Anal.: Theory, Methods Appl.* **74(5)** (2011), 1641-1652.

[104] A. Matei, Weak solvability via Lagrange multipliers for two frictional contact models, Proceedings of 11^{th} French-Romanian Conference on Applied Mathematics, 2012, Bucharest, *Annals of the University of Bucharest (mathematical series)*, **4(LXII)** (2013), 179-191.

[105] A. Matei, On the solvability of mixed variational problems with solution-dependent sets of Lagrange multipliers, *Proceedings of The Royal Society of Edinburgh, Section A Mathematics*, **143**(05) (2013), 1047-1059.

[106] A. Matei, A variational approach via bipotentials for unilateral contact problems, *Journal of Mathematical Analysis and Applications*, **397**(1) (2013), 371-380.

[107] A. Matei, An existence result for a mixed variational problem arising from Contact Mechanics, *Nonlinear Analysis Series B: Real World Application*, **20** (2014), 74-81, DOI: 10.1016/j.nonrwa.2014.01.010.

[108] A. Matei, A variational approach via bipotentials for a class of frictional contact problems, *Acta Applicanda Mathematicae*, DOI: 10.1007/s10440-014-9868-1, **134**(1) (2014), 45-59.

[109] A. Matei, A mixed variational formulation for a slip-dependent frictional contact model, Lecture Notes in Engineering and Computer Science: *Proceedings of The World Congress on Engineering 2014*, 2-4 July, 2014, London, U.K., pp 750-754 (ISBN: 978-988-19253-5-0, ISSN: 2078-0958).

[110] A. Matei, Weak solvability via Lagrange multipliers for contact problems involving multi-contact zones, *Mathematics and Mechanics of Solids*, DOI: 10.1177/1081286514541577.

[111] A. Matei, An evolutionary mixed variational problem arising from frictional contact mechanics, *Mathematics and Mechanics of Solids*, **19**(3) (2014), 223 - 239.

[112] A. Matei, Weak Solutions via Lagrange Multipliers for a Slip-dependent Frictional Contact Model *IAENG International Journal of Applied Mathematics*, **44**(3) (2014), 151-156.

[113] A. Matei, Two abstract mixed variational problems and applications in Contact Mechanics, http://dx.doi.org/10.1016/j.nonrwa.2014.09.014, in press.

[114] S. Migorski, A. Ochal and M. Sofonea, Analysis of a Dynamic Contact Problem for Electro-viscoelastic Cylinders, *Nonlinear Analysis, Series A : Theory, Methods and Applications*, **73** (2010), 1221-1238.

[115] S. Migorski, A. Ochal, M. Sofonea. Weak solvability of antiplane frictional contact problems for elastic cylinders. *Nonlinear Anal. Real World Appl.* **11** (2010), 172-183.

[116] S. Migorski, A. Ochal, M. Sofonea. Solvability of dynamic antiplane frictional contact problems for viscoelastic cylinders. *Nonlinear Anal. Theory, Methods Appl.* **70** (2009), 3738-3748.

[117] S. Migorski, A. Ochal, M. Sofonea. Modeling and analysis of an antiplane piezoelectric contact problem. *Math. Models Methods Appl. Sci.* **19** (2009), 1295-1324.

[118] R. Mignot, Contrôle dans les inéquations variationnelles elliptiques, *J. Func. Anal.*, **22** (1976), 130-185.

[119] R. Mignot and J.-P. Puel, Optimal control in some variational inequalities, *SIAM J. Control Optim.*, **22** (1984) 466-476.

[120] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A*, **462** (2006), 2625–2641.

[121] R. Mindlin, *Polarisation gradient in elastic dielectrics*, Internat. J. Solids Structures,**4** (1968), pp. 637–663.

[122] R. Mindlin, *Continuum and lattice theories of influence of electromechanical coupling on capacitance of thin dielectric films*, Internat. J. Solids Structures, **4** (1969), pp. 1197–1213.

[123] R. Mindlin, Elasticity, piezoelasticity and crystal lattice dynamics, *J. Elasticity*, **4** (1972), pp. 217–280.

[124] Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York, 1995.

[125] P. Neittaanmaki, J. Sprekels and D. Tiba, *Optimization of Elliptic Systems: Theory and Applications*, Springer Monographs in Mathematics, Springer, New York, 2006.

[126] J.W. Nunziato, S.C. Cowin, *A nonlinear theory of elastic materials with voids*, Arch. Rational Mech. Anal., **72**, 175-201, 1979.

[127] J.T. Oden and J.A.C. Martins, Models and computational methods for dynamic friction phenomena, *Computer Methods in Applied Mechanics and Engineering* **52** (1985), 527–634.

[128] P.D. Panagiotopoulos, *Hemivariational Inequalities: Applications in Mechanics and Engineering*, Springer-Verlag, New York/Boston/Berlin, 1993.

[129] A. Quarteroni, A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Series in Computational Mathematics, Springer, 2008.

[130] E. Rabinowicz, *Friction and Wear of Materials*, 2nd ed., Wiley, New York, 1995.

[131] P. A. Raviart and J.-M. Thomas, *Introduction à l'Analyse Numérique des Équations aux Dérivées Partielles*, Masson, Paris, 1988.

[132] B. D. Reddy, Mixed variational inequalities arising in elastoplasticity, *Nonlinear Analysis: Theory, Methods and Applications*, **19**(11) (1992), 1071–1089.

[133] G. de Saxcé: Une généralisation de l'inégalité de Fenchel et ses applications aux lois constitutives. *C. R. Acad. Sci., Paris,* Sér. II **314** (1992), 125–129.

[134] G. de Saxcé: *The bipotential method, a new variational and numerical treatment of the dissipative laws of materials.* In: Proc. 10th Int. Conf. on Mathematical and Computer Modelling and Scientific Computing, Boston, 1995.

[135] G. de Saxcé, Z.-Q. Feng, The Bipotential Method: A Constructive Approach to Design the Complete Contact Law with Friction and Improved Numerical Algorithms, *Mathl. Comput. Modelling*, **28**(4-8) (1998), 225-245.

[136] G. de Saxcé and L. Bousshine: *Implicit standard materials.* In: D. Weichert, G. Maier (eds.) Inelastic Behavior of Structures under Variable Repeated Loads. CISM Courses and Lectures, vol. **432**, Springer-Verlag, Wien, 2002.

[137] C.H. Scholz. *The mechanics of earthquakes and Faulting.* Cambridge University Press, Cambridge, 1990.

[138] M. Shillor, M. Sofonea and J.J. Telega, *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics, **655**, Springer, Berlin, 2004.

[139] M. Sofonea: *Problème non-lineaires dans la théorie de l'élasticité*, Cours de Magister de Mathématiques Appliquées, Université de Setif, Algérie, 1993.

[140] M. Sofonea and A. Matei, Elastic antiplane contact problem with adhesion, *Journal of Applied Mathematics and Physics (ZAMP)* **53** (2002), 962-972.

[141] M. Sofonea, W. Han, M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Chapman-Hall/CRC Pure and Applied Mathematics, 2005.

[142] M. Sofonea and A. Matei, An elastic contact problem with adhesion and normal compliance, *Journal of Applied Analysis*, **12**(1) (2006), 19-36.

[143] M. Sofonea, C. Niculescu and A. Matei, An antiplane contact problem for viscoelastic materials with long-term memory, *Mathematical Modelling and Analysis*, **11**(2) (2006), 213-228.

[144] M. Sofonea, C. Avramescu and A. Matei, A Fixed point result with applications in the study of viscoplastic frictionless contact problems, *Communications on Pure and Applied Analysis*, **7**(3), (2008), 645–658.

[145] M. Sofonea and A. Matei. *Variational Inequalities with Applications. A Study of Antiplane Frictional Contact Problems.* Advances in Mechanics and Mathematics, 18, Springer, New York, 2009.

[146] M. Sofonea and A. Matei, History-dependent Quasivariational Inequalities arising in Contact Mechanics, *European Journal of Applied Mathematics*, **22** (2011), 471-491.

[147] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society, Lecture Note Series, **398**, Cambridge University Press, 2012.

[148] M. Sofonea and A. Matei, History-dependent Mixed Variational Problems in Contact Mechanics, Journal of Global Optimization, DOI 10.1007/s10898-014-0193-z.

[149] J. Sokolowski and J.P. Zolesio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer, Berlin, 1991.

[150] L. Solymar and L.B. Au, *Solutions Manual for Lectures on the Electrical Properties of Materials,* 5^{th} Fifth Edition, Oxford University Press, Oxford, 1993.

[151] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

[152] E. Tarafdar, A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem, *J. Math. Anal. Appl.* **128** (2) (1987), 475-479.

[153] R. Toupin, *The elastic dielectrics*, J. Rat. Mech. Analysis, (1956), pp. 849–915.

[154] R. Toupin, *A dynamical theory of elastic dielectrics*, Internat. J. Engrg. Sci., (1963), pp. 101–126.

[155] C. Vallée, C. Lerintiu, D. Fortuné, M. Ban. and G. de Saxcé: *Hill's bipotential.* In: M. Mihailescu-Suliciu (ed.) New Trends in Continuum Mechanics. Theta Series in Advanced Mathematics, pp. 339–351, Theta Foundation, Bucharest, 2005.

[156] C. Vallée, C. Lerintiu, J. Chaoufi, D. Fortuné, M. Ban and K. Atchonouglo, A Class of Non-associated Materials: n-Monotone Materials - Hooke's Law of Elasticity Revisited, *Journal of Elasticity*, DOI 10.1007/s10659-012-9403-4, 112(2) (2013), 111-138.

[157] W. Voigt, *Lehrbuch der Kristall-Physik*, Teubner, Leipzig, 1910.

[158] J. Yang and J. S. Yang, *An Introduction to the Theory of Piezoelectricity*, Springer, New York, 2005.

[159] N. Zouian, I. Pontes Filho, L. Borges, L. Mouta da Costa, Plastic collapse in non-associated hardening materials with application to Cam-clay, *International Journal of Solids and Structures*, **44** (2007), 4382-4398.

[160] E. Zeidler, *Nonlinear Functional Analysis and its Applications. IV: Applications to Mathematical Physics*, Springer-Verlag, New-York, 1988.

[161] K. Willner, *Kontinuums- und Kontaktmechanik*, Springer-Verlag Berlin, 2003.

[162] B. Wohlmuth, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, Springer, 2001.

[163] B. Wohlmuth, A mortar finite element method using dual spaces for the Lagrange multiplier, *SIAM J. Numer. Anal.*, **38** (2000), 989-1012.

[164] B. Wohlmuth, Variationally consistent discretization schemes and numerical algorithms for contact problems, *Acta Numerica*, **20** (2011), 569-734.

[165] B. Wohlmuth and R. Krause, Monotone methods on non-matching grids for non linear contact problems, *SIAM Journal on Scientific Computing*, **25** (2003), 324-347.

[166] P. Wriggers, *Computational Contact Mechanics*, Wiley, Chichester, 2002.