Dan Coman: Stable algebras of entire functions

Abstract: Two entire functions f and g on $\mathbb{C}^n$ are algebraically dependent over $\mathbb{C}$ if there is a non-zero polynomial P in $\mathbb{C}[x,y]$ such that $P(f,g)=0$. It is easy to show that in this case there exists an entire function h so that $f,g \in \mathbb{C}[h]$ or $f,g \in \mathbb{C}[e^h, e^{-h}]$. A similar description holds in the case when f,g are meromorphic on $\mathbb{C}^n$.

The problem of characterizing when entire functions f,g are algebraically dependent over the ring $\mathcal{P}^n$ of polynomials on $\mathbb{C}^n$ seems difficult. The simple dependence relation $P(f,g) = P_1(f)g + P_0(f) = 0$, $P_1, P_0 \in \mathcal{P}^n[x]$, leads to the question of when the ratio $P_0(f)/P_1(f)$ is entire.

In general, a subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$ is called *stable in* $\mathcal{A}$ if $g,h \in \mathcal{B}$ and $h/g \in \mathcal{A}$ imply that $h/g \in \mathcal{B}$. Let $\mathcal{A}$ be the algebra of meromorphic functions on $\mathbb{C}^n$ which have algebraic polar variety, $\mathcal{R}^n \subset \mathcal{A}$ be the field of rational functions, and $\mathcal{B} = \mathcal{R}^n[f]$, where f is an entire function of finite order on $\mathbb{C}^n$. We show that either $\mathcal{B}$ is stable in $\mathcal{A}$ or $f = q_1 e^p + q_2$, where $p \in \mathcal{P}^n$ and $q_1, q_2 \in \mathcal{R}^n$. In the latter case, the algebra $\mathcal{R}^n[e^p, e^{-p}]$ is stable in $\mathcal{A}$.

The results are joint with Evgeny Poletsky.