Finite simplicial multicomplexes

Cimpoeaş Mircea - Junior Researcher
Institute of Mathematics of the Romanian Academy
(I. M. A. R.)

9th February 2005

Abstract

Simplicial multicomplexes are a very natural generalization of simplicial complexes. Indeed, instead to see a simplicial complex as a subset $\Delta \subset \mathcal{P}([n])$ we can think Δ as a subset of vectors in $\{0,1\}^n$, which satisfy the property: $(*)$ For any $F \in \Delta$ and any $G \in \{0,1\}^n$ such that $G \leq F$ it follows that $G \in \Delta$. Nothing can stop us to consider subsets $\Gamma \subset \mathbb{N}^n$ which have the property $(*)$. Such a set is called a simplicial multicomplex.

In this paper I will focus on the case of finite multicomplexes. More precisely, I will exploit the relation between a monomial ideal (which will correspond to a finite multicomplex) and its polarized ideal (which will correspond to a simplicial complex). Using this connexion, we can extend many construction and definitions in the category of simplicial complexes to the category of finite simplicial multicomplexes, as: homology, shellability, duality theories etc.

In the first section I introduce the main definitions and constructions of multicomplexes. In the second section, I present what I understand by a homology theory of multicomplexes. In the third section I extend the notion of shellability for simplicial multicomplexes and I prove a criterion of shellability (similar to the case of simplicial complexes) which allows us to see the duality with the case of ideals with linear quotients. This observation give us the idea to introduce the notion of co-shellable (multi)complexes. In the 5^{th} I define the base ring and the Erhart ring of a multicomplex. In the 6^{th} section I give some dual constructions in the category of multicomplexes and some results, which extends the case of simplicial complexes.

I wish to thank to my Professor Dorin Popescu for his support, encouragement and valuables observations on the contents of this paper...

1 Finite simplicial multicomplexes

First of all, let us fix some notations:

- k is an arbitrary field and $S = k[x_1, \ldots, x_n]$ is the ring of polynomials over k. For any monomial ideal $I \subset S$, we denote by $G(I)$ the set of minimal generators of I.
- A vector $u \in \mathbb{N}^n$ it will be written as $u = (u(1), \ldots, u(n))$. The module of u is the number $|u| := u(1) + \cdots + u(n)$.
- If $u, v \in \mathbb{N}^n$, we say that $u \leq v$ if $u(i) \leq v(i)$ for all $i = 1, \ldots, n$. Obvious, " $\leq$ " is a partial order on $\mathbb{N}^n$.
- We denote by $e_i = (0, \ldots, 1, 0, \ldots, 0)$ the canonical base of $\mathbb{N}^n$.
- If $u \in \mathbb{N}^n$, x^u is the monomial $x_1^{u(1)} x_2^{u(2)} \cdots x_n^{u(n)} \in S$.

Definition 1.1. *A finite subset $\Gamma \subset \mathbb{N}^n$ is called a* finite simplicial multicomplex *if for all $a \in \Gamma$ and all $b \in \mathbb{N}^n$ with $b \leq a$, it follows that $b \in \Gamma$. The elements of Γ are called* faces.

An element $m \in \Gamma$ is called a maximal facet *if there exists no $a \in \Gamma$ with $a > m$. In other words, if m is maximal with respect to "$\leq$". We denote $\mathcal{M}(\Gamma)$ the set of maximal facets of Γ.*

If $a \in \Gamma$ is a face, the dimension of a is the number $dim(a) = |a| - 1$. The dimension *of Γ is the number $dim(\Gamma) = \max\{dim(u) | u \in \Gamma\}$. A multicomplex Γ is called pure if all the maximal facets have the same dimension, equal to $dim(\Gamma)$.*

Remark 1.2. *An arbitrary intersection and a finite union of finite multicomplexes is again a multicomplex. Therefore, the set of all finite multicomplexes in $\mathbb{N}^n$ is the family of closed sets in a topology on $\mathbb{N}^n$, called the finite-simplicial topology. The continuous function in this topology are called finite-simplicial morphism of multicomplexes. This aspect will be not studied in this paper.*

Remark 1.3. *Any finite multicomplex is determined by its maximal facets set, $\mathcal{M}(\Gamma) = \{u_1, \ldots, u_r\}$. In fact,*

$$\Gamma = \{b \in \mathbb{N}^n | \ b \leq u_i, \ for \ some \ i \in \{1, \ldots, r\} \ \}.$$

We write $\Gamma = \langle u_1, \ldots, u_r \rangle$ and we say that Γ is the multicomplex spanned by the vectors $u_1, \ldots, u_r$. Obvious, Γ is the smallest multicomplex which contained $u_1, \ldots, u_r$.

Definition 1.4. *Let k be an arbitrary field. If $\Gamma \subset \mathbb{N}^n$ is a finite multicomplex, the ideal of* non-faces *of Γ is the monomial ideal, denoted by I_Γ, in $k[x_1, \ldots, x_n]$ spanned, as k-vector space, by all monomial x^a with $a \in \mathbb{N}^n \setminus \Gamma$. In particular, the monomial x^a with $a \in \Gamma$ form a k-basis of S/I_Γ.*

Obvious, I_Γ is an artinian ideal (i.e. S/I_Γ is an artinian ring). Conversely, if I is an artinian ideal, then $\Gamma_I = \{a \in \mathbb{N}^n | \ x^a \notin \Gamma\}$ is a finite multicomplex and moreover $I_{\Gamma_I} = I$.

Remark 1.5. *The ideal of non-faces of a simplicial multicomplex and the Stanley-Reisner ideal of a simplicial complex are different. More precisely, if Δ is a simplicial complexes and I its the Stanley-Reisner ideal of Δ and if J is the ideal of non-faces of Δ seed as a finite multicomplexes, then I is the ideal generated by the square-free minimal generators of J.*

For example, if $\Delta = \langle \{1,2\},\{2,3\}\rangle$, then the Stanley-Reisner ideal is $I = \langle x_1x_3\rangle$ and the non-faces ideal of Δ (as a multicomplex) is $J = \langle x_1^2, x_2^2, x_3^3, x_1x_3\rangle$.

Proposition 1.6. *1. Γ has only one maximal facet a, if and only if I_Γ is an irreducible artinian monomial ideal.*

2. Let $(\Gamma_j)_j$ be a finite family of multicomplexes. Then:

$$I_{\bigcap_j(\Gamma_j)} = \sum_j I_{\Gamma_j}, \quad I_{\bigcup_j(\Gamma_j)} = \bigcap_j I_{\Gamma_j}.$$

3. Let $\Gamma = \langle u_1, \ldots, u_r\rangle$ be a finite multicomplex. Then

$$I_\Gamma = P_{u_1} \cap P_{u_2} \cap \cdots \cap P_{u_r}$$

is the unique irredundant irreducible decomposition of $I(\Gamma)$.

Proof. 1. Since $\Gamma = \langle a\rangle$, it follows that $I_\Gamma = (x^b | b \in \mathbb{N}^n,\ b(i) > a(i)\ for\ some\ i) = (x_i^{a(i)+1} | i = 1, \ldots n)$. Conversely, if I is a irreducible monomial ideal, then I is generated by powers of variables (i.e $I = \left\langle x_1^{c(i)} | c(i) \geq 1\right\rangle$) and, thus $\Gamma_I = \langle a\rangle$, where $a = c - (1, \ldots, 1)$.

2. It is left as an exercise to reader. 3. It is obvious from 1. and 2. □

Definition 1.7. *Let $\Gamma \subset \mathbb{N}^n$ be a finite multicomplex. The ideal of maximal facets of Γ, denoted by $I(\Gamma) \subset S = k[x_1, \ldots, x_n]$ is the follow one:*

$$I = I(\Gamma) = \langle x^a | a\ is\ a\ maximal\ facet\ in\ \Gamma\rangle.$$

Conversely, if $I \subset S$ is an arbitrary monomial ideal we can asociate the multicomplex

$$\Gamma = \Gamma(I) = \langle a | x^a\ is\ a\ minimal\ generator\ of\ I\rangle.$$

Also, if I is a monomial ideal, we can associate the polarized ideal I^0 which is a square-free monomial ideal. The simplicial complexes of facets of I^0 is called the polarized simplicial complex *of Γ and its denoted by $\Delta^0(\Gamma)$. Obvious, I is Cohen-Macaulay (Gorenstein etc.) if and only if the same property holds for I^0.*

Remark 1.8. *If $\Gamma = \langle u_1, \ldots, u_r\rangle$ is a finite multicomplex and $m = \vee_{j=1}^r u_j$, then $\Delta^0(\Gamma)$ is a simplicial complex on a set of vertices labeled $\{v_1^1, \ldots, v_{m(1)}^1, \ldots, v_1^n, \ldots, v_{m(n)}^n\}$, There is a bijection between the faces of Γ and the sorted faces of Δ ($F \in \Delta$ is called sorted, if $v_i^j \in F \Rightarrow v_{i-1}^j \in F, \ldots, v_1^i \in F$). If $u \in \Gamma$ is a face, the corresponding face in $\Delta^0(\Gamma)$ is $F_u = \{v_1^1, \ldots, v_{u(1)}^1, \ldots, v_1^n, \ldots, v_{u(n)}^n\}$.*

If we make any change on Δ^0 (for example, if we take the complementary complex of Δ^0 or the Alexander dual complex etc.) using the above correspondence and renumbering the vertices, we can write down a new multicomplex which it will be called the complementar multicomplex of Γ (the Alexander dual of Γ etc.). This idea it will be explained later, in the section 3. Anyway, such a multicomplex is called the multicomplex of sorted faces of the corresponding simplicial complex.

Definition 1.9. *We say that the multicomplex $\Gamma' \subset \mathbb{N}^m$ is a subcomplex of $\Gamma \subset \mathbb{N}^n$ if there exists a ordering inclusion of $\mathbb{N}^m$ in $\mathbb{N}^n$ such that $\Gamma' \subset \Gamma \cap \mathbb{N}^m$. In particular, if $n = m$ we demand that $\Gamma' \subset \Gamma$. Obvious, any subcomplex of Γ' in $\mathbb{N}^m$ for $m < n$ correspond to a subcomplex of Γ in $\mathbb{N}^n$ but it can be more of such that subcomplexes.*

For example, if $\Gamma = \langle(1,2,2),(2,1,2)\rangle$ and $\Gamma' = \langle(1,1),(0,2)\rangle$ then Γ' is a subcomplex of Γ via the inclusions $(a,b) \mapsto (a,0,b)$ and $(a,b) \mapsto (b,a,0)$ of $\mathbb{N}^2$ in $\mathbb{N}^3$ (There are still more 3 posibilities. Find them!)

Definition 1.10. *Let $\Gamma \subset \mathbb{N}^n$ be a simplicial multicomplex and $a \in \Gamma$. The* link *of a in Γ is the set*

$$lk_\Gamma(a) = \{b \in \Gamma | a + b \in \Gamma\}.$$

Obvious, $lk_\Gamma(a)$ is also an simplicial multicomplex and a subcomplex of Γ.

The star *of Γ is the set*

$$star_\Gamma(a) = \{b \in \Gamma | a \vee b \in \Gamma\}.$$

which is also an subcomplex of Γ. Obvious, $lk_\Gamma(a) \subset star_\Gamma(a)$.

Let $\Gamma \subset \mathbb{N}^n$ $\Gamma' \subset \mathbb{N}^n$ be two finite multicomplexes. The join *of Γ with Γ', denoted by $\Gamma * \Gamma'$, is the multicomplex:*

$$\Gamma * \Gamma' = \{u + v |\ u \in \Gamma,\ v \in \Gamma'\}.$$

Note that it is not necessary that Γ and Γ' to be in the same $\mathbb{N}^n$. In general case, if $\Gamma \subset \mathbb{N}^n$ and $\Gamma \subset \mathbb{N}^m$ it is enough to choose two ordering inclusions $\mathbb{N}^n \subset \mathbb{N}^N$ and $\mathbb{N}^m \subset \mathbb{N}^N$ and to consider Γ and Γ' as multicomplexes in $\mathbb{N}^N$. Obvious, in that case, $\Gamma * \Gamma'$ depends on the choosed inclusions. How well, there is a canonical way to compute $\Gamma * \Gamma'$: It is suffice to take $N = n + m$ and $N^n \subset N^{n+m}$ to be $(a(1),\dots,a(n)) \mapsto (a(1),\dots,a(n),0,\dots,0)$, respectively $N^m \subset N^{n+m}$ to be $(b(1),\dots,b(m)) \mapsto (0,\dots,0,b(1),\dots,b(m))$.

In particular, if $\Gamma \subset \mathbb{N}^n$ is a multicomplex and $\Gamma' = \{0,1\} \subset \mathbb{N}$, then $\Gamma * \Gamma'$ in the sense of the last construction, is called the cone *over Γ.*

Example 1.11. *Let $\Gamma = \langle(3,1,2),(2,1,3),(3,2,1)\rangle$. Then*

$$lk_\Gamma(2,0,0) = \langle(1,1,2),(0,1,3),(1,2,1)\rangle.$$

Also, $lk_\Gamma(3,0,0) = \langle(0,1,2),(0,2,1)\rangle$ and $star_\Gamma(3,0,0) = \langle(3,1,2),(3,2,1)\rangle$.

Proposition 1.12. *Let Γ be a finite multicomplex, $u \in \Gamma$ and $v \in lk_\Gamma(u)$. Then:*

1. *$dim(\Gamma) = dim(lk_\Gamma(u)) + |u|$. If Γ is pure then $lk_\Gamma(u)$ is also pure.*
2. *$u \in lk_\Gamma(v)$ and $lk_{lk_\Gamma(u)}(v) = lk_{lk_\Gamma(v)}(u) = lk_\Gamma(u+v)$.*
3. *$\langle v \rangle * lk_{lk_v(\Gamma)}(u) \subset lk_{star_\Gamma(v)}(u)$.*
4. *If $\Gamma = \langle u_1, \ldots, u_r \rangle$ and $u \in \Gamma$ and $a \in \mathbb{N}^n$ then:*

$$star_\Gamma(u) = \langle u_i | u \leq u_i \rangle\,, \quad lk_\Gamma(u) = \langle u_i - u | u \leq u_i \rangle \quad \langle a \rangle * \Gamma = \langle u_1 + a, \ldots, u_r + a \rangle\,.$$

Proof. 1.This is obvious.

2.$v \in lk_\Gamma(u) \Rightarrow v + u \in \Gamma \Leftrightarrow u \in lk_\Gamma(v)$. Let $w \in lk_{lk_\Gamma(u)}(v)$. Then $w + v \in lk_\Gamma(u)$ so $w + v + u \in \Gamma$ which is equivalent to the fact that $w \in lk_\Gamma(u+v)$.

We can rewrite this proof, easier, as follows: $lk_{lk_\Gamma(u)}(v) = \{w \in \mathbb{N}^n | v + w \in lk_\Gamma(u)\} = \{w \in \mathbb{N}^n | v + w + u \in \Gamma\} = lk_\Gamma(u+v)$. Analog, $lk_{lk_\Gamma(v)}(u) = lk_\Gamma(u+v)$.

3. Let suppose that $w \in \langle v \rangle * lk_{lk_\Gamma(v)}(u)$. Then $w = w' + w''$ with $w' \leq v$ and $\eta = w'' + u + v \in \Gamma$. We have to proof that $(w+u) \vee v \in \Gamma$. Since $w' \leq w \wedge v \leq v$ and $w - w \wedge v \leq w''$, we can assume that $w' = w \wedge v$ and $w'' = w - w'$. Let $\eta := w - w \wedge v + u \in \Gamma$. It is enough to show that $(w+u) \vee v \in \Gamma$. We have

$$\eta(i) = \begin{cases} v(i) + u(i), & v(i) > w(i) \\ w(i) + u(i), & v(i) \leq w(i) \end{cases}.$$

Let $\xi := (w+u) \vee v$. If $v(i) \leq w(i)$ then $v(i) \leq w(i) + u(i)$, thus $\xi(i) = w(i) + u(i)$. When $v(i) > w(i)$ we cannot say that $v(i) \geq w(i) + u(i)$ but, anyway, $\xi(i) \leq v(i) + u(i)$. The conclusion is that $\xi \leq \eta \in \Gamma$, therefore $\xi \in \Gamma$ as required.

4.The proof is left as an exercise for reader. □

Example 1.13. *Let $\Gamma = \langle (3,4,4), (4,2,5) \rangle$, $u = (3,2,1)$ and $v = (0,1,2)$. Obvious, $star_\Gamma(v) = \Gamma$. Then $lk_{star_\Gamma(v)}(u) = lk_\Gamma(u) = \langle (0,2,3), (1,0,4) \rangle$. Since $lk_{lk_\Gamma(v)}(u) = lk_\Gamma(u+v) = lk_\Gamma(3,3,3) = \langle (0,1,1) \rangle$ we have $\langle v \rangle * lk_{lk_\Gamma(v)}(u) = \langle (0,2,3) \rangle$. This example show us that the inclusion $\langle v \rangle * lk_{lk_v(\Gamma)}(u) \subset lk_{star_\Gamma(v)}(u)$ can be strictly (in the case of simplicial complexes, allways, we have the equality).*

2 Geometrically description and homology of multicomplexes.

Definition 2.1. *Let $\Gamma = \langle u_1, \dots, u_r \rangle$ be an finite simplicial multicomplexes. Let $\Delta^0 = \Delta^0(\Gamma)$ be the polarized complex associate to Γ. Let $|\Delta^0|$ be the underlying topological space of Δ^0. As we already have seen, Δ^0 is a simplicial complex on a set of vertices labeled by $\{v_1^1, \dots, v_{m(1)}^1, \dots, v_1^n, \dots, v_{m(n)}^1\}$.*

The topological space associate to Γ, denoted by $|\Gamma|$ is the quotient topological space of $|\Delta^0|$ obtained by gluing the vertices $\{v_1^1, \dots, v_{m(1)}^1\}$, $\dots$, respectively $\{v_1^n, \dots, v_{m(n)}^1\}$.

Exercise 2.2. *If $\Gamma = \langle a \rangle$ with $a \geq (1, \dots 1)$ then $|\Gamma| \sim \vee_{i=1}^{s} S^1$ where $s = |a| - n$. (Hint, use induction on $|a|$.)*

Example 2.3. *Let $\Gamma = \langle (2,1), (1,2) \rangle \subset \mathbb{N}^2$. The polarized simplicial complex of Δ is $\Delta^0 = \langle \{v_1^1, v_2^1, v_1^2\}, \{v_1^1, v_1^2, v_2^2\} \rangle$. (In other language, $I = I_\Gamma = \langle x^2y, xy^2 \rangle$ and the polarized ideal of I is $I^0 = \langle x_1x_2y_1, x_1y_1y_2 \rangle$). For reasons of comprehensibility, we rewrite as $\Delta^0 = \langle \{1,2,3\}, \{2,3,4\} \rangle$.*

Note that $|\Delta^0|$ consist in two triangles with the common edge $\{2,3\}$. Therefore, $|\Gamma|$ is the topological space obtained from $|\Delta_0|$ by gluing the vertices 1 with 2 and 3 with 4 respectivelly. The topological space obtained $|\Gamma|$ is homotophic equivalent with $S^1 \vee S^1$.

In algebraic speak, the gluing "corresponds" to the factorizations with $x_1 - x_2$ and $y_1 - y_2$ which gives the isomorphism:

$$\frac{K[x_1, x_2, y_1, y_2]}{(x_1x_2y_1, x_1y_1y_2, x_1 - x_2, y_1 - y_2)} \cong \frac{k[x,y]}{(x^2y, xy^2)}.$$

Definition 2.4. *Let $\Gamma \subset \mathbb{N}^n$ be a finite simplicial multicomplex with $e_i \in \Gamma$, $(\forall)i = 1, \dots, n$. Let A be an arbitrary comutative ring with unity. Let Δ^0 the polarized simplicial complexul associate to Γ, and let $\{v_1^1, \dots, v_{m(1)}^1, \dots, v_1^n, \dots, v_{m(n)}^n\}$ be its vertices. Let $C_i(\Delta^0, A)$ be the free A-module spanned by the set of i-faces of Δ. (This is the A-modules complex which is used to compute the simplicial homology of Δ^0).*

Let $C_i(\Gamma, A) := C_i(\Delta^0, A)$ for $i \geq 1$ and let $C_0(\Gamma, A) := C_0(\Delta^0, A)/(e_j^i - e_k^i)$, where e_j^i is the base of $C_0(\Gamma, A)$ (more precisely, e_j^i correspond to the vertix v_j^i). It is obvious that $C_0(\Gamma, A) \cong A^n$.

Let $\partial_i : C_i(\Gamma, A) \to C_{i-1}(\Gamma, A)$, for $i \geq 2$ be the usual differentials and let $\partial_1 : C_1(\Gamma, A) \to C_0(\Gamma, A)$ be the composed map $C_1(\Gamma, A) = C_1(\Delta^0, A) \to C_0(\Delta^0, A) \to C_0(\Gamma, A)$. Let $\partial_0 := 0$. Obvious $\partial_{i-1} \circ \partial_i = 0$ for all $i \geq 1$.

The homology of $C_*(\Gamma, A)$ is the simplicial homology *of the simplicial multicomplex Γ and we denote it by $H_*(\Gamma, A)$. This means that $H_i(\Gamma, A) = Ker(\partial_i)/Im(\partial_{i+1})$.*

Remark 2.5. *Let Γ be a finite simplicial multicomplex. The i-*skeleton *of Γ, is the subcomplexul $\Gamma^{(i)} = \{a \in \Gamma | \ |a| \leq i\}$.*

Let Γ be a simplicial multicomplex. Then $|\Gamma^{(i+1)}|$ is obtained, topological means, by ataching some $i+1$-cells over $|\Gamma^{(i)}|$. Moreover, this gluing is compatible with the differentials ∂_i. In conclusion, $|\Gamma|$ has a structure of cellular complex which is identically with is simplicial structure. I.e. the complex $C_*(\Gamma, A)$ is exactly the cell complex of A-module which compute the homology for a cellular complex.

Therefore, we obtain the following corollary:

Corollary 2.6. *For any multicomplex Γ, $H_*(\Gamma, A) = H_*(|\Gamma|, A)$.*

Example 2.7.

- *Let $\Gamma = \langle (3) \rangle \subset \mathbb{N}$. $\Delta^0(\Gamma)$ is the 2-simplex, thus $|\Delta^0|$ is a triangle. Therefore, $|\Gamma|$ is obtained from the triangle by gluing its vertices. ($|\Gamma|$ looks as an "parachute"!). Obvious, $|\Gamma| \sim S^1 \vee S^1$. Let explained the structure of cell complex of $|\Gamma|$. 0-skeleton consist in a point. 1-skeleton consist in three circles glued in that point. (that means that we have atached three 1-cell over the 0-skeleton). At last, we atached one 2-cell over that three circles to obtain $|\Gamma|$. Let write down the simplicial homology (which is identically with the cell homology) of Γ:*

$$0 \longrightarrow A \xrightarrow{\partial_2} A^3 \xrightarrow{\partial_2} A \xrightarrow{\partial_0} 0.$$

Let denote $C_2(\Gamma, A) = e_{123}A$, $C_1(\Gamma, A) = e_{12}A + e_{13}A + e_{23}A$, $C_0(\Delta^0, A) = e_1A + e_2A + e_3A$ and $C_0(\Gamma, A) = C_0(\Delta^0, A)/(e_1 - e_2, e_1 - e_3) = eA$, where e_{123} corresponds to the face $\{1, 2, 3\}$ of Δ^0 etc.

We have $\partial_2(e_{123}) = e_{23} - e_{13} + e_{12}$. Also, $\partial_1(e_{ij}) = \widehat{e_j} - \widehat{e_i} = e - e = 0$. Thus $\partial_1 = 0$. Since ∂_2 is injective, $H^2(\Gamma, A) = 0$. Also, $H^1(\Gamma, A) = Ker(\partial_1)/Im(\partial_2) = A^3/A = A^2$ and $H^0(\Gamma, A) = Ker(\partial_0)/Im(\partial_1) = A^3/A^2 = A$. This is the well known homology of $S^1 \vee S^1$!

- *Let $\Gamma = \langle (2, 1), (1, 2) \rangle$ the multicomplex from the example* 1.17. *We have already seen that $\Delta^0 = \langle \{1, 2, 3\}, \{2, 3, 4\} \rangle$ and that $|\Gamma| \sim S^1 \vee S^1$. Write down the homology of Γ. We have $C_2(\Gamma, A) = A^2$, $C_1(\Gamma, A) = A^5$, $C_0(\Gamma, A) = A^2$, so:*

$$0 \longrightarrow A^2 \xrightarrow{\partial_2} A^5 \xrightarrow{\partial_2} A^2 \xrightarrow{\partial_1} 0.$$

The matrix of ∂_2 is $\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$ *and the matrix of ∂_1 is* $\begin{pmatrix} -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 \end{pmatrix}$

Obvious, $rank(\partial_2) = 2$ and $rank(\partial_0) = 0$. Then $H^2(\Gamma, A) = 0$, because ∂_2 is injective. $H^1(\Gamma, A) = Ker(\partial_1)/Im(\partial_2) = A^4/A^2 = A^2$ and, ofcourse $H^1(\Gamma, A) = A$. Exercise: Write down the structure of cell complex for Γ.

Remark 2.8. *(The reduced homology of a simplicial multicomplex) As in the case of the simplicial complexes, we can define the reduced homology for a multicomplex, to be the homology of the following A-module complex:*

$$\cdots \to C_i(\Gamma, A) \to C_{i-1}(\Gamma, A) \to \cdots \to C_0(\Gamma, A) \to C_{-1}(\Gamma, A) = A \to 0,$$

where the last map ∂_0 is given by the matrix $(1, \dots, 1)$. Obvious, we thing $C_{-1}(\Gamma, A)$ as the free A-module generated by the -1-faces of Γ, (i.e. by the (0,...,0)!). We denote by $\widetilde{H}_*(\Gamma, A)$ the reduced homology of Γ. Of course, $\widetilde{H}(\Gamma, A) = \widetilde{H}(|\Gamma|, A)$.

Exercise 2.9. *Prove that for any multicomplex Γ, the cone over Γ is acyclic. I.e. $\widetilde{H}_*(\Gamma, A) = 0$.*

Definition 2.10. *Let $\Gamma \in \mathbb{N}^n$ be a finite simplicial multicomplex and let A be an arbitrary comutative ring with unity. We consider the chains complex:*

$$\cdots \to C_i(\Gamma, A) \to C_{i-1}(\Gamma, A) \to \cdots \to C_0(\Gamma, A) \to 0.$$

Using the functor $Hom(-, A)$ on this complex, we obtained a cochains complex:

$$0 \to Hom(C_0(\Gamma, A), A) \to Hom(C_0(\Gamma, A), A) \to \cdots \to Hom(C_{i-1}(\Gamma, A), A) \to Hom(C_i(\Gamma, A), A) \to \cdots$$

Let $C^i(\Gamma, A) := Hom(C_i(\Gamma, A), A)$. We define the differentials $\delta_i : C^i(\Gamma, A) \to C^{i+1}(\Gamma, A)$ by $\delta_i(f)(x) := (-1)^i f(\partial_{i+1}(x))$, for any $x \in C^{i+1}(\Gamma, A)$.

The simplicial cohomology of Γ is, by definition, the cohomology of the cochain complex above, i.e. $H^i(\Gamma, A) := Ker(\delta_i)/Im(\delta_{i-1})$. Moreover, $H^*(\Gamma, A)$ has a structure of a graded A-algebra with the cup-product.

Of course, $H^*(\Gamma, A) = H^*(|\Gamma|, A)$ and, as in the homological case, we can define, similarly, the reduced cohomology of $|\Gamma|$.

Remark 2.11. *It would be interesting to compute the Euler characteristic $\chi(|\Gamma|)$ using only the combinatorial structure of $\Gamma = \langle u_1, \dots, u_r \rangle$. Of course, it is obvious that $\chi(|\Gamma|) = \chi(\Delta^0(\Gamma)) + n - |sup(\Gamma)|$, where $sup(\Gamma) = \vee_{i=1}^r u_i$. So, the problem is to compute $f_i(\Delta^0)$ using the combinatorial structure of Γ...*

3 Shellable finite multicomplexes

Let us recall that a simplicial complexes Δ is said to be connected if there exists a ordering on the facets set of Δ, $\{F_1, \dots, F_r\}$, such that $F_i \cap F_{i+1} \neq \emptyset$. Obvious, Δ is connected if and only if $|\Delta|$ is a connected space. In the case of multicomplexes, we have the following generalization:

Definition 3.1. *A finite simplicial multicomplex Γ is said to be connected, if there exists a ordering on $\mathcal{M}(\Gamma) = \{u_1, \dots, u_r\}$ such that $u_i \wedge u_{i+1} > (0, \dots, 0)$ for any $i = 1, \dots, r-1$. Obvious, Γ is connected, if and only if its topological underlying space is connected.*

Proposition 3.2. *Let Δ be a connected, pure simplicial complex. Let $F_1, \dots, F_r$ be a fixed orderion of the set of facets of Δ. The, the following assertions are equivalent:*

1. *Δ is shellable whit the ordering $F_1, \dots, F_r$: i.e $\langle F_1, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$ is generated by a set o proper maximal faces of $\langle F_i \rangle$.*

2. *The set $S_i = \{F | F \in \langle F_1, \dots, F_i \rangle, F \notin \langle F_1, \dots, F_{i-1} \rangle\}$ have only one minimal element, for any $i = 2, \dots, r$.*

3. *For any $j < i$, there exists a vertix $v \in F_i \setminus F_j$ and there exists $k < i$ such that $F_i \setminus F_k = \{v\}$.*

Definition 3.3. *Let $\Gamma \subset \mathbb{N}^n$ be a finite multicomplex. Let $b, a \in \Gamma$. We call a a* lower neighbour *of b if there exists an integer k such that $a(k) + 1 = b(k)$ and $a(i) = b(i)$ for any $i \neq k$. Equivalent, a is a lower neighbour of b if $a < b$ and $|a| = |b| - 1$.*

For example, $(4, 3, 0, 2)$ is a lower neighbour of $(4, 3, 1, 2)$.

Definition 3.4. *Let Γ be a finite connected pure multicomplex. We say that Γ is* shellable *as a finite multicomplex, if there is a order on the set of maximal facets of Γ, $u_1, \dots, u_r$ such that $\langle u_1, \dots, u_{i-1} \rangle \cap \langle u_i \rangle$ is generated by a set of lower neighbours of u_i.*

Our first aim is to give a characterization of shellability for a multicomplex using the scheme of the above proprosition.

Proposition 3.5. *Let Γ be a finite connected pure multicomplex. The following assertions are equivalents:*

1. *Γ is shellable with the order $u_1, \dots, u_r$ on $\mathcal{M}(\Gamma)$.*

2. *The set $S_i = \{v \in \mathbb{N}^n | v \leq u_i, v \not\leq u_j \text{ for } j < i\}$ has only one minimal element v, which, moreover has the property $v(j) = u_i(j)$ or $v(j) = 0$, $(\forall) j \in [n]$.*

3. *For any $j < i$, there exists an $m \in [n]$ with $u_i(m) > u_j(m)$ and a $k < i$ such that $u_i(m) = u_k(m) + 1$, and $u_i(s) \leq u_k(s)$ for $s \neq m \in [n]$.*

Proof. ($1 \Rightarrow 2$). Let suppose that $\langle u_1, \dots, u_{i-1}\rangle \cap \langle u_i\rangle$ is generated by the following lower neighbours ofu_i, $u_i - e_{i_1}, \dots, u_i - e_{i_k}$. Let $v := \begin{cases} u_i(j), & j \in \{i_1, \dots, i_k\} \\ 0, & j \notin \{i_1, \dots, i_k\} \end{cases}$. It is enough to prove that v is a minimal element of $S_i = \{a \in \mathbb{N}^n | a \leq u_i, a \nleq u_j \ pentru \ j < i\}$. Obvious, $v \leq u_i$. Also, from its definition, $v \nleq u_j$ for $j < i$ (because each u_j for $j \in \{i_1, \dots, i_k\}$ has at least one of its components " $<$ " than $u_i(j)$.)

Let suppose now that there exists a v' with $v' \leq u_i$ and $v' \nleq u_j$ for $j < i$. We have to show that $v \leq v'$.

Let notice that the maximal facets of $\langle u_1, \dots, u_{i-1}\rangle \cap \langle u_i\rangle$ are among $u_i \cap u_1, \dots, u_i \cap u_{i-1}$. Also, since Γ is shellable, it folows that the maximal facets have the dimension, $dim(u_i) - 1$.

For any $j \notin \{i_1, \dots, i_k\}$, we have $0 = v(j) \leq v'(j)$. Let suppose that $v'(i_1) < v(i_1) = u_i(i_1)$. I choose j such that $u_j \wedge u_i = u_i - e_{i_1}$. We have $u_j(i_1) = u_i(i_1) - 1$ and $u_j(t) \geq u_i(t)$, for any $t \neq i_1$. But then $v' \leq u_j$ which is a contradiction.

($2 \Rightarrow 3$). Before giving the proof in the general case, let study some particular cases. If $i = 1$ there is nothing to prove. If $i = 2$, I claim that there is only one nonzero component of f. Indeed, let suppose $v(1) = u_2(1) > 0, \dots, v(e) = u_2(e) > 0$. Obvious, there is an index such that $v(k) > u_1(k)$, or else $v \leq u_1$ which is absurd. Let suppose $v(1) > u_1(1)$. But then it is obvious that $v' = (v(1), 0, \dots, 0) \in S_2$! This force $e = 1$. From the uniqueness of v if follows that $u_1(k) \geq u_2(k)$ for any $k > 1$. Indeed, if $u_1(2) < u_2(2)$ for example, then $v' = (0, u_2(2), 0, \dots, 0) \in S_2$ and this in a contradiction! I claim that $u_1(1) = u_2(1) - 1$. Indeed, if $u_1(1) \leq u_2(1) - 1$ then $v' = (u_2(1) - 1, 0, \dots, 0) \in S_2$ and $v' < v$ which is again absurd. Since $|u_1| = |u_2|$, $u_1(1) = u_2(1) - 1$ and $u_1(k) \geq u_2(k)$ for any $k > 1$, it follows that there exists a $m > 1$ such that $u_1(m) = u_2(m) - 1$ and $u_1(k) = u_2(k)$ for any $k \neq 1, m$. The assertion 3 is now trivial.

The case $i = 3$ is more complicated. In the romanian version of this paper I'll give the proof, but since it is not need it at all, I prefere to skip it. Before get to the general case, we make some remarks:

- The condition 3 of the proposition can be replaced as follows: For any $j < i$ there exists a $k < i$ such that $u_j \wedge u_i \leq u_k \wedge u_i$ şi $d(u_i, u_k) = 1$.
- If $v \in S_i$ is the unique minimal element of S_i by reordering of vertices, we can assume that $v(1) = u_1(1) > 0, \dots, v(e) = u_i(e) > 0, v(e+1) = \cdots = v(n) = 0$.
- For any $m > e$ there exists a $j < i$ such that $u_i(m) \leq u_j(m)$. Indeed, on contrary, the vector $(0, \dots, u_i(m), \dots, 0)$ will be in S_i which is a contradiction whit the uniqueness of v.
- Also, I cannot have simultaneous $v(1) > max\{u_1(1), \dots, u_{i-1}(1)\}$ and $v(2) > max\{u_1(2), \dots, u_{i-1}(2)\}$ because in this case there are two minimal vector in S_i.
- Last but not least, let's notice that the vector u_j, for $j < i$ are obtained from a previous one be adding $+1$ to a component and substracting $+1$ to another. A posteriori,

this is clear from the definition of shellability. Anyway, this fact it is not use in the proof below.

Suppose $v = (u_i(1), \dots, u_i(e), 0, \dots, 0)$ is the unique minimal element of S_i. First of all, we want to prove that for any $j < i$, we have:

$$u_j \wedge u_i \leq (u_i(1) - 1, u_i(2), \dots, u_i(n)),\ or\ u_j \wedge u_i \leq (u_i(1), u_i(2) - 1, u_i(2), \dots, u_i(n))\ or$$

$$or\ \cdots\ or\ u_j \wedge u_i \leq (u_i(1), \dots, u_i(e-1), u_i(e) - 1, u_i(e+1), \dots, u_i(n)).$$

But this is almost obvious! Indeed, if the above condition fails for some j, if follows immediately that $v \leq u_j$.

Moreover, each inequality holds for some j. If, for example, $u_j \wedge u_i \nleq (u_i(1)-1, u_i(2), \dots, u_i(n))$ for any $j < i$ it follows that $(0, u_i(2), \dots, u_i(e), 0, \dots, 0) \in S_i$ which is a contradiction with the minimality of v.

Let $j < i$ with $u_j \wedge u_i \leq (u_i(1) - 1, u_i(2), \dots, u_i(n))$. I shall prove that there is a $k < i$ such that $u_k \wedge u_i = (u_i(1) - 1, u_i(2), \dots, u_i(n))$ and this, obvious, complete the proof. Let suppose that $u_j \wedge u_i \neq (u_i(1) - 1, u_i(2), \dots, u_i(n))$, for any $j < i$. Let $v' = (u_i(1) - 1, u_i(2), \dots, u_i(n))$. Obvious, $v' \leq u_i$. If there exists a $k < i$ such that $v' \leq u_k$ it follows that $u_k \wedge u_i = (u_i(1) - 1, u_i(2), \dots, u_i(n))$, a contradiction. On the other hand, if $v' \nleq u_j$ for any $j < i$ it follows that $v' \in S_i$, and this is again a contradiction, because $v \nleq v'$!

$(3 \Rightarrow 1)$. Let $v \in \langle u_1, \dots, u_{i-1} \rangle \cap \langle u_i \rangle$. Then $v \leq u_i \wedge u_j$ for some $j < i$. Let m as in assertion 3. Then, there exists an k such that $u_i(m) = u_k(m) + 1$ and $u_i(s) \leq u_k(s)$ for $s \neq m$. Obvious $v \leq u_k$, because $v \leq u_i$ (and then $v(s) \leq u_i(s) \leq u_k(s)$ for $s \neq m$) and $v(m) \leq u_j(m) \leq u_k(m)$. Thus $v \leq u_i \wedge u_k$. Also, it is clear that $|u_i \wedge u_k| = |u_i| - 1$. Then $u_i - e_m$ is a lower neighbour for u_i in $\langle u_1, \dots, u_{i-1} \rangle \cap \langle u_i \rangle$ cu $v \leq u_i - e_m$. But that means Γ is shellable. $\square$

Example 3.6. *Let* $\Gamma = (2, 1, 0), (1, 2, 0), (0, 2, 1)$. *Then* Γ *is shellable. Indeed,* $\langle (1, 2, 0) \rangle \cap \langle (2, 1, 0) \rangle = (1, 1, 0)$ *and* $\langle (0, 2, 1) \rangle \cap \langle (2, 1, 0), (1, 2, 0) \rangle = (0, 2, 0)$.

The minimal element of $S_2 = \{v | v \leq u_1, v \nleq u_2$ *is* $v = (0, 2, 0)$ *and the minimal element of* S_3 *is* $w = (0, 0, 1)$. *Obvious,* v *and* w *satisfies condition* 2 *of the proposition.*

Remark 3.7. *Let* Γ *be a simplicial multicomplex and let* $I(\Gamma)$ *be the ideal of maximal facets of* Γ. *Suppose that* Γ *is shellable. From the assertion* 3 *of the proposition, we have: for any* $j < i$ *there exists* $k < i$ *such that* $u_j \wedge u_i \leq u_k \wedge u_i$ *şi* $d(u_i, u_k) = 1$. *The translation of this assertion in algebraic language is:*

For any $j < i$, *there exists* $k < i$ *such that Evident* $I(\Gamma) = \langle m_1 = x^{u_1}, \dots, m_r = x^{u_r} \rangle$. *Traducerea sună astfel: Pentru fiecare* $1 \leq j < i \leq r$, *există un* $k < i$ *astfel încât* $gcd(m_i, m_j) | gcd(m_i, m_k)$ *şi* $m_i / gcd(m_i, m_k) = x_t$ *for some* t.

Note the similarity, but not coincidence, with the ideals with linear quotients!

Proposition 3.8. *Let Γ be a finite connected pure multicomplex. Then Γ is shellable if and only if $\Delta^0 = \Delta^0(\Gamma)$ is shellable.*

Proof. Suppose $\Gamma = \langle u_1, \ldots, u_r \rangle$. Then $\Delta^0 = \langle F_1, \ldots, F_r \rangle$, unde

$$F_i = \{v_1^1, \ldots, v_{u_i(1)}^1, v_1^2, \ldots, x_{u_i(2)}^2, \ldots, x_1^n, \ldots, x_{u_i(n)}^n\}.$$

Obvious Γ is pure, if and only if Δ^0 is pure. Assume that Γ is shellable. Using the above proposition, if follows that for any $j < i$, there exists m with $u_i(m) > u_j(m)$ and $k < i$ such that $u_i(m) = u_k(m) + 1$ and $u_i(s) \leq u_k(s)$ for $s \neq m$.

In terms of facets of Δ^0, the above fact is equivalent with the following one: For any $j < i$ there exists m with $x_{u_i(m)}^m \in F_i \setminus F_j$ and $k < i$ such that $F_i \setminus F_k = \{x_{u_i(m)}^m\}$. But this means that Δ^0 is shellable, as required. □

A well known property of shellable pure simplicial complexes is the following one:

Proposition 3.9. *If Δ is a pure shellable simplicial complex, then $|\Delta|$ has the homotopy type of a wedge of spheres of dimension d.*

Corollary 3.10. *If Γ is a pure shellable multicomplex, then $|\Gamma|$ has the homotopy type of a topological space obtained by a wedge of spheres of dimension d by gluing some points and therefore, it is a wedge of spheres of dimension d and 1.*

We can extend the notion of shellability for the simplicial complexes which are not pure, as follows:

Definition 3.11. *Let Δ be a simplicial complex. Δ is called* shellable *if there exists a ordering of facets of Δ, $F_1, \ldots, F_r$ such that $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure of dimension $dim(\Delta) - 1$.*

Obvious, this definition can be extended for multicomplexes:

Definition 3.12. *Let Γ be a finite multicomplex. Γ is called* shellable *if there exists a ordering of maximal facets of Γ, $u_1, \ldots, u_r$ such that $\langle u_1, \ldots, u_{i-1} \rangle \cap \langle u_i \rangle$ is generated by a set of lower neighbors of u_i.*

Lemma 3.13. *If Δ is shellable with the order $F_1, \ldots, F_r$, then: $|F_1| \geq |F_2|, \cdots |F_1| \geq |F_r|$. In particular, $dim(\Delta) = dim(F_1)$.*

Proof. We argue by induction on i. For $i = 2$, since $\langle F_1 \rangle \cap \langle F_2 \rangle$ has dimension $dim(F_2) - 1$, it follows that $|F_1| > |F_2| - 1$ and therefore $|F_1| \geq |F_2|$. Suppose $i > 2$. Then, by induction hypothesis, we have: $|F_1| \geq |F_2|, \cdots |F_1| \geq |F_{r-1}|$. Since $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ has the dimension $dim(F_i) - 1$, it follows that there exists a $k < i$, such that $|F_k \cap F_i| = |F_i| - 1$. But then $|F_k| > |F_i| - 1 \Rightarrow |F_k| \geq |F_i|$, and "a fortiori" $|F_1| \geq |F_i|$. □

This lemma can be written in language of multicomplexes:

Lemma 3.14. *If Γ is shellable with order $u_1, \ldots, u_r$, then: $|u_1| \geq |u_2|, \cdots, |u_1| \geq |u_r|$.*

Proof. We argue by induction on i. For $i = 2$, since $\langle u_1 \rangle \cap \langle u_2 \rangle$ has dimension $dim(u_2) - 1$, it follows that $|u_1| > |u_2| - 1$ and therefore $|u_1| \geq |u_2|$. Suppose $i > 2$. Then, by induction hypothesis, we have: $|u_1| \geq |u_2|, \cdots |u_1| \geq |u_{r-1}|$. Since $\langle u_1, \ldots, u_{i-1} \rangle \cap \langle u_i \rangle$ has the dimension $dim(u_i) - 1$, it follows that there exists a $k < i$, such that $|u_k \wedge u_i| = |u_i| - 1$. But then $|u_k| > |u_i| - 1 \Rightarrow |u_k| \geq |u_i|$, and "a fortiori" $|u_1| \geq |u_i|$. $\square$

Proposition 3.15. *Let Δ a shellable simplicial complex. Then there exists a shelling order $F_1, \ldots, F_r$ such that $|F_1| \geq |F_2| \geq \cdots \geq |F_r|$. Such a shelling is called a "good shelling".*

Proof. We use induction on r. For $r = 1, 2$ the assertion is obvious. Let us first prove the case $r = 3$. Suppose F_1, F_2, F_3 is a shelling with $|F_1| > |F_2|$ and $|F_3| > |F_2|$. From definition of shellability, it follows that $F_3 \cap F_2 \subsetneq F_1 \cap F_3$ and $|F_1 \cap F_3| = |F_3| - 1$. I claim that F_1, F_3, F_2 is a good shelling.

Indeed, F_1, F_3 satisfy the definition of shellability. But we have $F_2 \cap F_3 \subset F_1 \cap F_3$. Taking $\cap F_2$, we get: $F_2 \cap F_3 \subset F_1 \cap F_3 \cap F_2$, and therefore $F_2 \cap F_3 \subset F_1 \cap F_2$.

In general case, let suppose that we have a shelling such that $|F_1| \geq |F_2| \geq \cdots \geq |F_{r-1}|$ and $|F_r| > |F_{r-1}|$. We choose the greatest j such that $|F_j| \geq |F_r|$ (i.e. $|F_{j+1}| < |F_r|$). I claim that $F_1, \ldots, F_j, F_r, F_{j+1}, \ldots, F_{r-1}$ is a good shelling Δ. Ofcourse, the condition of shellability is satisfy from 1 to j. Let us show that $\langle F_1, \ldots, F_j \rangle \cap \langle F_r \rangle$ is generated by $dim(F_r) - 1$-facets. But this is almost obvious: From hypothesis, I know that $\langle F_1, \ldots, F_{r-1} \rangle \cap \langle F_r \rangle$ is generated by $dim(F_r) - 1$-facets. Those facets are between $F_1 \cap F_r, \ldots, F_{r-1} \cap F_r$. But $F_{j+1} \cap F_r, \ldots, F_{r-1} \cap F_r$ have at most dimension $|F_r| - 2$!

Let us show that $\langle F_1, \ldots, F_j, F_r \rangle \cap \langle F_{j+1} \rangle$ is generated by $dim(F_{j+1}) - 1$-facets. It is suffice to prove that $F_{j+1} \cap F_r$ is a subface of $F_{j+1} \cap F_t$ for some $t \leq j$. From the initial hypothesis ($F_1, \ldots, F_r$ ¸is a shelling), this is obvious, because $F_r \cap F_{j+1}$ cannot be a subface of $F_r \cap F_{j+s}$, $s > 1$ because it have to be included in a $dim(F_r) - 1$-face.

Similary, we prove the remains conditions. $\square$

This lemma can be written in language of multicomplexes:

Proposition 3.16. *Let Γ be a shellable multicomplex. Then there exists a "good" shelling (i.e. a shelling with $|u_1| \geq |u_2| \geq \cdots \geq |u_r|$).*

Proof. As is the case of simplicial complexeles, we argue by induction on r, the cases $r = 1, 2$ being trivial. Let suppose $r = 3$. I suppose $|u_1| > |u_2|$ and $|u_2| < |u_3|$. I claim that u_1, u_3, u_2 is a good shelling.

From definition of shellability, it follows that $F_3 \cap F_2 \subsetneq F_1 \cap F_3$ and $|F_1 \cap F_3| = |F_3| - 1$. I claim that F_1, F_3, F_2 is a good shelling. Indeed, u_1, u_3 satisfy the definition of shellability. But we have $u_2 \wedge u_3 \leq u_1 \cap u_3$. Taking $\wedge u_2$, we get: $u_2 \wedge u_3 \subset u_1 \cap u_3 \cap u_2$, and therefore $u_2 \cap u_3 \subset u_1 \cap u_2$.

In general case, let suppose that we have a shelling such that $|u_1| \geq |u_2| \geq \cdots \geq |u_{r-1}|$ and $|u_r| > |u_{r-1}|$. We choose the greatest j such that $|u_j| \geq |u_r|$ (i.e. $|u_{j+1}| < |u_r|$). I claim that $u_1, \ldots, u_j, u_r, u_{j+1}, \ldots, u_{r-1}$ is a good shelling on Γ. Ofcourse, the condition of shellability is satisfy from 1 to j. Let us show that $\langle u_1, \ldots, u_j\rangle \cap \langle u_r\rangle$ is generated by $dim(u_r) - 1$-maximal facets. But this is almost obvious: From hypothesis, I know that $\langle u_1, \ldots, u_{r-1}\rangle \cap \langle u_r\rangle$ is generated by $dim(u_r) - 1$-facets. Those facets are between $u_1 \cap u_r$, $\ldots$, $u_{r-1} \cap u_r$. But $u_{j+1} \cap u_r$,$\ldots$,$u_{r-1} \cap u_r$ have at most dimension $|u_r| - 2$!

Let us show that $\langle u_1, \ldots, u_j, u_r\rangle \cap \langle u_{j+1}\rangle$ is generated by $dim(u_{j+1}) - 1$-facets. It is suffice to prove that $u_{j+1} \cap u_r$ is a subface of $u_{j+1} \cap u_t$ for some $t \leq j$. From the initial hypothesis ($u_1, \ldots, u_r$ is a shelling), this is obvious, because $u_r \cap u_{j+1}$ cannot be a subface of $u_r \cap u_{j+s}$, $s > 1$ because it have to be included in a $dim(u_r) - 1$-face.

Similary, we prove the remains conditions. $\square$

4 Co-shellable multicomplexes

In this section, all the complexes and multicomplexes are supposed pure.

Definition 4.1. *A simplicial complex Δ is called* co-shellable*, if there exists a order of facets of Δ, $F_1, \ldots, F_r$ such that:*

$$(*)(\forall)j < i, (\exists)v \in F_j \setminus F_i, \ si\ k < i\ cu\ F_k \setminus F_i = \{v\}.$$

Proposition 4.2. *Let Δ be a simplicial complex on the vertex set $[n]$ and let $I = I(\Delta)$ be the facets ideal of Δ. Then I have linear quotient if and only if Δ is co-shellable.*

Since the ideal of the basis of a matroid have linear quotients, it follows that any matroid is a pure co-shellable simplicial complex.

Proof. Let $I = (m_1, \ldots, m_r)$ be a square-free monomial ideal. Let $\Delta = \langle F_1, \ldots, F_r \rangle$ be the correspondig simplicial complex (i.e. $F_i = supp(m_i) \subset [n]$). I want to prove that Δ is co-shellable with that gived order. Let $j < i$ and let $v = m_j / gcd(m_i, m_j)$. Obvious, v is a square-free monomial. Since $v \cdot m_i = lcm(m_i, m_j)$, which is a multiple of m_j, it follows that $v \in (m_1, \ldots, m_{i-1}) : m_i$. But I have linear quotient, and therefore, there exists a variable $x_t | v$ such that $x_t \in (m_1, \ldots, m_{i-1}) : m_i$. But that means there exists a monomial m_k with $m_k | x_t m_i$. Thus $F_k \setminus F_i = \{t\}$, and $t \in F_j \setminus F_i$. This complete the proof. The converse implication have a similar proof. □

Example 4.3.

- *There exists shellable complexes which are not co-shellable. This is the case, for example, when we give a shelling $F_1, \ldots, F_r$ such that $F_i \cap F_j = \emptyset$ for some $j < i$. For instance, let Δ be the complex of facets of the ideal $I = (abc, bcd, def, efg)$. Obvious, Δ is shellable, but I does not have linear quotients: $(abc, bcd, def) : efg = (d, abc)$.*

- *Even if we demand that Δ is* strong connected *(i.e. for any two facets F_i and F_j we have $F_i \cap F_j \neq \emptyset$) which is a very restrictive condition, there are shellable complexes which are not co-shellable. For example, if Δ is the facets complex of the ideal $I = (abc, bcd, cde, cef)$, then Δ is shellable but I does not have linear quotients: $(abc, bcd, cde) : cef = (d, ab)$.*

- *Also, there are co-shellable complexes which are not shellable. For instance, if $\Delta = \langle abc, bcd, acd, ade, bce \rangle$. It is easy to see that $I(\Delta)$ has linear quotients, but, also, Δ is not shellable since $\langle bce \rangle \cap \langle abc, bcd, acd, ade \rangle$ is not pure.*

The above definition can be extend for simplicial multicomplexes.

Definition 4.4. *A finite multicomplex Γ is called* co-shellable *if there exists a order of maximal facets of Γ such that for any $j < i$ there is a m and a $k < i$ such that $u_j(m) > u_i(m)$, $u_k(m) = u_i(m) + 1$ and $u_k(s) \leq u_i(s)$ pentru $s \neq m$.*

Proposition 4.5. *Any monomial ideal I, generated by monomial at the same degree, have linear quotients if and only if the simplicial multicomplex of maximal facets of I is co-shellable.*

In particular, any discrete polymatroid is a co-shellable finite multicomplex.

Proof. The proof is the same as in the square-free case. Let $I = (m_1, \dots, m_r)$ be a monomial ideal and let $\Gamma = \langle u_1, \dots, u_r \rangle$ be the corresponding simplicial complex (i.e. $m_i = x^{u_i}$). I want to prove that Γ is co-shellable with that given order. Let $j < i$ and let $v = m_j/gcd(m_i, m_j)$. Since $v \cdot m_i = lcm(m_i, m_j)$, which is a multiple of m_j, it follows that $v \in (m_1, \dots, m_{i-1}) : m_i$. But I have linear quotient, and therefore, there exists a variable $x_t|v$ such that $x_t \in (m_1, \dots, m_{i-1}) : m_i$. But that means there exists a monomial m_k with $m_k|x_t m_i$. Thus $u_k(t) = u_i(t) + 1$ and $u_k(s) \leq u_i(s)$ for $s \neq t$. Also, since $x_t|v = m_j/GCM(m_i, m_j) \Rightarrow u_j(t) > u_i(t)$. But this proves that Γ is co=shellable.

The converse implication have a similar proof. □

Proposition 4.6. *Let Δ be a simplicial complex. Then Δ is shellable if and only if Δ^c is co-shellable. (where Δ^c is the complementary simplicial complex of Δ)*

Proof. Suppose Δ is shellable, i.e. there exists a order $F_1, \dots, F_r$ on the set of facets of Δ such that: For each $j < i$, there exists a $v \in F_i \setminus F_j$ and there exists $k < i$ such that $F_i \setminus F_k = \{v\}$. I claim that $F_1^c, \dots, F_1^c$ is a co-shelling on Δ^c. But this is obvious, for the same choice of $k < i$ and v, since $F_j^c \setminus F_i^c = F_i \setminus F_j$ and $F_k^c \setminus F_i^c = F_i \setminus F_k$! □

Later, we will extend this property to multicomplexes.

5 The base ring and the Erhart ring of a multicomplex

Let Γ be a finite multicomplex with the maximal facets set $\mathcal{M}_\Gamma = \{u_1, \dots, u_r\}$. The *base ring* of Γ is the monomial subalgebra

$$K[\mathcal{M}(\Gamma)] := k[x^{u_1}, \dots, x^{u_r}] \subset k[x_1, \dots, x_n].$$

The *Erhart ring* of Γ is the monomial subalgebra:

$$K[\Gamma] := k[x^u t | u \in \Gamma] \subset k[x_1, \dots, x_n, t].$$

Obvious, $K[\Gamma]$ is the semigroup ring of the cone over Γ, $C(\Gamma) = \langle (u_1, 1), \dots, (u_r, 1) \rangle$.

Obvious, we have a natural epimorphism $\varphi : B = k[t_1, \dots, t_r] \to K[\mathcal{M}(\Gamma)]$, defined by $\varphi(t_i) := x^{u_i}$. If we take on B the grading, $deg(t_i) := deg(m_i)$, where $m_i = x^{u_i}$, then φ became a graded morphism. The kernel $Ker(\varphi) := P_{\mathcal{M}(\Gamma)}$ is called the toric ideal of $K[\mathcal{M}(\Gamma)]$. As well known, $P_{\mathcal{M}(\Gamma)}$ is a graded prime ideal generated by a finite set of binomial. Of course, the same construction can be made for the Erhart ring.

It would be a great interest to find combinatorial condition on Γ such that the base ring or the Erhart ring are normal, Cohen-Macaulay, Gorenstein etc. For example, if Γ is shellable, what can we say about $k[\mathcal{M}(\Gamma)]$ or $k[\Gamma]$?

6 Dual multicomplexes

Definition 6.1. *Let Δ be a simplicial complex. We called the* complementary complex *of Δ, and we denoted by Δ^c, the complex*

$$\Delta^c = \langle [n] \setminus F | F \ is \ a \ facet \ of \ \Delta \ \rangle \, .$$

Obvious, if we thing Δ as a subset of $\{0,1\}^n$, then

$$\Delta^c = \langle (1,1,\dots,1) - F | F \in \Delta \ facet \rangle \, .$$

We can, therefore, give the following generalization.

Definition 6.2. *Let $\Gamma \subset \mathbb{N}^n$ be a finite simplicial multicomplex with the set of maximal facets $\mathcal{M}(\Gamma) = \{u_1, \dots, u_r\}$. If $u \in \mathbb{N}^n$ is a "majorant" of Γ (i.e. $u \geq a$, for any $a \in \Gamma$; or equivalent: $\Gamma \subset \langle u \rangle$) the the complementary multicomplex of Γ with respect to u, denoted by Γ^c_u is the following one:*

$$\Gamma^c_u = \langle u - u_i | u_i \in \mathcal{M}(\Gamma) \rangle \, .$$

Obvious Γ^c_u depends on the choose of $u \in \mathbb{N}^n$. Of course, the minimal "majorant" of Γ, which will be denoted by $sup(\Gamma)$, is $sup(\Gamma) = \vee_{i=1}^r u_i$, where $\Gamma = \langle u_1, \dots, u_r \rangle$. We denoted $\Gamma^c_{sup(\Gamma)} = \Gamma^c$.

Remark 6.3. *Let Γ be a simplicial multicomplex and let $\Delta = \Delta^0(\Gamma)$ be the polarized simplicial complex of Γ. Let us consider Δ^c the complementary complex of Δ. Then, the multicomplex of ordered faces (see section 1) of Δ^c is Γ^c itself. The proof is left as an exercise, since it is obvious.*

Proposition 6.4. *If $\Gamma = \langle u_1, \dots, u_r \rangle$ is a simplicial multicomplex, and $u \in \mathbb{N}^n$ is a majorant of Γ, then u is a majorant of Γ^c_u too, and:*

$$(\Gamma^c_u)^c_u = \Gamma.$$

Proof. If $\Gamma = \langle u_1 \rangle$ and $u \geq u_1$, the assertion is obvious, even in the case $u = u_1$. Let suppose $\Gamma = \langle u_1, \dots, u_r \rangle$ with $r \geq 2$. I claim that the only thing we have to prove is: if $a, b \in \mathbb{N}^n$ are two incomparable vectors, and $u \in \mathbb{N}^n$ $u > a$, $u > b$ then $u - a$,$u - b$ are incomparable. If the claim is true, then it follows that Γ^c_u have the exactly r maximal facets $u - u_1, \dots u - u_r$ (and it is obvious that each of them is $\leq u$!) and therefore $(\Gamma^c_u)^c_u$ has the maximal facets $u_1, \dots, u_r$. Thus $(\Gamma^c_u)^c_u = \Gamma$, as required.

The claim is almost clear: Indeed, if $u - a \geq u - b$ it follows $u(i) - a(i) \geq u(i) - b(i)$ for any $i = 1, \dots, n$, so $a(i) \leq b(i)$ for any i so $a \leq b$, which is a contradiction. $b(i) \leq a(i)$ □

In monomial language, we can write down the following definition:

Definition 6.5. *Let $I = (m_1, \dots, m_r)$ be a monomial ideal and let $\Gamma = \Gamma(I) = \langle u_1, \dots, u_r \rangle$ be the multicomplex of maximal facets of I. Let $u \in \mathbb{N}^n$ be a majorant of Γ. (i.e. $lcm(m_1, \dots, m_r) | x^u$). The complementary ideal of I, with respect to x^u is the ideal $I^c_u := \langle x^u / m_i | i = 1, \dots, n \rangle$. Obvious, I^c_u is the ideal of maximal facets of Γ^c_u.*

Example 6.6. *If* $\Gamma = \langle (2,1,3),(1,2,3),(3,2,2)\rangle$ *and* $u = (4,4,3)$*, then*

$$\Gamma_u^c = \langle (2,3,0),(3,2,0),(1,2,1)\rangle\,.$$

In algebraic language, if $I = (x^2yz, xy^2z^3, x^3y^2z^2)$ *and* $m = x^4y^4z^3$*, then*

$$I_m^c = (x^2y^3, x^3y^2, xy^2z).$$

Exercise 6.7. *If* Γ *is a multicomplex and* $v \geq u \geq sup(\Gamma)$ *are two vectors in* $\mathbb{N}^n$*, then* $\Gamma_v^c = \langle v-u\rangle * \Gamma_u^c$*.*

Proposition 6.8. *Let* Γ *be a pure multicomplex and* $u \geq sup(\Gamma)$*. Then* Γ *is shellable if and only if* Γ_u^c *is co-shellable.*

Proof. The case $\Gamma = \langle u\rangle$ is trivial. Since Γ shellable, there exists a ordering of maximal facets of Γ, $u_1,\dots,u_r$ such that: for any $j < i$, there exists a m and a $k < i$ such that: $u_i(m) > u_j(m)$ and $u_i(m) = u_k(m)+1$ and $u_i(s) \leq u_k(s)$ for $s \neq m$. I claim that Γ_u^c is co-shellable with the ordering of maximal facets: $u-u_1,\dots,u-u_r$. Indeed, if we take m and $k<i$ as above, it is obvious that $(u-u_i)(m) = u(m)-u_i(m) < (u-u_j)(m) = u(m)-u_j(m)$ and $(u-u_i)(m) = (u-u_k)(m)-1$ and $(u-u_i)(s) \geq (u-u_k)(s)$ for any $s \neq m$, as required. □

We will discuse now on the very important notion of Alexander duality. First of all, lets see what is the Alexander dual for a simplicial complexes and how we can extend this concept in the case of multicomplexes.

Definition 6.9. *Let* Δ *be a simplicial complex. The* Alexander dual *of* Δ*, is the complex*

$$\Delta^{\vee} = \{[n]\setminus F | F \notin \Delta\}.$$

Thinking Δ *as a subset of* $\{0,1\}^n$*, we observe that* $\Delta^{\vee} = \{(1,\dots,1)-F|\ F \in \{0,1\}^n \setminus \Delta\}$*. This give us the idea of the following generalization:*

Definition 6.10. *Let* Γ *be a simplcial multicomplex and let* $u \in \mathbb{N}^n$ *be a majorant of* Γ*. The* Alexander dual *of* Γ*, w.r.t.* u *is the following multicomplex:*

$$\Gamma_u^{\vee} = \{u-v|v \leq u \ \ si \ \ v \notin \Gamma\}.$$

If $u = sup(\Gamma)$*, we denote* $\Gamma_u^{\vee} =: \Gamma^{\vee}$*.*

Let us remember some results on Alexander dual (in the case of simplicial complexes) which we will generalize in the case of multicomplexes.

Theorem 6.11. *Let Δ be a simplicial complex on the set of vertices $[n]$. Let I_Δ be the Stanley-Reisner ideal of Δ and $I(\Delta)$ be the ideal of facets of Δ. Then:*

1. $(\Delta^\vee)^\vee = \Delta$.
2. $I_{\Delta^\vee} = I(\Delta^c)$.
3. Δ *is shellable if and only if* $I_{\Delta^\vee}$ *has linear quotients.*

Proposition 6.12. *If Γ is a multicomplex and $u \in \mathbb{N}^n$ is a majorant for Γ then $(\Gamma_u^\vee)_u^\vee = \Gamma$.*

Proof. Let us first observe that we have a anti-ordering bijection between $\Gamma_u^\vee$ and the set $\{v \in \mathbb{N}^n | v \leq u$, and $v \notin \Gamma\}$. That means that we have a bijection between $(\Gamma_u^\vee)_u^\vee$ and $\{v \in \mathbb{N}^n | v \leq u$, and $v \notin \Gamma_u^\vee\}$. But this last set is obvious in bijection with Γ. Thus, $(\Gamma_u^\vee)_u^\vee = \Gamma$ as required. □

Proposition 6.13. *If Γ is a multicomplex and $u \in \mathbb{N}^n$ is a majorant for Γ then $I_\Gamma = I(\Gamma_u^\vee)_u^c$, where $u = sup(\Gamma) + (1,\dots,1)$. In particular, $I_{\Gamma_u^\vee} = I(\Gamma_u^c)$.*

Proof. Let us notice that $\Gamma_u^\vee$ is generated by $u - v$, where v is a minimal non-face of Γ. But the minimal non-faces of Γ are exactly the minimal generators of the ideal I_Γ. Writing this facts in algebraic language, we get: $I_\Gamma = \langle x^v | v\ is\ a\ minimal\ non-face\ of\ \Gamma\rangle$. Also, $I(\Gamma_u^\vee) = \langle x^{u-v} | v\ is\ a\ minimal\ non-face\ of\ \Gamma\rangle$, and therefore $I_\Gamma = I(\Gamma_u^\vee)_u^c$, as required.

The last identity is clear when we replace Γ by $\Gamma_u^\vee$. □

Example 6.14. *If $\Gamma = \langle(1,3),(4,2)\rangle$ and $u = (5,4) = sup(\Gamma) + (1,1)$, then*

$$\Gamma_{(5,4)}^\vee = \langle(5,0),(0,4),(3,1)\rangle .$$

(This is easy to compute if we figure Γ and $\Gamma_{(5,4)}^\vee$ on the same picture) Also,

$$(\Gamma_{(5,4)}^\vee)_{(5,4)}^c = \langle(5,0),(0,4),(2,3)\rangle .$$

$I_\Gamma = (x^5, y^4, x^2y^3)$. *Obvious,* $I(\Gamma_{(5,4)}^\vee)_{(5,4)}^c) = I_\Gamma$.

Corollary 6.15. *Let Γ be a multicomplex. Then Γ is shellable if and only if $I_{\Gamma_u^\vee}$ has linear quotients, where $u = sup(\Gamma) + (1,\dots,1)$.*

Proof. If is obvious from proposition 5.8 and proposition 5.13. □

References

[1] Jürgen Herzog,Dorin Popescu "Finite filtrations of modules and shellable multicomplexes", Preprint 2005.

[2] Jürgen Herzog,Winnfred Brunns "Cohen Macaulay Rings", Springer 1998.

[3] Ezra Miller,Bernd Stürmfels "Combinatorial commutative algebra", 2004.

[4] Anders Björner, Michelle L.Wachs "Shellable nonpure complexes and posets",1996

[5] R.Villareal "Monomial Algebras", 2001.

Contents