Metadata of the chapter that will be visualized online

Chapter Title	Thermoelastic Bar Theory	
Copyright Year	2013	
Copyright Holder	Springer Science+Business Media Dordrecht	
Corresponding Author	Family Name	**Făciu**
	Particle	
	Given Name	**Cristian**
	Suffix	
	Organization/University	“Simion Stoilow” Institute of Mathematics of the Romanian Academy, Research Unit No 6
	Postbox	1-764
	Postcode	014700
	City	Bucharest
	Country	Romania
	Phone	00 40 21 319 65 06
	Fax	00 40 21 319 65 05
	Email	Cristian.Faciu@imar.ro

T

Thermoelastic Bar Theory

Cristian Făciu
“Simion Stoilow” Institute of Mathematics of the
Romanian Academy, Research Unit No 6,
Bucharest, Romania

Overview

The simplest quasi-static and dynamic laboratory experiments concerning material behavior of solids are performed using uniaxial tension and compression tests. This requires a thermomechanical theory in one space dimension which corresponds to the motion of thin bars described by a single material coordinate. One considers here a general Lagrangian description of the thermodynamic bar theory which allows to take into account the heat exchanges across the lateral surface of a bar with its environment. Balance laws and entropy inequality in differential form and the corresponding jump conditions are described. For thermoelastic heat-conducting materials, thermodynamic restrictions, energy and entropy identities, and dissipative aspects are derived. One discusses additional restrictions in the form of thermostatic stability conditions versus dynamic stability conditions. The results are appropriate in solving one-dimensional problems in nonlinear thermo-elastodynamics and are related with central ideas in the analysis of quasilinear hyperbolic systems of conservation laws, for example, the need to consider discontinuous solutions, since even for smooth initial data, the thermoelastic system may develop discontinuous solutions within a finite time. The need to impose entropy conditions in order to select physical meaningful solutions and to ensure the uniqueness of weak solutions (as in ► Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials). It is shown that the thermoelastic model has a constitutive deficiency to describe solid-solid phase transitions. One way to overcome this inconvenience is to augment the thermoelastic model to include physical effects as viscosity or time of relaxation considered in ► Maxwellian rate-type thermo-viscoelastic bar theory – an approach to non-monotone thermoelasticity.

Thermodynamic Theory in One Space Dimension

One considers a thin bar $\mathcal{B}$ of length L in its unstressed reference configuration having a constant cross-sectional area A along its axis OX. Suppose that the motion of the bar is only longitudinal and is described by a function $x = \chi(X,t)$, $X \in [0,L]$, $t \geq 0$ with the property that $\chi(X,t)$ is injective and bicontinuous with respect to X for any fixed t. X is the initial coordinate (or Lagrangian coordinate), x is the actual axial coordinate (or Eulerian coordinate), and t denotes time. $\chi(\cdot,t)$ is called the *deformation* (or *configuration*) of the bar $\mathcal{B}$ at time t.

R. Hetnarski (ed.), *Encyclopedia of Thermal Stresses*, DOI 10.1007/978-94-007-2739-7,

Whenever χ is continuously differentiable, the functions $v(X,t) = \frac{\partial \chi}{\partial t}(X,t)$ and $\varepsilon(X,t) = \frac{\partial \chi}{\partial X}(X,t) - 1 > -1$ denote the particle velocity and the strain at point X and time t, respectively. Moreover, we assume that all the thermomechanical field quantities are uniform over a cross section, that is, they only depend on X and t.

If $\varrho = \varrho(X)$ denotes the mass density in the reference configuration, then in the Lagrangian description, the role of the conservation of mass is only to determine the current mass density $\varrho_a = \varrho_a(x,t)$, once the motion χ is known, through relation $\varrho(X) = \varrho_a(x,t)\frac{\partial \chi}{\partial X}(X,t)$.

The balance of linear momentum for a portion of the bar occupying the interval (X_1, X_2) requires

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} \varrho v(X,t) A\,\mathrm{d}X &= A\sigma(X,t)\big|_{X=X_1}^{X=X_2}\\ &+ \int_{X_1}^{X_2} \varrho b(X,t) A\,\mathrm{d}X,\end{aligned} \tag{1}$$

where $\sigma = \sigma(X,t)$ denotes the axial stress (force per unit area in the reference configuration) and $b = b(X,t)$ denotes a distributed longitudinal load per unit mass.

The balance of energy (the first law of thermodynamics) for the same piece of bar requires

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} \varrho\left(\frac{v^2}{2} + e\right) A\,\mathrm{d}X &= A(\sigma v - q)\big|_{X=X_1}^{X=X_2}\\ &+ \int_{X_1}^{X_2} \varrho(bv + r) A\,\mathrm{d}X + \int_{A_{lat}} \tilde{q}\,\mathrm{d}s,\end{aligned} \tag{2}$$

where $e = e(X,t)$ is the internal energy per unit mass, $q = q(X,t)$ is the axial heat flux per unit cross-sectional area of the bar, $\tilde{q} = \tilde{q}(X,t)$ is the heat flux per unit area across the lateral surface of the bar $\mathcal{A}_{lat}$ between X_1 and X_2, and $r = r(X,t)$ is the heat supply per unit mass, in the reference configuration.

The balance laws have to be supplemented with the Clausius-Duhem inequality (second law of thermodynamics) which for the interval (X_1, X_2) takes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} \varrho\eta A\,\mathrm{d}X &\geq -A\frac{q}{\theta}(X,t)\big|_{X=X_1}^{X=X_2}\\ &+ \int_{X_1}^{X_2} \frac{\varrho r}{\theta} A\,\mathrm{d}X + \int_{A_{lat}} \frac{\tilde{q}}{\theta}\,\mathrm{d}s,\end{aligned} \tag{3}$$

where $\eta = \eta(X,t)$ is the entropy per unit mass and $\theta = \theta(X,t)$ is the absolute temperature.

Traditionally one distinguishes two situations. The first one corresponds to the case when the thermomechanical variables are smooth (*smooth processes*). The second one corresponds to the case when the motion χ is continuous, but some of the quantities v, ε, θ, σ, e, η, and q have jump discontinuities across a smooth curve $X = S(t)$ in the $t - X$ plane, being smooth function of (X,t) everywhere else (*discontinuous processes*). This curve is called a *strong wave discontinuity* (for instance, shock wave, phase boundary) propagating with the speed $\dot{S}$. We name $X > S(t)$ as the $+$ side of the discontinuity and $X < S(t)$ as the $-$ side of the discontinuity.

Consider $f = f(X,t)$ one of the above quantities, and suppose there exists a single discontinuity $X = S(t)$ in the interval (X_1, X_2). Then, one shows that

$$\begin{aligned} f(X_2,t) - f(X_1,t) &= \int_{X_1}^{X_2} \frac{\partial f}{\partial X}\,\mathrm{d}X + [\![f]\!](t),\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} f\,\mathrm{d}X &= \int_{X_1}^{X_2} \frac{\partial f}{\partial t}\,\mathrm{d}X - \dot{S}[\![f]\!](t),\end{aligned} \tag{4}$$

where $[\![f]\!](t) = f^+(t) - f^-(t) = f(S(t)+0,t) - f(S(t)-0,t)$ denotes the jump across the discontinuity curve. Moreover, according to Hadamard's lemma ([1, 2, Section 173]), we have

$$\frac{\mathrm{d}[\![f]\!](t)}{\mathrm{d}t} = \left[\!\!\left[\frac{\partial f}{\partial t}\right]\!\!\right] + \dot{S}(t)\left[\!\!\left[\frac{\partial f}{\partial X}\right]\!\!\right]. \tag{5}$$

The definition of the particle velocity v and strain ε, on one side, and the continuity condition of the motion χ, on the other side, lead in the smooth case to the differential compatibility relation between v and ε and, in the discontinuous case, to the kinematic jump condition, respectively:

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial X}, \qquad \llbracket v \rrbracket + \dot{S}\llbracket \varepsilon \rrbracket = 0. \tag{6}$$

By using relations (4) in the integral forms (1)–(3), one gets, for the smooth case, the balance laws and Clausius-Duhem inequality in their differential form and, for the discontinuous case, the balance laws and entropy inequality across a wave discontinuity:

$$\varrho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial X} + \varrho b, \qquad \varrho \dot{S}\llbracket v \rrbracket + \llbracket \sigma \rrbracket = 0 \tag{7}$$

$$\begin{gathered}\varrho \frac{\partial}{\partial t}\left(\frac{v^2}{2} + e\right) = \frac{\partial}{\partial X}(\sigma v - q) + \varrho(bv + r) + \frac{2}{R}\tilde{q}, \\ \varrho \dot{S}\left\llbracket \frac{v^2}{2} + e \right\rrbracket + \llbracket \sigma v - q \rrbracket = 0\end{gathered} \tag{8}$$

$$\begin{gathered}\varrho \frac{\partial \eta}{\partial t} \geq -\frac{\partial}{\partial X}\left(\frac{q}{\theta}\right) + \frac{2}{R}\frac{\tilde{q}}{\theta} + \frac{\varrho r}{\theta}, \\ -\varrho \dot{S}\llbracket \eta \rrbracket + \left\llbracket \frac{q}{\theta} \right\rrbracket \geq 0.\end{gathered} \tag{9}$$

Here we have assumed that the bar $\mathcal{B}$ is circular with radius R.

By using relations (6) and (7) in (8), one can rewrite the PDEs system and the corresponding jump relations under the form

$$\begin{gathered}\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial X}, \qquad \varrho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial X} + \varrho b, \\ \varrho \frac{\partial e}{\partial t} = \sigma \frac{\partial \varepsilon}{\partial t} - \frac{\partial q}{\partial X} + \varrho r + \frac{2}{R}\tilde{q}\end{gathered} \tag{10}$$

$$\begin{gathered}\llbracket v \rrbracket + \dot{S}\llbracket \varepsilon \rrbracket = 0, \qquad \varrho \dot{S}\llbracket v \rrbracket + \llbracket \sigma \rrbracket = 0, \\ \varrho \dot{S}\llbracket e \rrbracket + < \sigma > \llbracket v \rrbracket - \llbracket q \rrbracket = 0,\end{gathered} \tag{11}$$

where $\langle f \rangle = \frac{1}{2}(f^+ + f^-)$, and we have used the identity $\llbracket ab \rrbracket = \langle a \rangle \llbracket b \rrbracket + \langle b \rangle \llbracket a \rrbracket$.

In thermodynamics it is useful to use the Helmholtz free energy $\psi = e - \theta\eta$ as a thermodynamic potential. Then the Clausius-Duhem inequality $(9)_1$ takes the form

$$-\varrho \frac{\partial \psi}{\partial t} + \sigma \frac{\partial \varepsilon}{\partial t} - \varrho \eta \frac{\partial \theta}{\partial t} - \frac{q}{\theta}\frac{\partial \theta}{\partial X} \geq 0, \tag{12}$$

and the jump entropy inequality $(9)_2$ becomes

$$\frac{\dot{S}}{\langle \theta \rangle}\left(\varrho[\psi] - \langle \sigma \rangle [\varepsilon] + \varrho[\theta]\langle \eta \rangle\right) - \frac{[\theta]}{\langle \theta \rangle}\left\langle \frac{q}{\theta} \right\rangle \geq 0. \tag{13}$$

It is useful to note here the role of the Clausius-Duhem inequality. For smooth thermomechanical fields, this expression of the second law of thermodynamics is used to restrict the form of the constitutive relations. On the other side, for the discontinuous thermomechanical fields, it becomes an additional constraint that weak solutions have to satisfy (see also ▶ Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials).

Heat-Conducting Thermoelastic Bars

Convective Heat Transfer with the Environment

In the one-dimensional bar theory, the heat absorption or emission through the lateral surface of the bar is by no means unimportant. For instance, in quasi-static loading or unloading tests on shape memory alloy bars, the latent heat of the phase transformation results in heating (or cooling) of the specimen in the neighborhood of the transformation front (see [3]). A test conducted in an air environment or water environment leads to different thermomechanical results (see also ▶ Pseudoelasticity and Shape Memory Effect – A Maxwellian Approach).

A good way to model such circumstances is to suppose that the heat flux $\tilde{q}$ across the lateral surface of the bar satisfies Newton's convective law of heat transfer

$$\tilde{q}(X,t) = -\omega(\theta(X,t) - \theta_{ext}(X,t)), \tag{14}$$

where θ_{ext} is the ambient temperature and $\omega =$ const. > 0 is a heat transfer coefficient, which depends upon both the constitution of the bar and the conditions of the environment. This law expresses the fact that the rate of heat loss/

gain of a body is proportional to the difference in temperatures between the body and its surroundings. Let us note that $\tilde{q}$ is zero in two situations: first, when the temperature of the environment coincides with the temperature of the bar at each X and t, which is not easy to be accomplished in practice when θ is not constant, and second, when placing the bar in an adiabatic environment, for example, a vacuum and $\omega = 0$.

The Thermoelastic Model

The reference density ϱ, the body force b, and the heat supply r are viewed as externally prescribed fields. The other thermomechanical variables are connected through constitutive relations that characterize the material response. A constitutive theory is determined by selecting a class of independent (prime) variables and a class of dependent variables, derived from the prime variables via constitutive relations.

One considers here thermomechanical theories in which χ and θ are prime variables. One advantage of this choice is that these quantities can be directly measured experimentally. The others such as the internal energy e, entropy η, axial force σ, axial heat flux q, and lateral flux $\tilde{q}$ are considered as dependent variables since they are derived from the prime variables by constitutive relations.

In determining the general form of phenomenological constitutive theories, one imposes on the constitutive relations some principles, like the principle of equipresence, the principle of material frame indifference, and the principle of compatibility with the Clausius-Duhem inequality (see [4] and [5] for the general form of constitutive theories in several space dimensions).

Thermodynamic Considerations

A *state* of the bar is any pair $S = (\chi(X), \theta(X))$, $X \in [0, L]$ where $\chi(X)$ and $\theta(X)$ are a deformation and temperature field over $\mathcal{B}$. One takes ε, θ, and the temperature gradient θ_X as independent variables of the constitutive theory. By including the temperature gradient, one incorporates the effect of heat conduction. The remaining thermomechanical variables are determined by constitutive relations satisfying the principle of equipresence:

$$\begin{aligned} &\sigma = \sigma_{eq}(\varepsilon, \theta, \theta_X), \quad \psi = \psi_{eq}(\varepsilon, \theta, \theta_X), \\ &\eta = \eta_{eq}(\varepsilon, \theta, \theta_X), \quad q = Q(\varepsilon, \theta, \theta_X). \end{aligned} \tag{15}$$

By anticipating that the constitutive functions of thermoelastic materials characterize the equilibrium states of some thermo-viscous materials, we have used consistently the index eq (see ► Maxwellian Rate-Type Thermo-Viscoelastic Bar Theory).

A smooth *process* of the bar will be any pair $S(t) = (\chi(X, t), \theta(X, t))$, $t \in [0, t_0]$ of smooth time-dependent fields over $\mathcal{B}$ which satisfies the balance equations (10) and the constitutive relations (14) and (15).

The Clausius-Duhem inequality must be satisfied for all smooth thermomechanical processes. Let us note that a process can be realized by a proper choice of the externally prescribed fields b and r. When the constitutive functions are brought into (12), one gets

$$\begin{aligned} &\left(\sigma_{eq} - \varrho\frac{\partial\psi_{eq}}{\partial\varepsilon}\right)\dot{\varepsilon} - \varrho\left(\eta_{eq} + \frac{\psi_{eq}}{\partial\theta}\right)\dot{\theta} \\ &\quad - \varrho\frac{\partial\psi_{eq}}{\partial\theta_X}\dot{\theta}_X - \frac{Q}{\theta}\theta_X \geq 0, \end{aligned} \tag{16}$$

where dot denotes time derivative. The values ε, $\theta, \theta_X, \dot{\varepsilon}, \dot{\theta}$, and $\dot{\theta}_X$ can be assigned independently. Since the inequality is to be satisfied for all independent variations of $\dot{\varepsilon}$, $\dot{\theta}$, and $\dot{\theta}_X$, which occur linearly, the coefficients of these terms have to vanish separately, that is,

$$\begin{aligned} &\frac{\psi_{eq}}{\partial\theta_X} = 0, \quad \sigma = \sigma_{eq}(\varepsilon, \theta) = \varrho\frac{\partial\psi_{eq}}{\partial\varepsilon}, \\ &\quad \eta = \eta_{eq}(\varepsilon, \theta) = -\frac{\partial\psi_{eq}}{\partial\theta}, \end{aligned} \tag{17}$$

and, in addition, a residual thermal dissipation inequality has to be satisfied

$$D_{th} = -\frac{1}{\theta} Q(\varepsilon, \theta, \theta_X)\theta_X \geq 0. \qquad (18)$$

Thus, the Clausius-Duhem inequality places severe restrictions on the form of the constitutive functions. The free energy of the thermoelastic material has to be independent of θ_X, and it is a potential for the stress and entropy functions.

Let us note that in a thermoelastic material *for any smooth fields* ε and θ, *the only dissipative mechanism is the thermal dissipation* (18). If one considers the Fourier law for axial heat conduction, that is, $Q = -\kappa(\varepsilon, \theta)\theta_X$, then (18) requires that the heat conduction coefficient κ has to be positive.

On the other side, when the motion of a bar involves a propagating discontinuity, then according to the jump entropy inequality $(9)_2$, strain and temperature discontinuous fields constitute a source of dissipation in thermoelastic materials (see also ▶ Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials).

Even in the isothermal case, a strain discontinuity constitutes a source of dissipation in an elastic nonlinear material defined by relation $\sigma = \sigma_{eq}(\varepsilon)$. Indeed, according to (13), the dissipation inequality takes the form

$$\dot{S}\left(\varrho\psi_{eq}(\varepsilon^+) - \varrho\psi_{eq}(\varepsilon^-) - \frac{1}{2}\left(\sigma_{eq}(\varepsilon^+) + \sigma_{eq}(\varepsilon^-)\right)(\varepsilon^+ - \varepsilon^-)\right) \geq 0. \qquad (19)$$

By using, in this isothermal case, the expression $(20)_1$ of the free energy determined below, one gets a useful geometrical meaning of the dissipation induced by a shock wave. That is, the coefficient of $\dot{S}$ in (19) is just the signed area between the graph of $\sigma = \sigma_{eq}(\varepsilon)$ and the chord which joins $(\varepsilon^-, \sigma_{eq}(\varepsilon^-))$ and $(\varepsilon^+, \sigma_{eq}(\varepsilon^+))$.

Let us note that if the stress response function $\sigma = \sigma_{eq}(\varepsilon, \theta)$ can be determined experimentally, then the free energy function $\psi = \psi_{eq}(\varepsilon, \theta)$, the entropy $\eta = \eta_{eq}(\varepsilon, \theta)$, the internal energy $e = e_{eq}(\varepsilon, \theta) = \psi_{eq} + \theta\eta_{eq}$, as well as the specific heat at constant strain $C = C_{eq}(\varepsilon, \theta)$ are uniquely determined, modulo an additive function of temperature $\phi = \phi(\theta)$ by relations

$$\psi_{eq}(\varepsilon, \theta) = \int_{\varepsilon_0}^{\varepsilon} \frac{1}{\varrho} \sigma_{eq}(s, \theta) \mathrm{d}s + \phi(\theta),$$
$$\eta_{eq}(\varepsilon, \theta) = -\int_{\varepsilon_0}^{\varepsilon} \frac{1}{\varrho} \frac{\partial \sigma_{eq}(s, \theta)}{\partial \theta} \mathrm{d}s - \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta} \qquad (20)$$

$$C_{eq}(\varepsilon, \theta) \equiv \frac{\partial e_{eq}}{\partial \theta} \equiv \theta \frac{\partial \eta_{eq}}{\partial \theta} \equiv -\theta \frac{\partial^2 \psi_{eq}(\varepsilon, \theta)}{\partial \theta^2} = -\theta \int_{\varepsilon_0}^{\varepsilon} \frac{1}{\varrho} \frac{\partial^2 \sigma_{eq}(s, \theta)}{\partial \theta^2} \mathrm{d}s - \theta \frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2}, \qquad (21)$$

where ε_0 is an arbitrary reference strain.

It is known that from calorimetric measurements, it is possible to determine the specific heat $C_{eq}(\varepsilon_0, \theta)$ at a constant strain ε_0 over an interval of temperature. Consequently, this information is sufficient to determine the additive function $\phi = \phi(\theta)$ as solution of the differential equation

$$\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2} = -\frac{C_{eq}(\varepsilon_0, \theta)}{\theta} \qquad (22)$$

up to an arbitrary linear function of θ, which can be established once the free energy and the entropy at a given state, respectively $\psi(\varepsilon_0, \theta_0)$ and $\eta(\varepsilon_0, \theta_0)$, are given.

Energy Identities

By using relations $(9)_1$, (14), and (17), one derives the following entropy identity for smooth fields of a thermoelastic bar:

$$\varrho \frac{\partial \eta_{eq}(\varepsilon, \theta)}{\partial t} + \frac{\partial}{\partial X}\left(\frac{q}{\theta}\right) + \frac{2\omega}{R} \frac{(\theta - \theta_{ext})}{\theta} - \frac{\varrho r}{\theta} = \frac{D_{th}}{\theta}, \qquad (23)$$

where the right term in (23) represents the total entropy production corresponding to a smooth process of the thermoelastic bar.

From (17) one derives also the following energy identity for smooth fields:

$$\varrho \frac{e_{eq}(\varepsilon, \theta)}{\partial t} = -\dot{W} + \dot{Q}, \qquad (24)$$

where $\dot{W} = -\sigma_{eq}(\varepsilon, \theta)\dot{\varepsilon}$ is the *rate of work* and $\dot{Q} = v_{eq}(\varepsilon, \theta)\dot{\varepsilon} + \varrho C_{eq}(\varepsilon, \theta)\dot{\theta}$ is the *rate of heat*. The term $v_{eq}(\varepsilon, \theta) = -\theta \frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} = -\varrho\theta \frac{\partial^2 \psi_{eq}(\varepsilon,\theta)}{\partial\theta\partial\varepsilon}$ denotes the *latent heat* with respect to strain and characterizes the heat released or absorbed by a body upon change of strain at constant temperature. On the other side, one sees from relation (24) that the *specific heat* C_{eq} characterizes the amount of heat required to change a body's temperature by a given amount by keeping the deformation fixed. While the sign of the latent heat depends on the sign of $\frac{\partial \sigma_{eq}}{\partial \theta}$, the sign of the specific heat at constant strain is supposed always positive. The Clausius-Duhem inequality says nothing about the sign of C_{eq}. Its positiveness, as we shall see below, is related to the stability of the body. Moreover, experience shows that there is no substance for which this condition is violated.

The *heat propagation equation* for a thermoelastic bar endowed with the Fourier's heat conduction law is then obtained from the balance of energy $(10)_3$ and the constitutive relations (14) and (20)–(21) as

$$\begin{aligned}\varrho C_{eq}(\varepsilon, \theta)\frac{\partial \theta}{\partial t} = \varrho\theta \frac{\partial^2 \psi_{eq}(\varepsilon, \theta)}{\partial\varepsilon\partial\theta}\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial X}\left(k\frac{\partial \theta}{\partial X}\right) \\ - \frac{2\omega}{R}(\theta - \theta_{ext}) + \varrho r.\end{aligned} \qquad (25)$$

Isentrope and Thermal Expansion

For a thermoelastic body, there is another natural choice of independent variables. Instead of using ε and θ as independent variables in the constitutive equations, one employs the strain ε and the entropy η. This is possible because C_{eq} is always supposed strictly positive and, according to (21), η_{eq} must be a strictly increasing function of θ for each fixed ε. Therefore, the equation $\eta = \eta_{eq}(\varepsilon, \theta)$ can be solved for θ in a unique manner as $\theta = \tilde{\theta}(\varepsilon, \eta)$. The internal energy is then defined by $e = \tilde{e}(\varepsilon, \eta) = e_{eq}(\varepsilon, \tilde{\theta}(\varepsilon, \eta))$ and the stress by $\sigma = \tilde{\sigma}(\varepsilon, \eta) = \sigma_{eq}(\varepsilon, \tilde{\theta}(\varepsilon, \eta))$. By using the identities $\eta = \eta_{eq}(\varepsilon, \tilde{\theta}(\varepsilon, \eta))$, for any pair (ε, η) and $\theta = \tilde{\theta}(\varepsilon, \eta_{eq}(\varepsilon, \theta))$, for any pair (ε, θ), and relations (17), one shows that, in this case, the internal energy is a thermodynamic potential for the stress and temperature, that is, $\sigma = \tilde{\sigma}(\varepsilon, \eta) = \varrho \frac{\tilde{e}(\varepsilon,\, \eta)}{\partial \varepsilon}$ and $\theta = \tilde{\theta}(\varepsilon, \eta) = \frac{\tilde{e}(\varepsilon,\eta)}{\partial \eta}$. The specific heat at constant strain is then given by $\tilde{C}(\varepsilon, \eta) = C_{eq}(\varepsilon, \tilde{\theta}(\varepsilon, \eta)) = \tilde{\theta}(\varepsilon, \eta)\left(\frac{\partial \tilde{\theta}(\varepsilon,\eta)}{\partial \eta}\right)^{-1}$.

Isentrope. Since we are using as independent variables the strain ε and the temperature θ, it is useful to remind here the equation of an isentrope. By differentiating the relation $\eta_{eq}(\varepsilon, \theta) = \eta^* =$ const. and by using the thermodynamic relations (17), one gets that an isentrope in the (θ, ε) plane is a solution $\theta = \theta_I(\varepsilon)$ of the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varepsilon} = \frac{\theta}{\varrho C_{eq}(\varepsilon, \theta)}\frac{\partial \sigma_{eq}(\varepsilon, \theta)}{\partial \theta}. \qquad (26)$$

If the initial condition is $\theta_I(\varepsilon^*) = \theta^*$, then this isentrope will be labelled with the value $\eta^* = \eta_{eq}(\varepsilon^*, \theta^*)$.

Let us note that if $\frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} < 0$, the temperature *decreases* along the isentrope, while if $\frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} > 0$, the temperature *increases* along the isentrope. Moreover, according to (24), an isentropic process occurs when the rate of heat $\dot{Q}$ is zero.

Some dimensionless combinations are often used. For instance, sometimes it is convenient to introduce the *Grüneisen coefficient* which is defined as

$$\Gamma(\varepsilon, \theta) = -\frac{1 + \varepsilon}{\varrho C_{eq}(\varepsilon, \theta)}\frac{\partial \sigma_{eq}(\varepsilon, \theta)}{\partial \theta}, \qquad (27)$$

characterizing the temperature changes along an isentrope. Indeed, according to (26), we have $\frac{\mathrm{d}\theta}{\theta} = -\Gamma(\varepsilon, \theta)\frac{\mathrm{d}\varepsilon}{1+\varepsilon}$, that is, it is the negative slope of the isentrope in the $\log\theta - \log(1 + \varepsilon)$ plane. The Grüneisen coefficient can have a profound effect on the existence and qualitative behavior of steady, structured shock waves (see ▶ Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials). Depending on its sign, a compressive shock discontinuity can be of heating type or of cooling type.

The coefficient of thermal expansion at constant stress is introduced as

$$\alpha(\varepsilon,\theta) = -\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}\left(\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon}\right)^{-1} \tag{28}$$

and characterizes the temperature changes along an isobar ($\sigma = \sigma_{eq}(\varepsilon,\theta) = \text{const.}$) in the $\theta - \varepsilon$ plane. Thus, α is positive when the material expands upon heating at constant pressure, which is true in most situations.

The Grüneisen coefficient and the coefficient of thermal expansion have the same sign if the thermostatic stability conditions, given below, are satisfied. Typically one expects to be positive. However, materials in which Γ or α changes sign are not uncommon, perhaps the most familiar being water, which contracts upon heating near its freezing point. Other materials which have a negative coefficient of thermal expansion are certain iron-nickel alloys or the near-equiatomic, cold-worked Nitinol exhibiting shape memory effect [6].

Thermostatic Stability Conditions

Let us note that the second law of thermodynamics places no restrictions on the sign of $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}$ and $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon}$. This high degree of generality may introduce into the constitutive equations improper effects for real materials. Therefore, in general, based on physical or experimental facts, additional restrictions, called a priori inequalities, are imposed on the constitutive function $\sigma = \sigma_{eq}(\varepsilon,\theta)$.

For instance, according to Gibbsian thermostatics (see [7]), a necessary condition for a point (ε,η) to be *thermostatically stable* is that

$$\begin{aligned}&\tilde{e}(\varepsilon^*,\eta^*) - \tilde{e}(\varepsilon,\eta) - \frac{\partial\tilde{e}(\varepsilon,\eta)}{\partial\varepsilon}(\varepsilon^*-\varepsilon)\\ &\quad - \frac{\partial\tilde{e}(\varepsilon,\eta)}{\partial\eta}(\eta^*-\eta) \ge 0,\end{aligned} \tag{29}$$

for any (ε^*,η^*) in the domain of $\tilde{e}(\cdot,\cdot)$. That means (ε,η) is a point of convexity for $e = \tilde{e}(\varepsilon,\eta)$, that is, the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2\tilde{e}}{\partial\eta^2} & \frac{\partial^2\tilde{e}}{\partial\eta\partial\varepsilon} \\ \frac{\partial^2\tilde{e}}{\partial\varepsilon\partial\eta} & \frac{\partial^2\tilde{e}}{\partial\varepsilon^2} \end{pmatrix} \tag{30}$$

is positive semi-definite at (ε,η). Therefore, the following restrictions have to be satisfied:

$$\frac{\partial^2\tilde{e}}{\partial\eta^2} \ge 0, \quad \frac{\partial^2\tilde{e}}{\partial\varepsilon^2} \ge 0, \quad \frac{\partial^2\tilde{e}}{\partial\eta^2}\frac{\partial^2\tilde{e}}{\partial\varepsilon^2} - \left(\frac{\partial^2\tilde{e}}{\partial\eta\partial\varepsilon}\right)^2 \ge 0 \tag{31}$$

or equivalently

$$\begin{aligned}&\frac{\partial\tilde{\theta}(\varepsilon,\eta)}{\partial\eta} \ge 0, \quad \frac{\partial\tilde{\sigma}(\varepsilon,\eta)}{\partial\varepsilon} \ge 0,\\ &\frac{\partial\tilde{\theta}(\varepsilon,\eta)}{\partial\eta}\frac{\partial\tilde{\sigma}(\varepsilon,\eta)}{\partial\varepsilon} - \left(\frac{\partial\tilde{\sigma}(\varepsilon,\eta)}{\partial\eta}\right)^2 \ge 0.\end{aligned} \tag{32}$$

One can show that the necessary and sufficient conditions which ensure the *Gibbsian thermostatic stability* (29) are the positiveness of the specific heat and the positiveness of the derivative of the stress with respect to ε along an isotherm:

$$C_{eq}(\varepsilon,\theta) \ge 0 \quad \text{and} \quad \frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon} \ge 0. \tag{33}$$

Indeed, this assertion follows from relations

$$\begin{aligned}
&\frac{\partial\tilde{\theta}(\varepsilon,\eta)}{\partial\eta}=\frac{\tilde{\theta}(\varepsilon,\eta)}{\tilde{C}(\varepsilon,\eta)}\geq 0\\
&\frac{\partial\tilde{\sigma}(\varepsilon,\eta)}{\partial\varepsilon}=\frac{\partial\sigma_{eq}(\varepsilon,\tilde{\theta}(\varepsilon,\eta))}{\partial\varepsilon}+\frac{\theta}{\varrho C_{eq}(\varepsilon,\tilde{\theta}(\varepsilon,\eta))}\\
&\quad\left(\frac{\partial\sigma_{eq}(\varepsilon,\tilde{\theta}(\varepsilon,\eta))}{\partial\theta}\right)^2\geq 0,\\
&\frac{\partial^2\tilde{e}}{\partial\eta^2}\frac{\partial^2\tilde{e}}{\partial\varepsilon^2}-\left(\frac{\partial^2\tilde{e}}{\partial\eta\partial\varepsilon}\right)^2=\frac{\tilde{\theta}(\varepsilon,\eta)}{\varrho\tilde{C}(\varepsilon,\theta)}\frac{\partial\sigma_{eq}(\varepsilon,\tilde{\theta}(\varepsilon,\eta))}{\partial\varepsilon}\geq 0,
\end{aligned}\tag{34}$$

obtained by using the chain rule and differentiating the identity $\eta=\eta_{eq}(\varepsilon,\tilde{\theta}(\varepsilon,\eta))$ with respect to ε and η.

Another way to prove these assertions is the following. By using the free energy function $\psi=\psi_{eq}(\varepsilon,\theta)$, one derives from (29) that a necessary condition for a point (ε,θ) to be thermostatically stable is that the following inequality

$$\begin{aligned}
\psi_{eq}(\varepsilon^*,\theta^*)-\psi_{eq}(\varepsilon,\theta)-\frac{\partial\psi_{eq}(\varepsilon,\theta)}{\partial\varepsilon}(\varepsilon^*-\varepsilon)&\\
-\frac{\partial\psi_{eq}(\varepsilon^*,\theta^*)}{\partial\theta}(\theta^*-\theta)\geq 0&
\end{aligned}\tag{35}$$

be satisfied for all (ε^*,θ^*) in the domain of definition of ψ_{eq}.

The consequences of this inequality are the following. Let (ε,θ) be fixed and $(\varepsilon^*,\theta^*)=(\varepsilon+\lambda\bar{\varepsilon},\theta+\lambda\bar{\theta})$ a point in the domain of definition of ψ_{eq} for sufficiently small λ. Let us introduce the function

$$\begin{aligned}
\delta(\lambda)=\psi_{eq}\big(\varepsilon+\lambda\bar{\varepsilon},\theta+\lambda\bar{\theta}\big)-\psi_{eq}(\varepsilon,\theta)&\\
-\frac{\partial\psi_{eq}(\varepsilon,\theta)}{\partial\varepsilon}\lambda\bar{\varepsilon}-\frac{\partial\psi_{eq}\big(\varepsilon+\lambda\bar{\varepsilon},\theta+\lambda\bar{\theta}\big)}{\partial\theta}\lambda\bar{\theta}&
\end{aligned}\tag{36}$$

which has the following properties: $\delta(0)=0$ and $\delta'(0)=0$. According to (35), one gets

$$\begin{aligned}
\delta''(0)=\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\varepsilon^2}\bar{\varepsilon}^2-\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\theta^2}\bar{\theta}^2\geq 0,&\\
\text{for any pair}(\bar{\varepsilon},\bar{\theta}).&
\end{aligned}\tag{37}$$

Immediate consequences of this inequality are

$$\begin{aligned}
\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\varepsilon^2}&=\frac{1}{\varrho}\frac{\partial\sigma(\varepsilon,\theta)}{\partial\varepsilon}\geq 0,\\
-\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\theta^2}&=\frac{C_{eq}(\varepsilon,\theta)}{\theta}\geq 0.
\end{aligned}\tag{38}$$

Therefore, the Gibbsian stability condition reads in the following terms: the free energy function $\psi_{eq}(\varepsilon,\theta)$ has to be convex in ε for each θ and concave in θ for each ε.

Dynamic Stability Conditions

A natural physical condition to be imposed on the constitutive functions is to require the existence of real sound speeds (or, equivalently, acceleration waves) with finite propagation velocity in the adiabatic case. This is called *dynamic stability condition* since it ensures in this case the stability of the solutions of the equations of motion.

The system composed by $(10)_{1,2}$ and (25) describing the motion of an isolated ($r=0$, $\omega=0$) thermoelastic bar in the absence of axial heat conduction ($\kappa=0$) is called the *adiabatic thermoelastic system* and can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix}v\\ \varepsilon\\ \theta\end{pmatrix}-\begin{pmatrix}0 & \frac{1}{\varrho}\frac{\partial\sigma_{eq}}{\partial\varepsilon} & \frac{1}{\varrho}\frac{\partial\sigma_{eq}}{\partial\theta}\\ 1 & 0 & 0\\ \frac{\theta}{\varrho C_{eq}}\frac{\partial\sigma_{eq}}{\partial\theta} & 0 & 0\end{pmatrix}\frac{\partial}{\partial X}\begin{pmatrix}v\\ \varepsilon\\ \theta\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\tag{39}$$

This system is suitable for the description of wave propagation since the heat conductivity, or other physical mechanisms like viscosity, can be ignored outside the narrow transition zones corresponding to wave discontinuity. The type of this system is given by the eigenvalues and the right eigenvectors of the above matrix.

The eigenvalues are solution of the equation $\lambda\left[\lambda^2-\left(\frac{1}{\varrho}\frac{\partial\sigma_{eq}}{\partial\varepsilon}+\frac{\theta}{\varrho^2 C_{eq}}\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2\right)\right]=0$. This system is *strictly hyperbolic* if the three eigenvalues are real and distinct and the corresponding right eigenvectors are linearly independent. One shows that this happens if and only if

$$U^2(\varepsilon,\theta) \equiv \frac{1}{\varrho}\frac{\partial \sigma_{eq}}{\partial \varepsilon} + \frac{\theta}{\varrho^2 C_{eq}}\left(\frac{\partial \sigma_{eq}}{\partial \theta}\right)^2 > 0. \quad (40)$$

In this case, the eigenvalues, which correspond to the characteristic directions of the hyperbolic system, are $\lambda_1(\varepsilon,\theta) = -U(\varepsilon,\theta)$, $\lambda_2(\varepsilon,\theta) = 0$, and $\lambda_3(\varepsilon,\theta) = U(\varepsilon,\theta)$, and $U(\varepsilon,\theta)$ is called the *adiabatic sound speed* at the state (ε,θ). It is obvious that the hyperbolicity condition, that is, the dynamic stability condition (40), is satisfied when the thermostatic stability conditions (33) are fulfilled. The reverse statement is not true. The dynamic stability condition ensures that initial and boundary value problems for the adiabatic thermoelastic system (39) are well posed, or correctly set, in the sense of Hadamard (see [8]).

Phase Transitions in Thermoelastic Solids On the other side, circumstances when $\frac{\partial \sigma_{eq}}{\partial \varepsilon} < 0$ on certain intervals of strain and temperature cannot be excluded since they contain the essence of the physics of phase transformations. Indeed, a classical constitutive viewpoint in phenomenological modelling of phase-transforming materials is based on non-monotone thermoelasticity theories and started at the end of nineteenth century with the equation of van der Waals for thermoelastic fluids. The van der Waals fluid is characterized by non-monotone pressure-specific volume relations and is a prototype of continuum-mechanical models of two-phase materials. In solid mechanics such an approach based on non-monotone stress-strain relation for certain ranges of temperature has been initiated by Ericksen [9] in 1975 in an isothermal and one-dimensional context and has been followed by numerous and important studies (see, for instance, [10] and the references therein).

When $\frac{\partial \sigma_{eq}}{\partial \varepsilon}$ is *negative*, and sufficiently large, the adiabatic thermoelastic system (39) has complex eigenvalues over some regions in the $\theta - \varepsilon$ plane called elliptic regions of state space, and the system is of mixed type. The system does not have more the nature of wave propagation problem, and the initial and boundary value problems are ill posed. In general, the solution fails to be unique or is continuously dependent on initial data. The multiplicity of solutions at the continuum level can be viewed as arising from a constitutive deficiency, reflecting the need to specify additional pieces of constitutive information. Indeed, phase transitions are strongly dissipative phenomena, but for thermoelastic materials, the only source of dissipation as we have seen is the thermal dissipation (18). Therefore, there is a need to find simple and appropriate dissipative mechanisms to be included in the constitutive description of thermoelastic materials such that the model will be able to describe the process of phase transformation, that is, the nucleation and propagation of phases.

One way to introduce a dissipative mechanism is the thermodynamical framework developed by Abeyaratne and Knowles presented extensively in [10]. A different approach to the problem of phase transformations is to augment the thermoelastic constitutive equation $\sigma = \sigma(\varepsilon,\theta)$ by introducing rate-type effects as in [11]. This formulation is considered in ▶ Maxwellian rate-type thermo-viscoelastic bar theory – an approach to non-monotone thermoelasticity.

Cross-References

- ▶ Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials
- ▶ Maxwellian Rate-Type Thermo-Viscoelastic Bar Theory – An Approach to Non-Monotone Thermoelasticity
- ▶ Pseudoelasticity and Shape Memory Effect – A Maxwellian Rate-Type Approach

References

1. Hadamard J (1903) Leçons sur la propagation des ondes et les équations de l'hydrodynamique. A. Hermann, Paris
2. Truesdell C, Toupin R (1961) The classical fields theories. In: Handbuch der Physik, vol III/1. Springer, Berlin/Heidelberg/New York

3. Shaw JA, Kyriakides S (1997) On the nucleation and propagation of phase transformation fronts in a NiTi alloy. Acta Mater 45:683–700
4. Truesdell C, Noll W (1965) The nonlinear field theories of mechanics. In: Handbuch der Physik, vol III/3. Springer, Berlin
5. Cemal AE (1980) Mechanics of continua. Wiley, New York
6. Uchil J, Mohanchandra K, Ganesh Kumara K, Mahesh KK, Murali TP (1999) Thermal expansion in various phases of Nitinol using TMA. Physica B 270:289–297
7. Coleman BD, Noll W (1959) On the thermostatics of continuous media. Arch Ration Mech Anal 4:97–128
8. LeFloch PG (2002) Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves. Birkhäuser, Basel/Boston/Berlin
9. Ericksen JL (1975) Equilibrium of bars. J Elast 5:191–201
10. Abeyaratne R, Knowles JK (2006) Evolution of phase transformation. A continuum theory. Cambridge University Press, Cambridge/New York
11. Făciu C, Mihăilescu-Suliciu M (2002) On modelling phase propagation in SMAs by a Maxwellian thermo-visco – elastic approach. Int J Solids Struct 39:3811–3830