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Maxwellian Rate-Type Thermo-viscoelastic Bar Theory: An Approach to Non-monotone Thermoelasticity

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Overview

Thermoelastic constitutive equations discussed in ▶ thermoelastic bar theory are not able to fully determine the discontinuous solutions arising in nonlinear elastodynamic problems as well as the process of phase transition in solid bodies with non-monotone stress-strain relations. That is due to the fact that some small-scale physical mechanisms such as, for example, viscosity, capillarity, or relaxation time are neglected. One introduces here a class of materials called Maxwellian materials with the property that the stress, the strain, and the temperature are related with the strain rate and the stress rate. This approach allows to introduce a linear instantaneous response, and rate-type effects like, relaxation, and pseudo-creep, appropriate to describe viscoelastic and viscoplastic behavior of a one-dimensional body (see [3]). Moreover, it can describe the process of solid-solid phase transition for non-monotone thermoelasticity like in ▶ pseudoelasticity and shape memory effect – a Maxwellian rate-type approach. The compatibility with the second law of thermodynamics of this heat conducting constitutive structure is analyzed. In the thermo-viscoelastic case, the thermodynamic potentials are explicitly determined starting from the constitutive information available from the thermoelastic equilibrium stress-strain-temperature relation, the instantaneous response of the material, and the heat capacity at a constant strain. The adiabatic Maxwellian thermo-viscoelastic system is a relaxation system with stiff source terms which allows an approach of the thermoelastodynamic system in the adiabatic case even for non-monotone stress-strain relations.

Maxwellian Rate-Type Constitutive Equations

The thermodynamic theory in one space dimension has been described in ▶ thermoelastic bar theory. A class of dissipative materials for a one-dimensional body which has as prime variables the strain ε, the absolute temperature θ, and the stress σ can be introduced through the Maxwellian rate-type constitutive equation

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = G(\varepsilon, \sigma, \theta) \tag{1}$$

where E = const. > 0 is called the dynamic Young modulus, G is called relaxation function, and t denotes time. The class of Maxwellian rate-type materials is a subclass of the simple

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materials with fading memory [1]. It is attractive in its simplicity and includes some interesting viscoelastic and viscoplastic models [3]. The first constitutive relation of this type with $G = -\sigma$ has been proposed by Maxwell in 1867, and this explains why materials described by such relations are often called Maxwellian materials.

General Thermodynamic Considerations

In this case a *state* of a bar of length L is any triplet $S = (\chi(X), \sigma(X), \theta(X))$, $X \in [0, L]$ where $\chi(X)$ is a deformation of the bar, $\sigma(X)$ and $\theta(X)$ are a stress and temperature field over the body in its reference configuration. Unlike the thermoelastic bar theory for the rate-type constitutive equation (1), we have to introduce σ as independent variable together with ε, θ, and the temperature gradient θ_X. The remaining dependent thermomechanical variables are the free energy ψ_{mx}, the entropy η_{mx} (or, equivalently, the internal energy $e_{mx} = \psi_{mx} + \theta_{mx}$), and the axial heat flux

$$\begin{aligned}&\psi_{mx} = \psi_{mx}(\varepsilon, \sigma, \theta, \theta_X), \quad \eta_{mx} = \eta_{mx}(\varepsilon, \sigma, \theta, \theta_X),\\ &\quad q = Q(\varepsilon, \sigma, \theta, \theta_X).\end{aligned} \tag{2}$$

In order to satisfy the principle of equipresence, G would have to depend on θ_X, too. Due to the examples considered below and for simplicity reasons, one ignores this case.

It is useful to observe that if we denote by $\varepsilon^i = \varepsilon - \frac{\sigma}{E}$ the inelastic strain and by $\varepsilon^e = \frac{\sigma}{E}$ the elastic strain at state S, then the constitutive relations (1) and (2) may be seen as a constitutive structure for a body with one internal variable, namely, the inelastic strain (for the theory of materials with internal state variables, see [2]).

A smooth *process* of the bar is any triplet $S(t) = (\chi(X,t), \sigma(X,t), \theta(X,t))$, $t \in [0, t_0]$, of smooth time-dependent fields over the body, which satisfies the balance equations

$$\begin{aligned}&\frac{\partial\varepsilon}{\partial t} = \frac{\partial v}{\partial X}, \qquad \varrho\frac{\partial v}{\partial t} = \frac{\partial\sigma}{\partial X} + \varrho b,\\ &\varrho\frac{\partial e_{mx}}{\partial t} = \sigma\frac{\partial\varepsilon}{\partial t} - \frac{\partial Q}{\partial X} + \varrho r - \frac{2\omega}{R}(\theta - \theta_{ext})\end{aligned} \tag{3}$$

and the constitutive relations (1) and (2). Here, ϱ is the mass density in the reference configuration, v denotes the particle velocity, b is the body force per unit mass, r is the heat supply per unit mass, θ_{ext} is the temperature of the surroundings, R is the radius of the bar, and $\omega = \text{const.} > 0$ is a heat transfer coefficient through the lateral surface of the bar.

Thermomechanical processes for the rate-type theory can be realized by a proper choice of body force b and heat supply r so as to fulfill (1)–(3). By introducing the constitutive structures (1) and (2) into the Clausius-Duhem inequality

$$-\varrho\frac{\partial\psi}{\partial t} + \sigma\frac{\partial\varepsilon}{\partial t} - \varrho\eta\frac{\partial\theta}{\partial t} - \frac{q}{\theta}\frac{\partial\theta}{\partial X} \geq 0, \tag{4}$$

one gets

$$\begin{aligned}&-\left(\varrho\frac{\partial\psi_{mx}}{\partial\varepsilon} + \varrho E\frac{\partial\psi_{mx}}{\partial\sigma} - \sigma\right)\dot{\varepsilon} - \varrho\left(\eta_{mx} + \frac{\psi_{mx}}{\partial\theta}\right)\dot{\theta}\\ &\qquad -\varrho\frac{\partial\psi_{mx}}{\partial\theta_X}\dot{\theta}_X - \varrho\frac{\partial\psi_{mx}}{\partial\sigma}G - \frac{Q}{\theta}\theta_X \geq 0,\end{aligned} \tag{5}$$

where dot denotes time derivative. For fixed local values of ε, σ, θ, θ_X, the variations of $\dot{\varepsilon}$, $\dot{\theta}$, θ_X, which appear linearly in (5), can be assigned independently by a proper choice of the thermomechanical processes. Requiring that this inequality be satisfied for all thermomechanical processes, one obtains that the rate-type constitutive structure is compatible with the second law of thermodynamics if and only if

$$\frac{\psi_{mx}}{\partial\theta_X} = 0, \frac{\partial\psi_{mx}}{\partial\varepsilon} + E\frac{\partial\psi_{mx}}{\partial\sigma} = \frac{\sigma}{\varrho}, \eta_{mx}(\varepsilon, \theta) = -\frac{\partial\psi_{mx}}{\partial\theta} \tag{6}$$

$$-\varrho\frac{\partial\psi_{mx}}{\partial\sigma}G(\varepsilon, \sigma, \theta) - \frac{1}{\theta}Q(\varepsilon, \sigma, \theta, \theta_X)\theta_X \geq 0 \tag{7}$$

for any ε, σ, θ, θ_X. Thus, the same as in the thermoelastic case, the free energy ψ_{mx} has to be independent of θ_X, and it is a potential for the entropy. The first term in the inequality (7) is called the *internal dissipation* or *intrinsic dissipation*, and the second one is called the *thermal dissipation*.

Since G does not depend on θ_X, one gets from the reduced dissipation inequality (7) that

$$D_{mx} = -\varrho \frac{\partial \psi_{mx}(\varepsilon, \sigma, \theta)}{\partial \sigma} G(\varepsilon, \sigma, \theta) \geq 0 \qquad (8)$$

for any ε, σ, θ.

Rate-Type Thermo-Viscoelasticity

The kernel of the function G defines the *thermoelastic set* or the *equilibrium set* of the rate-type constitutive equation. One says that the constitutive equation (1) describes a rate-type *thermo-viscoelastic* material if the following two conditions are satisfied: (a) the equilibrium set is a single surface $\sigma = \sigma_{eq}(\varepsilon, \theta)$, that is,

$$G(\varepsilon, \sigma, \theta) = 0 \text{ if and only if } \sigma = \sigma_{eq}(\varepsilon, \theta), \qquad (9)$$

and (b) the function G is stable from above and from below the equilibrium set, that is,

$$\begin{aligned}(\sigma - \sigma_{eq}(\varepsilon, \theta))G(\varepsilon, \sigma, \theta) \leq 0,\\ \text{for any } \varepsilon, \sigma, \theta.\end{aligned} \qquad (10)$$

This latter constraint represents the necessary and sufficient condition that the equilibrium set be attractive with respect to any homogeneous constant-strain process (or *relaxation process*). Indeed, according to (1) and (10), each homogeneous process $(\varepsilon = \varepsilon_0 = \text{const.}, \sigma(t), \theta(t))$ starting at a given initial strain-stress-temperature state $(\varepsilon_0, \sigma_0, \theta_0)$ such that $\sigma_0 > \sigma_{eq}(\varepsilon_0, \theta_0)$ (respectively, $\sigma_0 < \sigma_{eq}(\varepsilon_0, \theta_0)$) moves in the direction of decreasing (respectively, increasing) stress. Thus, a relaxation process will always end on $\sigma = \sigma_{eq}(\varepsilon, \theta)$. This requirement reflects some experimental evidence.

If the kernel of the function G is an open set in the space $\varepsilon - \sigma - \theta$, then the constitutive equation (1) may describe a *thermo-viscoplastic* material (for isothermal examples, see [3]).

By exploiting the residual inequality (7) for a heat-conducting thermo-viscoelastic material, one gets

$$D_{th} = -\frac{1}{\theta} Q(\varepsilon, \sigma_{eq}(\varepsilon, \theta), \theta, \theta_X)\theta_X \geq 0 \qquad (11)$$

for any ε, θ, θ_X. Thus, if Q is given by the Fourier law for heat conduction $Q = -k(\varepsilon, \sigma, \theta)\theta_X$, then (11) requires that the heat conduction coefficient at equilibrium be positive, that is, $k = k(\varepsilon, \sigma_{eq}(\varepsilon, \theta), \theta) \geq 0$ for any ε, θ.

Example: Processes Far from Equilibrium

An example enough general of a rate-type viscoelastic constitutive equation is

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = -\frac{E}{\mu}|\sigma - \sigma_{eq}(\varepsilon, \theta)|^{\lambda - 1}(\sigma - \sigma_{eq}(\varepsilon, \theta)). \qquad (12)$$

Here, $\lambda = \text{const.} > 0$ is a *rate sensitivity parameter*, and $\mu = \text{const.} > 0$ is a *viscosity coefficient*. For $\lambda = 1$, μ is a *Newtonian viscosity coefficient*, $\frac{\mu}{E}$ is a *relaxation time* of the material, while $k = \frac{E}{\mu}$ is called *Maxwellian viscosity coefficient*. For $\lambda = 1$ this equation has been considered in [4].

When $\mu \to 0$ (or, $k \to \infty$), this rate-type constitutive equation can be seen as a rate-type approach of the thermoelastic model $\sigma = \sigma_{eq}(\varepsilon, \theta)$. From physical point of view, it introduces a mechanism of energy dissipation. Unlike the thermoelastic model, the thermo-viscoelastic constitutive equation can describe the way the body may deviate from equilibrium. Indeed, this model has the pseudo-creep property, the relaxation property, and the capability to describe a linear instantaneous response.

We exemplify these properties for the isothermal case $\theta = \theta_0$. For instance, let us consider an initial homogeneous state of strain, stress, and temperature $(\varepsilon_0, \sigma_0, \theta_0)$ and a strain history $\varepsilon = \varepsilon(t)$ having a jump from $\varepsilon(0) = \varepsilon_0$ to $\varepsilon(0+) = \varepsilon_1$. According to (1), the *instantaneous* *response in stress* relative to the initial state is the solution $\varepsilon \to \sigma_I(\varepsilon)$ of the differential equation $\frac{d\sigma_I(\varepsilon)}{d\varepsilon} = E, \quad \sigma_I(\varepsilon) = \sigma_0,$ that is, $\sigma(0+) = \sigma_I(\varepsilon_1) = \sigma_0 + E(\varepsilon_1 - \varepsilon_0)$.

An isothermal homogeneous constant-strain process (or *relaxation process*) $(\varepsilon(t) = \varepsilon_0, \sigma(t), \theta(t) = \theta_0)$ is solution of the problem

$$\begin{aligned}\dot{\sigma}(t) = & -\frac{E}{\mu}|\sigma - \sigma_{eq}(\varepsilon_0, \theta_0)|^{\lambda-1}(\sigma - \sigma_{eq}(\varepsilon_0, \theta_0)),\\ \sigma(0) = & \ \sigma_0.\end{aligned} \tag{13}$$

One verifies that any isothermal relaxation process starting at $(\varepsilon_0, \sigma_0, \theta_0)$ ends on the equilibrium set after a finite time interval if $\lambda \in (0, 1)$ and an infinite time interval if $\lambda \in [1, \infty)$. It follows that the equilibrium set of the rate-type constitutive equation is attractive, or stable, for any relaxation process irrespective of the monotonicity of $\sigma = \sigma_{eq}(\varepsilon, \theta)$ with respect to ε (see Fig. 1).

An isothermal homogeneous constant-stress process (or *pseudo-creep process*) $(\varepsilon(t), \sigma(t) = \sigma_0, \theta(t) = \theta_0)$ is solution of the problem

$$\begin{aligned}\dot{\varepsilon}(t) = & \frac{1}{\mu}|\sigma_0 - \sigma_{eq}(\varepsilon, \theta_0)|^{\lambda-1}(\sigma_0 - \sigma_{eq}(\varepsilon, \theta_0)),\\ \varepsilon(0) = & \ \varepsilon_0.\end{aligned} \tag{14}$$

Let us consider an equilibrium strain-stress-temperature state $\left(\varepsilon^*, \sigma_0 = \sigma_{eq}(\varepsilon^*, \theta_0), \theta_0\right)$ and the initial strain ε_0 of the constant-stress process (14) a perturbation of ε^*. One verifies easily that the equilibrium set is *attractive* for any pseudo-creep process if $\frac{\partial\sigma_{eq}(\varepsilon,\theta_0)}{\partial\varepsilon} \geq 0$ and it is *repulsive* for any pseudo-creep process if $\frac{\partial\sigma_{eq}(\varepsilon,\theta_0)}{\partial\varepsilon} < 0$. This is one reason why the equilibrium set $\sigma = \sigma_{eq}(\varepsilon, \theta)$ is called *stable* if $\frac{\partial\sigma_{eq}}{\partial\varepsilon} \geq 0$ and it is called *unstable* if $\frac{\partial\sigma_{eq}}{\partial\varepsilon} < 0$. Therefore, any solution of the problem (14) ends always on the stable parts of the equilibrium set (see Fig. 1).

It is useful to note that the thermo-viscoelastic rate-type constitutive equation (12) includes as a limiting case for $E \to \infty$ the generalized *Kelvin-Voigt thermo-viscoelastic model*

$$\dot{\varepsilon} = \frac{1}{\mu}|\sigma - \sigma_{eq}(\varepsilon, \theta)|^{\lambda-1}(\sigma - \sigma_{eq}(\varepsilon, \theta)) \tag{15}$$

or equivalently

$$\sigma = \sigma_{eq}(\varepsilon, \theta) + \mu^{\frac{1}{\lambda}}|\dot{\varepsilon}|^{\frac{1}{\lambda}-1}\dot{\varepsilon}. \tag{16}$$

For $\lambda = 1$ this equation has been considered in [5] in relation with phase transformations in solids. It is useful to note that Kelvin-Voigt's viscoelastic constitutive equation can describe only pseudo-creep processes which are governed by the same equation (14), while any instantaneous process has an infinite slope.

Thermodynamic Potentials for the Rate-Type Thermo-viscoelastic Model

We have seen that the second law of thermodynamics imposes restrictions (6) and (8) on the general Maxwellian rate-type constitutive structure (1)–(2). The response of a thermo-viscoelastic model is characterized by two constitutive functions: one describing the thermoelastic equilibrium, that is, $\sigma = \sigma_{eq}(\varepsilon, \theta)$, and the other describing the instantaneous response of the material, that is, the dynamic Young modulus E. The question is to investigate the consequences of the assumptions (9) and (10) on the existence of a free energy function.

One shows that the rate-type thermo-viscoelastic model admits a *unique* free energy function $\psi = \psi_{mx}(\varepsilon, \sigma, \theta)$ (modulo an additive function of θ) if and only if the slope of the straight line connecting two points of an equilibrium isotherm is bounded from above by the instantaneous Young modulus E (see also [4]).

We assume in the following more than that, namely, there exist two positive constants E_* and E^* such that

$$-E_* \leq \frac{\sigma_{eq}(\varepsilon_1, \theta) - \sigma_{eq}(\varepsilon_2, \theta)}{\varepsilon_1 - \varepsilon_2} \leq E^* < E, \tag{17}$$

for any $\varepsilon_1, \varepsilon_2$ and any θ.

Indeed, starting from the internal dissipation inequality (8) and condition (10), one gets that

$$\frac{\partial\psi_{mx}(\varepsilon,\sigma,\theta)}{\partial\sigma}(\sigma-\sigma_{eq}(\varepsilon,\theta))\leq 0, \tag{18}$$

wherefrom it follows that $\frac{\partial\psi_{mx}}{\partial\sigma}=0$ must vanish for $\sigma=\sigma_{eq}(\varepsilon,\theta)$. Consequently, the free energy function has to satisfy the following Cauchy problem for a first-order PDE

$$\frac{\partial\psi_{mx}}{\partial\varepsilon}+E\frac{\partial\psi_{mx}}{\partial\sigma}=\frac{\sigma}{\varrho},\quad \frac{\partial\psi_{mx}}{\partial\sigma}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta)=0. \tag{19}$$

The general form of the free energy equation is

$$\varrho\psi_{mx}(\varepsilon,\sigma,\theta)=\frac{\sigma^2}{2E}+\varphi(\sigma-E\varepsilon,\theta), \tag{20}$$

where $\varphi=\varphi(\tau,\theta)$ satisfies relation

$$\frac{\partial\varphi}{\partial\tau}(\sigma_{eq}(\varepsilon,\theta)-E\varepsilon,\theta)=-\frac{\sigma_{eq}(\varepsilon,\theta)}{E}. \tag{21}$$

In order to determine the function φ, it is necessary that function $h(\varepsilon,\theta)=\sigma_{eq}(\varepsilon,\theta)-E\varepsilon$ be invertible with respect to ε for any fixed θ. That requires $h(\varepsilon_1,\theta)\neq h(\varepsilon_2,\theta)$ for any ε_1, ε_2, $\varepsilon_1\neq\varepsilon_2$ and any θ. In fact, this condition requires that $\sigma=\sigma_{eq}(\varepsilon,\theta)$ be a noncharacteristic for the Cauchy problem (19). If one denotes by $h^{-1}(\cdot,\theta)$ this inverse function, then φ will be the unique solution (modulo an additive function of θ) of the following equation

$$\frac{\partial\varphi}{\partial\tau}(\tau,\theta)=-\frac{\sigma_{eq}(h^{-1}(\tau,\theta),\theta)}{E} \tag{22}$$

Therefore, for any triplet $(\varepsilon,\sigma,\theta)$, there is a unique $\tilde\varepsilon=\tilde\varepsilon(\varepsilon,\sigma,\theta)=h^{-1}(\sigma-E\varepsilon,\theta)$ such that

$$\sigma-E\varepsilon=h(\tilde\varepsilon,\theta)=\sigma_{eq}(\tilde\varepsilon,\theta)-E\tilde\varepsilon. \tag{23}$$

By using (23), one shows that $\sigma-\sigma_{eq}(\varepsilon,\theta)=h(\tilde\varepsilon,\theta)-h(\varepsilon,\theta)$ and from (20) one gets

$$\varrho E\frac{\partial\psi_{mx}}{\partial\sigma}(\varepsilon,\sigma,\theta)=\sigma-\sigma_{eq}(\tilde\varepsilon,\theta)=E(\varepsilon-\tilde\varepsilon). \tag{24}$$

Consequently, the inequality (18) requires $(h(\varepsilon,\theta)-h(\tilde\varepsilon,\theta))(\varepsilon-\tilde\varepsilon)\leq 0$ for any ε and $\tilde\varepsilon$, which combined with the injectivity condition of function $h(\cdot,\theta)$ leads to condition (17). Moreover, these are the necessary and sufficient conditions for the existence of a free energy function, that is, a solution of the Cauchy problem (19).

By integrating equation (22), one derives the explicit form of the free energy function for the Maxwellian thermo-viscoelastic model

$$\begin{aligned}\varrho\psi_{mx}(\varepsilon,\sigma,\theta)&=\frac{\sigma^2}{2E}-\frac{\sigma_{eq}^2(\tilde\varepsilon,\theta)}{2E}\\&\quad+\int_{\varepsilon_0}^{\tilde\varepsilon}\sigma_{eq}(s,\theta)\,\mathrm{d}s+\varrho\phi(\theta),\end{aligned} \tag{25}$$

where $\phi(\theta)$ is a smooth function. The entropy function is given by

$$\begin{aligned}\varrho\eta_{mx}(\varepsilon,\sigma,\theta)&=-\varrho\frac{\partial\psi_{mx}}{\partial\theta}\\&=-\int_{\varepsilon_0}^{\tilde\varepsilon}\frac{\partial\sigma_{eq}(s,\theta)}{\partial\theta}\,\mathrm{d}s-\varrho\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}\end{aligned} \tag{26}$$

and the specific heat by

$$C_{mx}(\varepsilon,\sigma,\theta) = \theta\frac{\partial\eta_{mx}}{\partial\theta} = -\frac{\theta}{\varrho}\left(\int_{\varepsilon_0}^{\tilde{\varepsilon}}\frac{\partial^2\sigma_{eq}(s,\theta)}{\partial\theta^2}\,\mathrm{d}s + \frac{\left(\frac{\partial\sigma_{eq}(\tilde{\varepsilon},\theta)}{\partial\theta}\right)^2}{E-\frac{\partial\sigma_{eq}(\tilde{\varepsilon},\theta)}{\partial\varepsilon}} + \varrho\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2}\right) \tag{27}$$

where $\tilde{\varepsilon} = h^{-1}(\sigma - E\varepsilon,\theta)$.

Let us note that the free energy, entropy, and internal energy of the Maxwellian thermo-viscoelastic model *at equilibrium* are just the free energy, entropy, and internal energy of the thermoelastic model $\sigma = \sigma_{eq}(\varepsilon,\sigma)$, that is,

$$\begin{aligned}\psi_{mx}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta) &= \psi_{eq}(\varepsilon,\sigma),\\ \eta_{mx}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta) &= \eta_{eq}(\varepsilon,\sigma).\end{aligned} \tag{28}$$

Indeed, from (23) one gets that $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon,\ \sigma_{eq}(\varepsilon,\theta),\theta) = h^{-1}(\sigma_{eq}(\varepsilon,\theta) - E\varepsilon,\theta) = \varepsilon$, wherefrom by using (25) and (26), one retrieves the relations obtained in ▶ thermoelastic bar theory.

Not the same property is true for the specific heat. Indeed, the specific heat (27) of the Maxwellian model does not coincide at equilibrium with the specific heat of the thermoelastic model. Indeed, one can show that

$$C_{mx}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta) = C_{eq}(\varepsilon,\theta)\frac{E-\varrho U^2(\varepsilon,\theta)}{E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon,\theta)} \tag{29}$$

where $U(\varepsilon,\theta)$ is the adiabatic sound speed of the thermoelastic material defined in ▶ thermoelastic bar theory. By requiring that the specific heat of the Maxwellian model be positive, one derives an important physical condition called *the subcharacteristic condition*

$$\varrho U^2(\varepsilon,\theta) = \frac{\partial\sigma_{eq}}{\partial\varepsilon} + \frac{\theta}{\varrho C_{eq}(\varepsilon,\theta)}\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2 < E \tag{30}$$

whose meaning is explained below. Let us note that this condition is stronger than condition (17).

The additive function $\phi(\theta)$ can be determined from calorimetric measurements. Indeed, it is reasonable to assume that the specific heat of the Maxwellian model at the equilibrium state, that is, $C_{mx}(\varepsilon_0,\sigma(\varepsilon_0,\theta),\theta)$, can be determined experimentally over an interval of temperature. Therefore, $\phi(\theta)$ is the solution of the equation $\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2} = -\frac{C_{mx}(\varepsilon_0,\sigma(\varepsilon_0,\theta)\theta,)}{\theta}$ up to an arbitrary linear function of θ, which can be established once the free energy and the entropy at a given state (ε_0,θ_0) are given.

Remark 1. The free energy of the Maxwellian viscoelastic model is completely determined by the equilibrium response function $\sigma = \sigma_{eq}(\varepsilon,\theta)$, the dynamic Young modulus E, and the specific heat at an equilibrium state $C_{mx}(\varepsilon_0,\sigma(\varepsilon_0,\theta),\theta)$ over an interval of temperature.

Remark 2. The free energy of the Maxwellian rate-type thermo-viscoelastic model defined by equation (1) and conditions (9)–(10) corresponds for an entire class of functions $G = G(\varepsilon,\sigma,\theta)$ having the same equilibrium $\sigma = \sigma_{eq}(\varepsilon,\theta)$.

Remark 3. The rate sensitivity parameter λ and the viscosity coefficient μ influence only the rate effects, that is, how quickly or slowly the stability or unstability phenomena of the thermomechanical processes take place and how close or far are these processes with respect to the equilibrium set.

Remark 4. Condition (17) which ensures the existence of a unique free energy function for the Maxwellian thermo-viscoelastic model allows to include materials for which the equilibrium response function has the property that $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon} < 0$, which arise when studying materials that can undergo a phase change considered in ► pseudoelasticity and shape memory effect – a Maxwellian rate-type approach.

Energetic Estimates and Energetic Identities

By using relation (24) and restriction (17), one derives the bounds

$$\begin{aligned}
&\frac{E}{E+E_*}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2\\
&\quad\le E\varrho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma,\theta)(\sigma-\sigma_{eq}(\varepsilon,\theta))\\
&\quad\le\frac{E}{E-E^*}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2 \qquad (31)
\end{aligned}$$

wherefrom one obtains the following energetic estimate: the intrinsic dissipation (8) for the Maxwellian thermo-viscoelastic constitutive equation (12) satisfies relation

$$\begin{aligned}
&\frac{E}{\mu(E+E_*)}\left|\sigma-\sigma_{eq}(\varepsilon,\theta)\right|^{\lambda+1}\le D_{mx}(\varepsilon,\sigma,\theta)\\
&\qquad\le\frac{E}{\mu(E-E^*)}\left|\sigma-\sigma_{eq}(\varepsilon,\theta)\right|^{\lambda+1}. \qquad (32)
\end{aligned}$$

We also note that by investigating the properties of the thermodynamic potentials corresponding to the Maxwellian thermo-viscoelastic model when $E\to\infty$, one gets from relations (25)–(27) using (23)

$$\begin{aligned}
&\lim_{E\to\infty}\psi_{mx}(\varepsilon,\sigma,\theta)=\psi_{eq}(\varepsilon,\theta),\\
&\lim_{E\to\infty}\ \eta_{mx}(\varepsilon,\sigma,\theta)=\eta_{eq}(\varepsilon,\theta), \qquad (33)\\
&\lim_{E\to\infty}\ C_{mx}(\varepsilon,\sigma,\theta)=C_{eq}(\varepsilon,\theta),
\end{aligned}$$

that is, the free energy, the entropy, and the specific heat of the Kelvin-Voigt model (16) coincide with the free energy, entropy, and specific heat of the thermoelastic model described in ► thermoelastic bar theory. Moreover, the internal dissipation generated in a smooth process by the Kelvin-Voigt model (16) is obtained from (32) and (15) as

$$\begin{aligned}
D_{kv}(\varepsilon,\sigma,\theta)&=\lim_{E\to\infty}D_{mx}(\varepsilon,\sigma,\theta)\\
&=\frac{1}{\mu}\left|\sigma-\sigma_{eq}(\varepsilon,\theta)\right|^{\lambda+1}\\
&=\mu^{\frac{1}{\lambda}}\left|\dot{\varepsilon}\right|^{\frac{1}{\lambda}+1}. \qquad (34)
\end{aligned}$$

The thermodynamic relations (19) and (25)–(27) lead to the following energetic identities for the Maxwellian model endowed with Fourier heat conduction law

$$\begin{aligned}
\varrho\frac{\partial e_{mx}(\varepsilon,\sigma,\theta)}{\partial t}&=-\dot{W}+\dot{Q},\\
\varrho\frac{\partial\eta_{mx}(\varepsilon,\sigma,\theta)}{\partial t}+\frac{\partial}{\partial X}\left(\frac{q}{\theta}\right)&=\frac{1}{\theta}(D_{mx}+D_{th})\ge 0
\end{aligned} \qquad (35)$$

where the *rate of work* $\dot{W}$ and the *rate of heat* $\dot{Q}$ are given by

$$\begin{aligned}
\dot{W}=&-\sigma\dot{\varepsilon},\ \dot{Q}=\varrho\frac{\partial\psi_{mx}}{\partial}G(\varepsilon,\sigma,\theta)\\
&-\varrho\theta\frac{\partial^2\psi_{mx}}{\partial\sigma\partial\theta}G(\varepsilon,\sigma,\theta)+\varrho C_{mx}(\varepsilon,\sigma,\theta)\dot{\theta}.
\end{aligned} \qquad (36)$$

Let us note that the first term in the expression of $\dot{Q}$ is always negative, being the internal dissipation D_{mx} with minus sign. The second term represents the heating or the cooling of the material due to the latent heat released or absorbed by the body during a thermomechanical process. One can show that

$$\varrho\frac{\partial^2\psi_{mx}}{\partial\sigma\partial\theta}(\varepsilon,\sigma,\theta)=-\frac{\partial\sigma_{eq}}{\partial\theta}(\tilde{\varepsilon},\theta)\left(E-\frac{\partial\sigma_{eq}(\tilde{\varepsilon},\theta)}{\partial\varepsilon}\right)^{-1} \qquad (37)$$

where $\tilde{\varepsilon}=\tilde{\varepsilon}(\varepsilon,\sigma,\theta)=h^{-1}(\sigma-E\varepsilon,\theta)$ according to (23). Therefore, the latent heat depends essentially, like in the thermoelastic case, on the variation of the equilibrium stress with respect to the temperature. Its role is illustrated numerically in ► pseudoelasticity and shape memory effect – a Maxwellian rate-type approach. The third term represents the rate of heat due to the specific heat. One sees that the specific heat C_{mx}

characterizes the amount of heat required to change a body's temperature by a given amount by keeping the inelastic deformation $\varepsilon^i = \varepsilon - \frac{\sigma}{E}$ fixed.

The *heat propagation equation* for a Maxwellian thermo-viscoelastic bar allowing lateral heat transfer to the environment and axial Fourier heat conduction is obtained from the balance of energy (3)$_3$ and the thermodynamic relations (25)–(27) as

$$\begin{aligned}\varrho C_{mx}\frac{\partial\theta}{\partial t} = & -\varrho\frac{\partial\psi_{mx}}{\partial\sigma}G + \varrho\theta\frac{\partial^2\psi_{mx}}{\partial\sigma\partial\theta}G + \frac{\partial}{\partial X}\left(k\frac{\partial\theta}{\partial X}\right)\\ & -\frac{2\omega}{R}(\theta-\theta_{ext}) + \varrho r.\end{aligned} \tag{38}$$

For the thermo-viscoelastic Kelvin-Voigt's material (16) with Fourier heat conduction law, one gets the following energetic identities:

$$\begin{aligned}&\varrho\frac{\partial e_{eq}(\varepsilon,\theta)}{\partial t} = -\dot W + \dot Q,\\ &\varrho\frac{\partial\eta_{eq}(\varepsilon,\sigma,\theta)}{\partial t} + \frac{\partial}{\partial X}\left(\frac{q}{\theta}\right) = \frac{1}{\theta}(D_{kv} + D_{th}) \geq 0\end{aligned} \tag{39}$$

where the *rate of work* $\dot W$ and the *rate of heat* $\dot Q$ are given by

$$\begin{aligned}\dot W = & -\sigma_{eq}(\varepsilon,\theta),\dot\varepsilon\\ \dot Q = & -\mu^{\frac{1}{\lambda}}\left|\dot\varepsilon\right|^{\frac{1}{\lambda}+1} - \varrho\theta\frac{\partial^2\psi_{eq}}{\partial\varepsilon\partial\theta}\dot\varepsilon + \varrho C_{eq}(\varepsilon,\theta)\dot\theta.\end{aligned} \tag{40}$$

We note that the first term in the expression of $\dot Q$ is always negative, being the internal dissipation D_{kv} with minus sign. The second and the third term are identically with those in the thermoelastic case and represent the contribution to the rate of heat of the latent heat and of the specific heat, respectively.

The *heat propagation equation* for the Kelvin-Voigt thermo-viscoelastic bar (16) allowing lateral heat transfer and axial Fourier heat conduction is

$$\begin{aligned}\varrho C_{eq}\frac{\partial\theta}{\partial t} = & \mu^{\frac{1}{\lambda}}\left|\frac{\partial\varepsilon}{\partial t}\right|^{\frac{1}{\lambda}+1} + \varrho\theta\frac{\partial^2\psi_{eq}}{\partial\varepsilon\partial\theta}\frac{\partial\varepsilon}{\partial t} + \frac{\partial}{\partial X}\left(k\frac{\partial\theta}{\partial X}\right)\\ & -\frac{2\omega}{R}(\theta-\theta_{ext}) + \varrho r.\end{aligned} \tag{41}$$

Let us note that the right-hand terms in (35)$_2$ and (39)$_2$ represent the *entropy production due to intrinsic dissipation* and the *entropy production due to thermal dissipation* through axial heat conduction.

The Adiabatic Thermo-viscoelastic System: A Relaxation System with Stiff Source Terms

The adiabatic Maxwellian thermo-viscoelastic system is composed by the balance of momentum, the compatibility equation between strain and velocity, the heat propagation equation (38), written for an isolated bar ($\omega = 0,\ r = 0$) in the absence of axial heat conduction ($\kappa = 0$) and by the thermo-viscoelastic constitutive equation (1). For the example (12), it takes the form of a relaxation system

$$\frac{\partial}{\partial t}\begin{pmatrix} v\\ \varepsilon\\ \theta\\ \sigma\end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\varrho}\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ E & 0 & 0 & 0\end{pmatrix}\frac{\partial}{\partial X}\begin{pmatrix} v\\ \varepsilon\\ \theta\\ \sigma\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ s_3(\varepsilon,\sigma,\theta)\\ s_4(\varepsilon,\sigma,\theta)\end{pmatrix} \tag{42}$$

with the stiff sources

$$\begin{aligned}s_3 = & \frac{E}{\mu}\left|\sigma - \sigma_{eq}(\varepsilon,\theta)\right|^{\lambda-1}(\sigma - \sigma_{eq}(\varepsilon,\theta))\\ & \frac{1}{C_{mx}}\left(\frac{\partial\psi_{mx}}{\partial\sigma} - \theta\frac{\partial^2\psi_{mx}}{\partial\sigma\partial\theta}\right)\\ s_4 = & -\frac{E}{\mu}\left|\sigma - \sigma_{eq}(\varepsilon,\theta)\right|^{\lambda-1}(\sigma - \sigma_{eq}(\varepsilon,\theta)).\end{aligned}$$

One expects that when $\mu \to 0$ solutions of the rate-type system (42) "approach" solutions of the adiabatic thermoelastic system described in ▶ thermoelastic bar theory, in the sense that the stress σ is rapidly driven back to the equilibrium

$\sigma_{eq}(\varepsilon, \theta)$, except perhaps in narrow zones where σ, ε, θ, and v have a very steep variation.

It is easy to verify that this system is always *hyperbolic semilinear* as long as the dynamic Young's modulus E is strictly positive and finite. Indeed, the system is semilinear since all nonlinear terms are included in the right part of (42), that is, in the source terms, and the eigenvalues of the matrix are given by $r_{1,2} = \pm\sqrt{E/\varrho}$ and $r_{3,4} = 0$. In other words, the information for the adiabatic thermo-viscoelastic system propagates with the characteristic speed $\sqrt{E/\varrho}$ and the source terms do not change this fact. On the other side, we have seen that "the approximated system," or the so-called reduced system, that is, the adiabatic thermoelastic system, has the characteristic speed $U(\varepsilon, \theta)$ given by (30). Usually, for systems involving relaxation, one requires a priori that the characteristic speed of the reduced system cannot exceed the characteristic speed of the system with relaxation, that is, in our case condition (30). That means, the information for the reduced system can not propagate faster than the speed $\sqrt{E/\varrho}$. That is why it is called *subcharacteristic condition*. The terminology was introduced by Liu in [6] for relaxation systems (see also [7]). For the adiabatic thermo-viscoelastic system, the subcharacteristic condition appears naturally when studying the consistency with the second law of thermodynamics of the rate-type constitutive equation. We have seen that requirement (30) is a necessary condition for the existence of a positive specific heat for the Maxwellian thermo-viscoelastic model.

An important consequence of the fact that the adiabatic thermo-viscoelastic system is always hyperbolic irrespective of the sign of $\frac{\partial\sigma_{eq}}{\partial\varepsilon}$ is that the initial-boundary value problems for system (42) are now well-posed. Therefore, this rate-type approach proves to be appropriate to describe phase transition phenomena and has been used in ▶ pseudoelasticity and shape memory effect – a Maxwellian rate-type approach.

If one considers instead of the Maxwellian thermo-viscoelastic constitutive equation (12) the Kelvin-Voigt thermo-viscoelastic model (16), then the corresponding adiabatic system can be viewed as a limiting case of the adiabatic Maxwellian rate-type system (42) for $E \to \infty$. In this case, the characteristic directions of the hyperbolic system in the *t*-*X* plane tend to be infinite, that is, the hyperbolic system (42) transforms into a parabolic one.

There is a considerable interest in the study of relaxation systems and on the conditions under which convergence of solutions occur as the stiff parameter goes to zero (see, for instance, [8], [7], and their references). We illustrate some aspects for the case of isothermal motions.

Approach to Equilibrium in the Isothermal Case

Let us consider the isothermal motions of a Maxwellian viscoelastic material described by the relaxation system

$$\varrho\frac{\partial v}{\partial t}-\frac{\partial\sigma}{\partial X}=0,\frac{\partial\varepsilon}{\partial t}-\frac{\partial v}{\partial X}=0,\frac{\partial\sigma}{\partial t}-E\frac{\partial\varepsilon}{\partial t} = -\frac{E}{\mu}\left|\sigma-\sigma_{eq}(\varepsilon)\right|^{\lambda-1}(\sigma-\sigma_{eq}(\varepsilon)). \qquad (43)$$

In the formal zero limit $\mu \to 0$, the system yields the equations of isothermal elastodynamics:

$$\varrho\frac{\partial v}{\partial t}-\frac{\partial\sigma_{eq}(\varepsilon)}{\partial X}=0, \qquad \frac{\partial\varepsilon}{\partial t}-\frac{\partial v}{\partial X}=0. \qquad (44)$$

This relaxation system has been studied from thermodynamical point of view in [9] for monotone equilibrium curves $\sigma = \sigma_{eq}(\varepsilon)$. The result has been extended in [10] for the non-monotone case. For the case $\lambda = 1$, the relaxation system has been used to describe the behavior of a phase-transforming material, like shape memory alloys, in the quasistatic case in [11, 12] and in the dynamic case in [13].

By using the properties (19) of the free energy function in the isothermal case, one can establish the following energy identity for the smooth solutions of the system (43)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\varrho\frac{v^2}{2}+\psi_{mx}(\varepsilon,\sigma)\right)&=\frac{\partial}{\partial X}(\sigma v)\\ -\frac{E}{\mu}\varrho\frac{\partial\psi_{mx}}{\partial\sigma}&\left|\sigma-\sigma_{eq}(\varepsilon)\right|^{\lambda-1}(\sigma-\sigma_{eq}(\varepsilon)).\end{aligned}\tag{45}$$

If we denote the total energy of a bar of length L by

$$\mathcal{E}_{mx}(t)=\int_0^L\left(\varrho\frac{v^2}{2}+\psi_{mx}(\varepsilon,\sigma)\right)(X,t)dX,\tag{46}$$

one gets the energy identity

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}_{mx}(t)}{\mathrm{d}t}&=-\frac{E}{\mu}\int_0^L\varrho\frac{\partial\psi_{mx}}{\partial\sigma}\left|\sigma-\sigma_{eq}(\varepsilon)\right|^{\lambda-1}\\ &(\sigma-\sigma_{eq}(\varepsilon))dX+(\sigma v)(L,t)-(\sigma v)(0,t).\end{aligned}\tag{47}$$

Then, for an isolated body problem, that is, a boundary value problem for which

$$(\sigma v)(L,t)=0 \qquad \text{and} \qquad (\sigma v)(0,t)=0,\tag{48}$$

one derives, by using the consequence (18) of the dissipation inequality, that

$$\frac{\mathrm{d}\mathcal{E}_{mx}(t)}{dt}\leq 0.\tag{49}$$

Therefore, for an isolated body problem, the total energy of a rate-type bar is a nonincreasing function of time. Moreover, by integrating (47) with respect to the time t and using the inequality (31), one gets the following estimates for the solutions of an isolated body problem (48) for the rate-type system (43):

$$\begin{aligned}&\int_0^t\int_0^L\left|\sigma-\sigma_{eq}(\varepsilon)\right|^{\lambda+1}\\ &\leq\mu\frac{E-E^*}{E}(\mathcal{E}_{mx}(0)-\mathcal{E}_{mx}(t))\\ &\leq\mu\frac{E-E^*}{E}\mathcal{E}_{mx}(0).\end{aligned}\tag{50}$$

Consequently, when $\mu\to 0$, we get an $L^{\lambda+1}$ approach to the equilibrium set of the solutions of the system with relaxation for an isolated body problem. This result is valid irrespective of the slope of the equilibrium curve $\sigma=\sigma_{eq}(\varepsilon)$, that is, even when the semilinear hyperbolic system "approximates" a mixed hyperbolic-elliptic quasilinear system.

Let us note that for smooth initial data the rate-type viscoelastic system admits smooth solutions, while the nonlinear elastodynamic system may develop discontinuous solutions (shock waves) in finite time. In fact, the rate-type viscoelastic system can smooth the shock waves caused by the physically nonlinear elastic constitutive equation $\sigma=\sigma_{eq}(\varepsilon)$. The problem of constructing weak solutions of the equations of isothermal elasticity as limits of the relaxation system when $\mu\to 0$ has been addressed in [14]. The convergence problem of viscosity approximations for conservation laws is by no means trivial. It has been considered in [14] where it has been discussed the uniform stability and compactness for solutions of the relaxation system (43) (when $\lambda=1$) in the zero-relaxation limit. The convergence to the system of isothermal elastodynamics has been established by using compensated compactness. The result is obtained *only* for the monotonous case when $\frac{\partial\sigma_{eq}}{\partial\varepsilon}>0$, that is, only when one approximates a hyperbolic quasilinear system with the solutions of a hyperbolic semilinear system.

Cross-References

- ► Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials
- ► Pseudoelasticity and Shape Memory Effect – A Maxwellian Rate-Type Approach
- ► Thermoelastic Bar Theory

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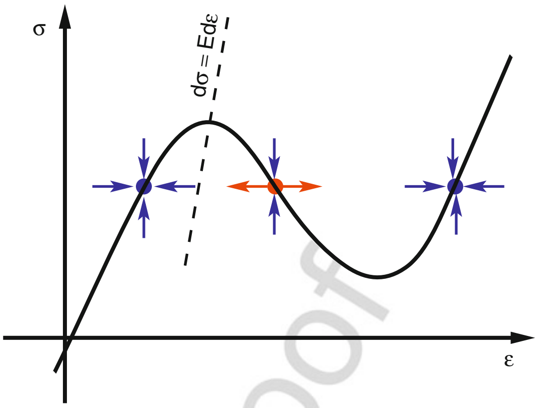

Maxwellian Rate-Type Thermo-viscoelastic Bar Theory: An Approach to Non-monotone Thermoelasticity, Fig. 1 Instantaneous process relative to an equilibrium state and behavior of relaxation processes and pseudo-creep processes near the equilibrium isotherm $\sigma = \sigma_{eq}(\varepsilon, \theta_0)$