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# H

## Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials

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### Overview

The dynamics of a nonlinear thermoelastic bar is governed in the adiabatic case by a system of conservation laws (see ► Thermoelastic Bar Theory) leading to the need to consider discontinuous solutions. These are usually called shock waves or phase boundaries and represent essential mathematical tools in investigating solutions of quasi-linear hyperbolic PDE systems. Initial value problems for such systems may have multiple discontinuous solutions, hence the necessity to impose selection criteria, frequently called "entropy" conditions. In reality, shock waves do not exist. In solids, the role of viscous-like effects on the shock structure has been experimentally identified in the 1960s by velocity interferometry techniques, and a shock thickness has been put into evidence (see, for instance, [1, 2]). Thus, the study of steady, structured shock waves or traveling waves is an important subject in the theory of waves both from practical and theoretical point of view. Such an analysis provides admissibility criteria for discontinuous solutions of the adiabatic thermoelastic theories which derive from associated dissipative systems. For instance, thermoviscous fluids have been considered in [3, 4, 5], while thermo-viscous fluids with capillarity effects in [6, 7]. One considers here a thermoviscous heat-conducting bar whose equilibrium constitutive setting is described by a nonlinear thermoelastic relation. Both the viscous dissipation and the heat conduction produce structure in steady waves. One shows that the admissibility criterion for viscous, heat-conducting shock layers is, in general, equivalent with a geometrical criterion, namely, the chord criterion with respect to the Hugoniot locus in the stress–strain space.

### Shock Waves in Thermoelastic Bars

#### Balance Laws for One-Dimensional Bodies

One considers a one-dimensional body whose particles in a fixed reference configuration are labeled by $X$, $X \in (-\infty, \infty)$. According to the ► thermoelastic bar theory, the differential form of the kinematic compatibility condition, the balance laws of linear momentum and energy, as well as the Clausius–Duhem inequality in the absence of external body forces and radiating heating can be written as



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$$\begin{aligned} &\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial X}, \quad \varrho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial X}, \\ &\varrho \frac{\partial e}{\partial t} = \sigma \frac{\partial \varepsilon}{\partial t} - \frac{\partial q}{\partial X}, \quad \varrho \frac{\partial \eta}{\partial t} \geq -\frac{\partial}{\partial X}\left(\frac{q}{\theta}\right) \end{aligned} \tag{1}$$

where at the particle $X$ and time $t$, $\varepsilon$ is the strain, $v$ is the particle velocity, $\sigma$ is the axial stress ($\sigma< 0$ in compression), $e$ is the specific internal energy, $q$ is the axial flux, $\eta$ is the specific entropy, $\theta$ is the (positive) absolute temperature, and $\varrho =$const. is the density.

Let us suppose that across a curve $X = S(t)$ in the $t - X$ plane, called wave discontinuity (or strong discontinuity), the quantities mentioned above may have jumps. The most common examples are the shock waves and phase boundaries. Then, the continuity of the motion, the balance laws of momentum and energy, and the Clausius–Duhem inequality across this discontinuity take the form

$$\begin{aligned} &[\![v]\!] + \dot{S}[\![\varepsilon]\!] = 0, \quad \varrho\dot{S}[\![v]\!] + [\![\sigma]\!] = 0, \\ &\varrho\dot{S}[\![e]\!] \quad + \langle\sigma\rangle[\![v]\!] - [\![q]\!] = 0, \\ &-\varrho\dot{S}[\![\eta]\!] + \left[\!\!\left[\frac{q}{\theta}\right]\!\!\right] \geq 0 \end{aligned} \tag{2}$$

Here, $\dot{S}(t)$ denotes the speed of propagation of the discontinuity, and for any quantity $f = f(X,t)$, we have used the notations $[\![f]\!](t) = f^{+}(t) - f^{-}(t) = f(S(t)+,t) - f(S(t)-,t)$ and $\langle f\rangle(t) = \frac{1}{2}(f^{+}(t) + f^{-}(t))$. We name $X > S(t)$ as the + side and $X < S(t)$ as the $-$ side of the discontinuity.

## The Adiabatic Thermoelastic System

For many applications, like wave propagation theory, physical effects such as viscosity, heat conduction, and relaxation either are neglected or are not the focus of attention. In such situations it is found to be sufficient to consider the response of the material described only by the equilibrium stress response function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ and to take the heat flux $q = 0$. From the ► thermoelastic bar theory, it is known that the thermodynamic restrictions imposed by the Clausius–Duhem inequality $(1)_4$ requires that the free energy function $\psi_{eq} = \psi_{eq}(\varepsilon,\theta)$ be a potential for the stress and for the entropy function, that is,

$$\begin{aligned} \sigma_{eq}(\varepsilon,\theta) &= \varrho \frac{\partial \psi_{eq}(\varepsilon,\theta)}{\partial \varepsilon}, \\ \eta_{eq}(\varepsilon,\theta) &= -\frac{\psi_{eq}(\varepsilon,\theta)}{\partial \theta} \end{aligned} \tag{3}$$

Hence, the free energy function $\psi = \psi_{eq}(\varepsilon,\theta)$, the entropy function $\eta_{eq} = \eta_{eq}(\varepsilon,\theta)$, and the specific heat of the thermoelastic material $C_{eq}(\varepsilon,\theta) = -\theta\frac{\partial^2\psi_{eq}}{\partial\theta^2}$ are uniquely determined by the stress response function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ modulo, an additive function of temperature $\phi = \phi(\theta)$. This function is determined experimentally by measuring the specific heat at constant strain $\varepsilon_0$ over an interval of temperature, that is, $C_{eq}(\varepsilon_0,\theta)$, as it has been explained in the ► thermoelastic bar theory.

Therefore, the system $(1)_{1-3}$ supplemented with the equilibrium stress response function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ and the corresponding internal energy $e = e_{eq}(\varepsilon,\theta) = \psi_{eq} + \theta\eta_{eq}$ in the absence of heat conduction takes the form

$$\begin{aligned} &\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial X}, \quad \varrho \frac{\partial v}{\partial t} = \frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial X}, \\ &\frac{\partial \theta}{\partial t} = \frac{\theta}{\varrho C_{eq}(\varepsilon,\theta)} \frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} \frac{\partial v}{\partial X} \end{aligned} \tag{4}$$

This nonlinear PDE system is called the *adiabatic thermoelastic system*. It is a strictly hyperbolic system if

$$\lambda^2(\varepsilon,\theta) = \frac{1}{\varrho}\frac{\partial \sigma_{eq}}{\partial \varepsilon} + \frac{\theta}{\varrho^2 C_{eq}}\left(\frac{\partial \sigma_{eq}}{\partial \theta}\right)^2 > 0 \tag{5}$$

and its characteristic directions are $\frac{\mathrm{d}X}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}X}{\mathrm{d}t} = \pm\lambda(\varepsilon,\theta)$.

If one investigates, for example, the impact of two semi-infinite thermoelastic bars, that is, if we consider a Riemann problem for the system (4), then discontinuous solutions of the form $\frac{X}{t} = \dot{S}$ =const. may be generated by the initial data (see [8]). Moreover, it is well known that for this quasi-linear hyperbolic system even from smooth initial data, the solution may develop discontinuities in a finite time (see, for instance, [9]). Usually such a discontinuous solution is called a *shock wave* and the jump conditions (2) tell us how the shock wave changes the strain, the stress, the velocity, and the temperature.

## Rankine–Hugoniot Conditions

If $\dot{S} > 0$, that is, the shock wave propagates in the positive-X direction, one calls the material at the + side to be in the front of the wave, while the material at the – side to be in back of the wave. The shock wave is said to be compressive if the deformation decreases after the passage of the wave ($\varepsilon^- < \varepsilon^+$) and expansive if the deformation increases ($\varepsilon^- > \varepsilon^+$). If $\dot{S} < 0$, one changes only + to − and correspondingly the terminology.

According to jump relations (2), the discontinuous solutions of the adiabatic thermoelastic system have to satisfy the following front state–back state relations:

$$\begin{aligned} &v^- - v^+ = -\dot{S}(\varepsilon^- - \varepsilon^+),\\ &\sigma_{eq}(\varepsilon^-, \theta^-) - \sigma_{eq}(\varepsilon^+, \theta^+) = \varrho\, \dot{S}^2\, (\varepsilon^- - \varepsilon^+), \end{aligned} \tag{6}$$

$$\begin{aligned} &\varrho(e_{eq}(\varepsilon^-, \theta^-) - e_{eq}(\varepsilon^+, \theta^+))\\ &\quad = \frac{1}{2}(\sigma_{eq}(\varepsilon^-, \theta^-) + \sigma_{eq}(\varepsilon^+, \theta^+))(\varepsilon^- - \varepsilon^+), \end{aligned} \tag{7}$$

$$\dot{S}(\eta_{eq}(\varepsilon^-, \theta^-) - \eta_{eq}(\varepsilon^+, \theta^+)) \geq 0 \tag{8}$$

Relations (6), (7) are usually referred to as the *Rankine–Hugoniot conditions*, while (7) is the famous *Rankine–Hugoniot equation*. Let us suppose that the *front state* $(\varepsilon^+, \theta^+, v^+)$ is known. Then, relations (6), (7) represent an algebraic nonlinear system for the unknown *back state* $(\varepsilon^-, \theta^-, v^-)$ and the speed of the discontinuity $\dot{S}$. Depending on the thermoelastic constitutive assumptions, this system may generally be solved if one of these four quantities is prescribed. In addition, such a weak solution has to satisfy the constraint imposed by the entropy inequality (8), which asserts that after the passage of a strong discontinuity, the entropy of a particle will not decrease. This condition has been firstly stated in gas dynamics by Jouguet [10].

It is useful to note that the Rankine–Hugoniot equation (7) provides only restrictions, on the back states $(\varepsilon, \theta)$ which can be reached in a shock process which has $(\varepsilon^+, \theta^+)$ as a front state. Moreover, this restriction does not depend on the shock speed $\dot{S}$. One denotes by

$$\begin{aligned} H(\varepsilon, \theta; \varepsilon^+, \theta^+) =& \varrho e_{eq}(\varepsilon, \theta) - \varrho e^+\\ & - \frac{1}{2}(\sigma_{eq}(\varepsilon, \theta) + \sigma^+)(\varepsilon - \varepsilon^+) \end{aligned} \tag{9}$$

the *Hugoniot function based at* $(\varepsilon^+, \theta^+)$ where $e^+ = e_{eq}(\varepsilon^+, \theta^+)$ and $\sigma^+ = \sigma_{eq}(\varepsilon^+, \theta^+)$. The set $\{(\varepsilon, \theta) \quad | H(\varepsilon, \theta; \varepsilon^+, \theta^+) = 0\}$ is called the *Hugoniot set (locus) based* at $(\varepsilon^+, \theta^+)$ in the $\varepsilon - \theta$ plane. It is obvious that $(\varepsilon^+, \theta^+)$ belongs to the Hugoniot set.

Let us suppose for simplicity that the equation $H(\varepsilon, \theta; \varepsilon^+, \theta^+) = 0$ can be solved uniquely with respect to $\varepsilon$. That is, there exists a function

$$\theta = \Theta_H(\varepsilon; \varepsilon^+, \theta^+), \tag{10}$$

with the properties that $\theta^+ = \Theta_H(\varepsilon^+; \varepsilon^+, \theta^+)$ and $H(\varepsilon, \Theta_H(\varepsilon; \varepsilon^+, \theta^+); \varepsilon^+, \theta^+) = 0$ on its domain of definition. This is called the *temperature–strain Hugoniot curve (locus) based at* $(\varepsilon^+, \theta^+)$ and describes all those states in the $\varepsilon - \theta$ plane that are potentially attainable as back states in a shock process which has $(\varepsilon^+, \theta^+)$ as a front state. Situations when the Hugoniot set is not curve-like and can bifurcate have been considered in [11].

The image of the curve (10) through the function $\sigma = \sigma_{eq}(\varepsilon, \theta)$ in the $\varepsilon - \sigma$ plane is denoted by

$$\sigma = \sigma_H(\varepsilon; \varepsilon^+, \theta^+) \equiv \sigma_{eq}(\varepsilon, \Theta_H(\varepsilon; \varepsilon^+, \theta^+)) \tag{11}$$

and is called the *stress–strain Hugoniot curve (locus) based at* $(\varepsilon^+, \sigma^+)$. This function describes all reachable $(\varepsilon, \sigma)$ back states in a wave discontinuity which has $(\varepsilon^+, \sigma^+)$ as a front state.

The concept of the shock wave is an extremely useful tool that has allowed the study of a great variety of wave phenomena. However, the adiabatic thermoelastic system (4) may admit weak solutions that do not resemble physical solutions, so the system must be supplemented with conditions that exclude the nonphysical solutions. These extra conditions should mimic the physical effects that are not fully modeled by system (4). In some situations, in gas dynamics or in elastodynamics, simple rules such as the Lax characteristics criterion [12], or the requirement that the entropy should not decrease, suffice to isolate physically reasonable solutions. In general, more complex admissibility criteria are needed, such as requiring the existence of viscous profiles.

## Shock Layers and Admissibility Conditions

### A Thermo-viscous Heat-Conducting Material

The nonuniqueness of discontinuous solutions of the adiabatic thermoelastic system (4) can be resolved by requiring that shock waves arise as limits of solutions of more complete equations. When heat conduction and viscosity are included, physical shock waves are limits of traveling wave profiles.

One considers in the following an augmented theory of the thermoelastic material by including dissipative mechanisms described by Kelvin–Voigt constitutive equation and by Fourier heat conduction law, that is,

$$\sigma = \sigma_{eq}(\varepsilon,\theta) + \mu\frac{\partial\varepsilon}{\partial t}, \quad \text{and} \quad q = -\kappa\frac{\partial\theta}{\partial X} \tag{12}$$

where $\mu =$ const. $> 0$ is a Newtonian viscosity coefficient and $\kappa =$ const. $> 0$ is the heat conduction coefficient.

This is a simple linear model for viscosity. In metals, according to the experimental results obtained by Barker [1] (see also [11, 2]), the relation between the impact pressure and the strain rate is a power law which corresponds to the generalized Kelvin–Voigt model considered in ► Maxwellian rate-type thermo-viscoelastic bar theory. Here it has been shown by investigating the compatibility of the Kelvin–Voigt model with the second law of thermodynamics that the free energy, the entropy, the internal energy, and the specific heat of this viscous model are the same as those of the thermoelastic model, that is, they satisfy relations (3).

The system governing the motion of a viscous, heat-conducting bar is obtained from $(1)_{1-3}$, by adding the Kelvin–Voigt constitutive relation, the Fourier heat conduction law (12), and the corresponding internal energy $e = e_{eq}(\varepsilon,\theta)$ of the thermoelastic material, that is,

$$\frac{\partial\varepsilon}{\partial t} = \frac{\partial v}{\partial X}, \qquad \varrho\frac{\partial v}{\partial t} = \frac{\partial\sigma}{\partial X}, \qquad \sigma = \sigma_{eq}(\varepsilon,\theta) + \mu\frac{\partial v}{\partial X} \tag{13}$$

$$\varrho C_{eq}(\varepsilon,\theta)\frac{\partial\theta}{\partial t} = \theta\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}\frac{\partial v}{\partial X} + \mu\left(\frac{\partial v}{\partial X}\right)^2 + \kappa\frac{\partial^2\theta}{\partial X^2} \tag{14}$$

It is a parabolic PDE system. When $\mu \to 0$ and $\kappa \to 0$, one retrieves the adiabatic thermoelastic system (4).

The total dissipation, that is, the intrinsic dissipation and the thermal dissipation, generated in a smooth process by the heat-conducting Kelvin–Voigt material is given by

$$D_{tot} = \mu\left(\frac{\partial\varepsilon}{\partial t}\right)^2 + \frac{\kappa}{\theta}\left(\frac{\partial\theta}{\partial X}\right)^2 \geq 0 \tag{15}$$

### Traveling Wave Solutions

Independent of any constitutive assumption, a *traveling wave solution* for a one-dimensional body is a set of smooth functions $(\varepsilon,\sigma,\theta,v,q,e,\eta)$ satisfying (1) and which depends on $(X,t)$ through the variable $\xi = X - \dot{S}t$, where $\dot{S} = \text{const.}$. The functions $(\varepsilon,\sigma,\theta,v,q,e,\eta)(X,t) = (\hat{\varepsilon},\hat{\sigma},\hat{\theta},\hat{v},\hat{q},\hat{e},\hat{\eta})(\xi)$ represent a smooth profile with constant shape propagating with a constant velocity $\dot{S}$. That is why often they are referred as *steady, structured waves*. According to (1), the following relations are verified

$$\hat{v}'(\xi) + \dot{S}\hat{\varepsilon}'(\xi) = 0, \quad \hat{\sigma}'(\xi) + \varrho\dot{S}\hat{v}'(\xi) = 0,$$
$$\dot{S}(\varrho\hat{e}'(\xi) - \hat{\sigma}(\xi)\hat{\varepsilon}'(\xi)) = \hat{q}'(\xi), \quad \varrho\dot{S}\eta' \leq \left(\frac{q}{\theta}\right)' \tag{16}$$

where prime denotes the derivative with respect to $\xi$. The limiting values correspond to the thermomechanical equilibrium states of the augmented theory, that is,

$$\lim_{\xi\to\pm\infty}(\hat{\varepsilon},\hat{\sigma},\hat{\theta},\hat{v},\hat{q},\hat{e},\hat{\eta})(\xi)=(\varepsilon^{\pm},\sigma^{\pm}=\sigma_{eq}(\varepsilon^{\pm},\theta^{\pm}),\theta^{\pm},\hat{v}^{\pm},0,e_{eq}(\varepsilon^{\pm},\theta^{\pm}),\eta_{eq}(\varepsilon^{\pm},\theta^{\pm})) \tag{17}$$

where $\varepsilon^+$, $v^+$, $\theta^+$, $\varepsilon^-$, $v^-$, and $\theta^-$ are given values.

By integrating relations (16) between $\xi$ and $+\infty$, one gets

$$\hat{v}(\xi) = v^+ - \dot{S}(\hat{\varepsilon}(\xi) - \varepsilon^+) \tag{18}$$

$$\hat{\sigma}(\xi) = \sigma_R(\hat{\varepsilon}(\xi)) \equiv \sigma^+ + \varrho\,\dot{S}^2\,(\hat{\varepsilon}(\xi) - \varepsilon^+) \tag{19}$$

$$\hat{q}(\xi) = \dot{S}(\varrho\hat{e}(\xi) - \varrho e^+ - \frac{1}{2}(\hat{\varepsilon}(\xi) - \varepsilon^+)(\hat{\sigma}(\xi) + \sigma^+)) \tag{20}$$

$$\hat{q}(\xi) \le \varrho\dot{S}\hat{\theta}(\xi)(\hat{\eta}(\xi) - \eta^+) \tag{21}$$

If we set $\xi \to -\infty$, we recover the Rankine–Hugoniot relations (6), (7) and the entropy jump inequality (8) for the adiabatic thermoelastic system. Therefore, if $\dot{S} > 0$ and $(\varepsilon^+, \theta^+)$ is a given front state of a wave discontinuity, then the pair $(\varepsilon^-, \theta^-)$ has to belong to the Hugoniot set based at $(\varepsilon^+, \theta^+)$ given by (9), that is, $H(\varepsilon^-, \theta^-; \varepsilon^+, \theta^+) = 0$ or equivalently $\theta^- = \Theta_H(\varepsilon^-; \varepsilon^+, \theta^+)$. The constant steady wave speed $\dot{S}$ is determined by the equilibrium states to be connected through relation

$$\varrho\,\dot{S}^2 = \frac{\sigma_{eq}(\varepsilon^+, \theta^+) - \sigma_{eq}(\varepsilon^-, \theta^-)}{\varepsilon^+ - \varepsilon^-}. \tag{22}$$

Let us note that relation (19) asserts that in a steady structured wave, the strain–stress pairs $(\hat{\varepsilon}(\xi), \hat{\sigma}(\xi))$ belong to a straight line of slope $\varrho\,\dot{S}^2$ in the $\varepsilon - \sigma$ plane. This is called the *Rayleigh line construction*. That is why the function $\sigma = \sigma_R(\varepsilon)$ defined above is called the *Rayleigh line*.

By using the Kelvin–Voigt constitutive equation and the Fourier law (12), one gets that $\varepsilon = \hat{\varepsilon}(\xi)$ and $\theta = \hat{\theta}(\xi)$ have to satisfy the nonlinear autonomous system with boundary conditions

$$\hat{\varepsilon}' = -\frac{1}{\mu\dot{S}}R(\hat{\varepsilon}, \hat{\theta}), \qquad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi) = \varepsilon^{\pm} \tag{23}$$

$$\hat{\theta}' = -\frac{\dot{S}}{\kappa}H_{KV}(\hat{\varepsilon}, \hat{\theta}), \qquad \lim_{\xi\to\pm\infty}\hat{\theta}(\xi) = \theta^{\pm} \tag{24}$$

where if $\dot{S} > 0$,

$$R(\varepsilon, \theta; \varepsilon^+, \theta^+, \varepsilon^-) \equiv \sigma_R(\varepsilon) - \sigma_{eq}(\varepsilon, \theta) = \sigma^+ + \varrho\,\dot{S}^2\,(\varepsilon - \varepsilon^+) - \sigma_{eq}(\varepsilon, \theta) \tag{25}$$

$$H_{KV}(\varepsilon, \theta; \varepsilon^+, \theta^+, \varepsilon^-) \equiv \varrho e_{eq}(\varepsilon, \theta) - \varrho e^+ - \frac{1}{2}(\varepsilon - \varepsilon^+)(\sigma_R(\varepsilon) + \sigma^+) \tag{26}$$

**Admissibility Condition**

One says that a shock wave is an *admissible* weak solution for the adiabatic thermoelastic system if there exists a unique traveling wave solution $(\varepsilon(\xi), \theta(\xi), v(\xi))$ provided by the augmented constitutive approach which connects the limit values $(\varepsilon^{\pm}, \theta^{\pm}, v^{\pm})$. Such a traveling wave displays the character of a shock wave (for small viscosity $\mu$ and heat conductivity $\kappa$) because it differs sensibly from their end states at $\xi = \pm\infty$ only in a small interval of rapid transition. This behavior explains why it is usually called a *shock layer*.

The Isothermal Case

For nonlinear isothermal elasticity, the elastodynamic system is composed by $(1)_{1,2}$ and the equilibrium stress–strain relation $\sigma = \sigma_{eq}(\varepsilon)$. If one considers the isothermal Kelvin–Voigt viscoelastic constitutive equation $(12)_1$, then $\hat{v}(\xi)$ and $\hat{\sigma}(\xi)$ satisfy relations (18) and (19), and $\hat{\varepsilon}(\xi)$ is a solution of the differential equation with boundary conditions

$$\hat{\varepsilon}'(\xi) = -\frac{1}{\mu\dot{S}}(\sigma_R(\hat{\varepsilon}(\xi)) - \sigma_{eq}(\hat{\varepsilon}(\xi)),\qquad \lim_{\xi\to\pm\infty}(\hat{\varepsilon})(\xi) = \varepsilon^{\pm} \tag{27}$$

It has been shown (see [6, 13]) that a unique solution of the problem (27) exists or, in other words, a viscous shock layer exists, if and only if the following criterion is satisfied.

### Chord Criterion With Respect to the Elastic Constitutive Equation $\sigma = \sigma_{eq}(\varepsilon)$

A *compressive wave discontinuity*, that is, $(\varepsilon^+ - \varepsilon^-)\dot{S} > 0$, is admissible iff the chord $\sigma = \sigma_R(\varepsilon)$ which joins $(\varepsilon^+, \sigma^+ = \sigma_{eq}(\varepsilon^+))$ to $(\varepsilon^-, \sigma^- = \sigma_{eq}(\varepsilon^-))$ lies *below* the graph of the function $\sigma = \sigma_{eq}(\varepsilon)$ for $\varepsilon$ between $\varepsilon^+$ and $\varepsilon^-$, while an *expansive wave discontinuity*, that is, $(\varepsilon^+ - \varepsilon^-)\dot{S} < 0$, is admissible if the chord $\sigma = \sigma_R(\varepsilon)$ lies *above* the graph in the same interval.

This result shows that a *viscosity admissibility criterion* is equivalent with a geometrical criterion with respect to the elastic constitutive equation $\sigma = \sigma_{eq}(\varepsilon)$. In addition, it is a simple and extremely practical criterion since it allows to detect directly admissible discontinuous solutions for the nonlinear elastodynamic system and thus to build solutions for initial step data problems like Riemann problem, for instance.

It has been shown in [14] that the Maxwellian rate-type viscoelastic constitutive equation, investigated in ▶ Maxwellian rate-type thermoviscoelastic bar theory, leads to the same admissibility criterion. Moreover, this criterion is valid even for phase transforming thermoelastic bars for which the equilibrium stress–strain relation $\sigma = \sigma_{eq}(\varepsilon)$ is non-monotone (see also [6, 13]). In this case when $\varepsilon^{\pm}$ belongs to different stable phases of the material, then the corresponding discontinuous solution is called propagating phase boundary. In fact, mathematically, propagating phase boundaries are discontinuities separating states in one hyperbolic domain from states in another hyperbolic domain.

### Non-isothermal Case: Only Viscous Dissipation as Structuring Mechanism

Let us consider the viscosity as the only structuring mechanism for a steady, structured shock wave. By taking $\kappa = 0$ in relation (24), it follows that the strain–temperature structured solutions $(\hat{\varepsilon}(\xi), \hat{\theta}(\xi))$ have to satisfy an algebraic equation and a differential equation with boundary conditions, that is,

$$H_{KV}(\hat{\varepsilon},\hat{\theta}) = 0,\quad \hat{\varepsilon}' = -\frac{1}{\mu\dot{S}}R(\hat{\varepsilon},\hat{\theta}),\quad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi) = \varepsilon^{\pm} \tag{28}$$

The set $\{(\varepsilon,\theta)\,|\,H_{KV}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-) = 0\}$ describes the trajectory in the $\varepsilon - \theta$ plane of the traveling wave governed by the Kelvin–Voigt dissipative mechanism in the absence of heat conduction. Since $\frac{\partial H_{KV}}{\partial\theta}(\varepsilon,\theta) = \varrho\frac{\partial e_{eq}}{\partial\theta}(\varepsilon,\theta) = \varrho C_{eq}(\varepsilon,\theta) > 0$, it follows that the implicit equation $H_{KV}(\varepsilon,\theta) = 0$ is locally uniquely representable as a single valued function of $\varepsilon$. Let us suppose there exists a unique function

$$\theta = \Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-), \tag{29}$$

with the properties that $H_{KV}(\varepsilon,\Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)) = 0$ for any $\varepsilon$ belonging to an interval containing $\varepsilon^-$ and $\varepsilon^+$, and $\Theta_{KV}(\varepsilon^{\pm};\varepsilon^+,\theta^+,\varepsilon^-) = \theta^{\pm}$. Its image through the equilibrium stress response function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ in the $\varepsilon - \sigma$ plane is given by

$$\sigma = \sigma_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-) \equiv \sigma_{eq}(\varepsilon,\Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)), \tag{30}$$

which connects the states $(\varepsilon^{\pm},\sigma^{\pm})$. It is useful to note that $\sigma^{\pm} = \sigma_{KV}(\varepsilon^{\pm}) = \sigma_H(\varepsilon^{\pm}) = \sigma_{eq}(\varepsilon^{\pm},\theta^{\pm})$. By using the previous notations, one gets from (28) that $\varepsilon = \hat{\varepsilon}(\xi)$ is solution of the problem

$$\hat{\varepsilon}' = -\frac{1}{\mu\dot{S}}(\sigma_R(\hat{\varepsilon}(\xi)) - \sigma_{KV}(\hat{\varepsilon};\varepsilon^+,\theta^+,\varepsilon^-)),\qquad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi) = \varepsilon^{\pm} \tag{31}$$

Taking into account the result described in the isothermal case, it follows that a solution of this problem exists if and only if the previous chord criterion is satisfied, but now *with respect to the curve* $\sigma = \sigma_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$.

### Chord Criterion With Respect to the Hugoniot Locus

It can be shown that the chord criterion with respect to the curve $\sigma = \sigma_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ is *equivalent* with *the chord criterion with respect to the Hugoniot locus* $\sigma = \sigma_H(\varepsilon; \varepsilon^+, \theta^+)$ defined by (11). The proof uses the reduction to the absurd and relies on the relation $H(\varepsilon, \theta) = H_{KV}(\varepsilon, \theta) + \frac{1}{2}(\varepsilon - \varepsilon^+)R(\varepsilon, \theta)$ which exists between the Hugoniot function based at $(\varepsilon^+, \theta^+)$ and the functions (25) and (26).

This result is extremely useful in practice since, like in the isothermal case, it reduces the problem of existence of a shock layer to a geometrical criterion which depends only on the properties of the adiabatic thermoelastic system, that is, on the stress–strain Hugoniot locus based $(\varepsilon^+, \theta^+)$.

### Non-isothermal Case: Viscous Dissipation and Heat Conduction as Structuring Mechanisms

Let us investigate the traveling wave solutions when the viscosity and the heat conduction are coupled. The method used here has been initiated by Gilbarg [3] for the study of shock profiles in fluid dynamics. To analyze this system, it is important to characterize its critical points. The linearization of (23)–(24) in a neighborhood of $(\varepsilon^\pm, \theta^\pm)$ leads to the system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\hat{\varepsilon}\\ \hat{\theta}\end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\mu\dot{S}}\left(\varrho\dot{S}^2 - \dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\right) & \dfrac{1}{\mu\dot{S}}\dfrac{\partial\sigma_{eq}}{\partial\theta} \\ \dfrac{\dot{S}}{\kappa}\theta^\pm\dfrac{\partial\sigma_{eq}}{\partial\theta} & -\dfrac{\dot{S}}{\kappa}\varrho C_{eq} \end{pmatrix}\begin{pmatrix}\hat{\varepsilon}\\ \hat{\theta}\end{pmatrix} \quad (32)$$

The characteristic equation of the linearized system at the critical points $(\varepsilon^\pm, \theta^\pm)$ is

$$r^2 + r\left\{\frac{1}{\mu\dot{S}}(\varrho\dot{S}^2 - \frac{\partial\sigma_{eq}}{\partial\varepsilon}) + \frac{\dot{S}}{\kappa}\varrho C_{eq}\right\} + \frac{1}{\kappa\mu}\left\{\varrho C_{eq}\left(\varrho\dot{S}^2 - \frac{\partial\sigma_{eq}}{\partial\varepsilon}\right) - \theta^\pm\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2\right\} = 0 \quad (33)$$

The discriminant of this equation

$$\Delta(\varepsilon^\pm, \theta^\pm) = \left\{\frac{1}{\mu\dot{S}}\left(\varrho\dot{S}^2 - \frac{\partial\sigma_{eq}}{\partial\varepsilon}\right) - \frac{\dot{S}}{\kappa}\varrho C_{eq}\right\}^2 + \frac{4}{\kappa\mu}\theta^\pm\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2 \quad (34)$$

is positive and then both eigenvalues $r_{1,2}(\varepsilon^\pm, \theta^\pm)$ are real. Let us note that their product and their sum are

$$r_1 r_2 = \frac{\varrho^2}{\mu\kappa}C_{eq}(\dot{S}^2 - \lambda^2) \quad (35)$$

$$r_1 + r_2 = -\frac{1}{\dot{S}}\left\{\frac{\varrho}{\mu}(\dot{S}^2 - \lambda^2) + \frac{1}{\mu\varrho C_{eq}}\theta^\pm\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2 + \frac{\dot{S}^2}{\kappa}\varrho C_{eq}\right\} \quad (36)$$

where $\lambda^2(\varepsilon^\pm, \theta^\pm)$ is given by (5) and represents the square of the nonzero characteristic directions of the adiabatic thermoelastic system (4) at the critical points. Let us note that the sign of the product of the eigenvalues is positive or negative according to whether the speed of the propagating discontinuity $\dot{S}$ is larger or smaller than the adiabatic sound speed at the critical point. Thus, if $r_1 r_2 < 0$, that is, $\dot{S}^2 < \lambda^2(\varepsilon, \theta)$, (subsonic case) the eigenvalues have opposite signs and the critical point is a *saddle point*. If $r_1 r_2 > 0$, that is, $\dot{S}^2 > \lambda^2(\varepsilon, \theta)$, (supersonic case) the eigenvalues have the same sign. The sign of $r_1 + r_2$ is equal to the sign of $-\dot{S}$. Thus, if $\dot{S} > 0$, then both eigenvalues are negative and the critical point is an *attractive node*, while if $\dot{S} < 0$, both eigenvalues are positive and the critical point is a *repulsive* node. If $r_1 = 0$, that is, $\dot{S}^2 = \lambda^2(\varepsilon, \theta)$, then the sign of $r_2$ is equal to the sign of $-\dot{S}$.

Trajectories of the Shock Layers in the $\varepsilon - \theta$ Plane

**The Compressive Case: $\dot{S} > 0$ and $\varepsilon^- < \varepsilon^+$** One shows that in the compressive case *the chord criterion with respect to the curve* $\sigma = \sigma_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$, and hence, *the chord criterion with respect to the Hugoniot locus* $\sigma = \sigma_H(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$, is also a necessary and sufficient condition for the existence and uniqueness of a solution of the nonlinear autonomous system (23), (24).

In this case, the chord criterion with respect to the curve $\sigma = \sigma_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ requires that

$$s(\varepsilon) = \sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-) < 0, \quad \text{for any } \varepsilon \in (\varepsilon^-, \varepsilon^+) \tag{37}$$

By using the thermodynamic properties (3) and the definitions of the functions introduced through relations (26), (29), and (30), one shows that

$$\frac{d\Theta_{KV}(\varepsilon)}{d\varepsilon} = \frac{1}{\varrho C_{eq}(\varepsilon, \Theta_{KV}(\varepsilon))}\Big(\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon) + \Theta_{KV}(\varepsilon)\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon, \Theta_{KV}(\varepsilon))\Big) \tag{38}$$

wherefrom one gets that $\frac{d\sigma_{KV}}{d\varepsilon}(\varepsilon^\pm) = \varrho\lambda^2(\varepsilon^\pm, \theta^\pm)$ and, consequently, $s'(\varepsilon^\pm) = \varrho(\dot{S}^2 - \lambda^2(\varepsilon^\pm, \theta^\pm))$. Because $s(\varepsilon^\pm) = 0$, as a direct consequence of the chord condition (37), we have $s'(\varepsilon^-) \le 0$ and $s'(\varepsilon^+) \ge 0$. That means the chord criterion in the *compressive case* requires

$$\dot{S}^2 - \lambda^2(\varepsilon^-, \theta^-) \le 0, \quad \dot{S}^2 - \lambda^2(\varepsilon^+, \theta^+) \ge 0 \tag{39}$$

If the inequalities are strict, from (35), (36) one gets that $(\varepsilon^-, \theta^-)$ is a *saddle node (subsonic critical point)*, while $(\varepsilon^+, \theta^+)$ is an *attractive node (supersonic critical point)*. On the other side, one sees that the chord criterion is consistent with the shock inequalities of Lax [12] which for a right-facing wave discontinuity read $0 < \lambda(\varepsilon^+, \theta^+) < \dot{S} < \lambda(\varepsilon^-, \theta^-)$. Geometrically, this criterion requires that the characteristics from the same family impinge on the shock front as time advances. In gas dynamics, it requires the flow to be supersonic ahead and subsonic behind the wave discontinuity. The degenerate case when $\dot{S} = \lambda(\varepsilon^\pm, \theta^\pm)$ should be considered separately.

We suppose in the following, as usual, that $\frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} < 0$. This assumption involves that the coefficient of thermal expansion coefficient and the Grüneisen coefficient are positive (see ▶ Thermoelastic Bar Theory). Moreover, this assumption coupled with the chord condition (37) involves, according to (38), that in the compressive case, the function $\theta = \Theta_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ is monotonically decreasing for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$. Consequently, after the passage of a compressive shock wave, the Hugoniot back state temperature has to be larger than the front state temperature, that is, $\theta^- > \theta^+$. One says that *the compressive discontinuity is of heating type*.

Let also note that since $\frac{\partial R(\varepsilon,\theta)}{\partial \theta} = -\frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} > 0$, the implicit equation $R(\varepsilon, \theta) = 0$ is locally uniquely representable as a single valued function of $\varepsilon$. We suppose there exists a function denoted $\theta = \Theta_R(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ for $\varepsilon$ belonging to an interval which contains $\varepsilon^\pm$ such that $R(\varepsilon, \Theta_R(\varepsilon)) = 0$ and $\theta^\pm = \Theta_R(\varepsilon^\pm; \varepsilon^+, \theta^+, \varepsilon^-)$. Its image through the function $\sigma = \sigma_{eq}(\varepsilon, \theta)$ in the $\varepsilon - \sigma$ plane is just the Rayleigh line, that is, $\sigma_R(\varepsilon) = \sigma_{eq}(\varepsilon, \Theta_R(\varepsilon))$. Therefore, one can show that

$$\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon) = \frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon, \bar{\theta}(\varepsilon))(\theta_R(\varepsilon) - \theta_{KV}(\varepsilon)), \quad \text{for any } \varepsilon \in (\varepsilon^-, \varepsilon^+) \tag{40}$$

where $\bar{\theta}(\varepsilon)$ lies between $\theta_R(\varepsilon)$ and $\theta_{KV}(\varepsilon)$. From here and from the chord condition (37), it follows that $\theta_R(\varepsilon) > \theta_{KV}(\varepsilon)$ for any $\varepsilon \in (\varepsilon^-, \varepsilon^+)$. Moreover, one can show the inequalities $\frac{d\Theta_R(\varepsilon^+)}{d\varepsilon} < \frac{d\Theta_{KV}(\varepsilon^+)}{d\varepsilon} < 0$ and $\frac{d\Theta_R(\varepsilon^-)}{d\varepsilon} > \frac{d\Theta_{KV}(\varepsilon^-)}{d\varepsilon}$, which require only that $\theta = \Theta_R(\varepsilon)$ is a decreasing function of $\varepsilon$ in the neighborhood of $\varepsilon^+$ (Fig. 1a). Therefore, unlike the function $\theta = \Theta_{KV}(\varepsilon)$, the function $\theta = \Theta_R(\varepsilon)$ can be non-monotone.

The existence of a connecting orbit, that is, of a shock layer, results now from the following topological considerations, which follow the analysis made by Gilbarg [3]. The closed curve formed by $\theta = \Theta_{KV}(\varepsilon)$ and $\theta = \Theta_R(\varepsilon)$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$, bounds a simply connected region $P$ in the plane $\varepsilon - \theta$. Since $H_{KV} > 0$ on $R = 0$ and $R < 0$ on $H_{KV} = 0$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$, one concludes that everywhere in $P$, $H_{KV} > 0$ and $R < 0$. Let us note that on the boundaries $H_{KV} = 0$ and $R = 0$, all vector fields of the flow (23), (24) point toward the region $P$, horizontally and vertically, respectively. Since $\frac{d\theta}{d\varepsilon} = \frac{\mu \dot{S}^2}{\kappa}\frac{H_{KV}}{R}$, all integral curves must be monotone decreasing in $P$, and because they cannot leave $P$ and there is no critical point in this region, they must tend to the attractive point $(\varepsilon^+, \theta^+)$. Taking into account that $(\varepsilon^-, \theta^-)$ is a saddle point, one obtains that a trajectory connecting $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$ exists and lies inside the region $P$. One can prove, by reduction to the absurd, that the chord criterion is also a necessary condition for the existence of a shock layer. The uniqueness of this shock layer is based on the fact that a trajectory connecting $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$ cannot lie outside $P$.

Therefore, for any $\mu > 0$ and $\kappa > 0$, there exists a unique shock layer $(\hat{\varepsilon}(\xi; \mu, \kappa), \hat{\theta}(\xi; \mu, \kappa))$ joining $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$. Its limit behavior as $\mu, \kappa \to 0$ can be studied as in the case of viscous, heat-conducting fluids considered by Gilbarg [3]. One can prove the existence of iterated limits and their equality with the double limit. The limit is just a shock wave with the same end states. This study points up a basic difference in the effects of viscosity and heat conduction on the structure of the shock layers. Thus, if one considers a fixed viscosity $\mu = \bar{\mu}$ and $\kappa \to 0$, the trajectories in the $\varepsilon - \theta$ plane of the shock layer $(\hat{\varepsilon}(\xi; \bar{\mu}, \kappa), \hat{\theta}(\xi; \bar{\mu}, \kappa))$ are increasingly close to the decreasing curve $\theta = \theta_{KV}(\varepsilon)$ and approach the smooth solution of the reduced system (28). This limit solution describes a *viscous, heat-nonconducting shock layer*.

If $\theta = \theta_R(\varepsilon)$ is monotone decreasing and one considers a fixed conductivity $\kappa = \bar{\kappa}$ and $\mu \to 0$, the shock layers $(\hat{\varepsilon}(\xi; \mu, \bar{\kappa}), \hat{\theta}(\xi; \bar{\mu}, \bar{\kappa}))$ are increasingly close to the curve $\theta = \theta_R(\varepsilon)$ and approach the solutions of the reduced system

$$R(\hat{\varepsilon}, \hat{\theta}) = 0, \quad \hat{\theta}' = -\frac{\dot{S}}{\kappa} H_{KV}(\hat{\varepsilon}, \hat{\theta}), \qquad \lim_{\xi \to \pm\infty} \hat{\theta}(\xi) = \theta^{\pm} \tag{41}$$

This limit solution describes a *nonviscous, heat-conducting shock layer*.

A significant difference appears when $\theta = \Theta_R(\varepsilon)$ is *non-monotone*. Since the integral curves of the system (23), (24) are monotone decreasing in $P$, one shows that as $\mu \to 0$, the trajectories in $\varepsilon - \theta$ plane of the shock layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ are increasingly close to the monotone decreasing curve $\theta = \bar{\Theta}_R(\varepsilon)$ defined by

$$\theta = \bar{\Theta}_R(\varepsilon) = \min_{\zeta \in [\varepsilon^-, \varepsilon]} \Theta_R(\zeta), \quad \text{for } \varepsilon \in [\varepsilon^-, \varepsilon^+] \tag{42}$$

This function is the maximum among all monotone decreasing curves bounded from above by the curve $\theta = \Theta_R(\varepsilon)$. It is represented with dotted line on those parts which do not coincide with $\theta = \Theta_R(\varepsilon)$ in Fig. 1a. If $\theta = \Theta_R(\varepsilon)$ has a finite number of minima, then $\theta = \bar{\Theta}_R(\varepsilon)$ has at most a finite number of intervals on which $\theta$ is constant, which correspond to what are called *isothermal jumps in strain inside the profile layer*. Therefore, in this case, as $\mu \to 0$, the profile layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ approach a pair of functions denoted by $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu = 0, \bar{\kappa})$ with the property that $\hat{\varepsilon}(\xi; \mu = 0, \bar{\kappa})$ is *discontinuous* and $\hat{\theta}(\xi; \mu = 0, \bar{\kappa})$ is continuous and piecewise smooth. Thus, the notion of traveling wave solution must be enlarged in order to admit such discontinuous solutions for the reduced system (41).

**The Expansive Case: $\dot{S} > 0$ and $\varepsilon^- > \varepsilon^+$** In the expansive case, the chord criterion with respect to the curve $\sigma = \sigma_{KV}(\varepsilon)$ requires that $s(\varepsilon) = \sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-) > 0$, for any $\varepsilon \in (\varepsilon^+, \varepsilon^-)$. Since $s(\varepsilon^\pm) = 0$, the chord condition results in $s'(\varepsilon^+) = \varrho(\dot{S}^2 - \lambda^2(\varepsilon^+, \theta^+)) \geq 0$ and $s'(\varepsilon^-) = \varrho(\dot{S}^2 - \lambda^2(\varepsilon^-, \theta^-)) \leq 0$. If the inequalities are strict, one obtains again that $(\varepsilon^-, \theta^-)$ is a saddle node and $(\varepsilon^+, \theta^+)$ is an attractive node. From (38) one gets that $\frac{d\Theta_{KV}}{d\varepsilon}(\varepsilon^\pm) = \frac{\theta^\pm}{\varrho C_{eq}(\varepsilon^\pm, \theta^\pm)} \frac{\partial \sigma_{eq}(\varepsilon^\pm, \theta^\pm)}{\partial \theta} < 0$. Therefore, $\theta = \Theta_{KV}(\varepsilon)(\varepsilon)$ is a monotone decreasing function in the neighborhood of $\varepsilon^\pm$, but one cannot say anything without additional constitutive assumptions, neither about its monotonicity nor about the order relation between $\theta^-$ and $\theta^+$. By using relation (40), one gets that $\theta_R(\varepsilon) < \theta_{KV}(\varepsilon)$ for any $\varepsilon \in (\varepsilon^+, \varepsilon^-)$ (Fig. 1b).

Let us consider in Fig. 1b the phase portrait of the system (23), (24) for the case when $\theta^- < \Theta_{KV}(\varepsilon) < \theta^+$, for any $\varepsilon \in (\varepsilon^+, \varepsilon^-)$, and functions $\theta = \theta_R(\varepsilon)$ and $\theta = \Theta_{KV}(\varepsilon)$ are nonmonotone. A similar phase portrait analysis like in the compressive case shows that the chord criterion ensures the existence of a unique trajectory which connects the states $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$ and lies between the curves $\theta = \theta_R(\varepsilon)$ and $\theta = \Theta_{KV}(\varepsilon)$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$. In this case, the expansive shock is of cooling type since the Hugoniot back state temperature is lower than the front state temperature, that is, $\theta^- - < \theta^+$.

For the unusual case, when $\theta^- > \theta^+$, Pego [5] has constructed an equation of state with the property that there may exist a shock wave discontinuity satisfying the chord criterion, but for which a profile layer does not exist if the heat conduction dominates the viscosity. Thus, in the expansive case, the chord criterion is no longer a necessary and sufficient condition for the existence of a profile layer.

Remark. In fluid dynamics, Liu [4] has proved that a compressive viscous shock profile exists if and only if the chord condition with respect to the Hugoniot locus is satisfied. When both the viscosity and the heat conduction are present, Gilbarg's [3] result and Liu's [4] chord criterion have been extended and discussed by Pego [5].

Traveling wave solutions for a heat conducting Maxwellian rate-type approach to thermoelastic materials have been analyzed in [15].

## The Entropy Production in a Viscous, Thermally Conducting Shock Layer

The entropy production due to the intrinsic and thermal dissipation in a smooth process for a heat-conducting Kelvin–Voigt material (see ▸ Maxwellian Rate-Type Thermo-viscoelastic Bar Theory) is

$$P = \frac{D_{tot}}{\theta} = \frac{1}{\mu\theta}\left(\frac{\partial \varepsilon}{\partial t}\right)^2 + \frac{\kappa}{\theta^2}\left(\frac{\partial \theta}{\partial X}\right)^2 \geq 0. \quad (43)$$

If $(\hat{\varepsilon}(\xi), \hat{\theta}(\xi))$ is a traveling wave solution of the system (23), (24), the total entropy production in a profile layer structured by Kelvin–Voigt viscosity and heat conduction is

$$P_{trav} = \int_{-\infty}^{\infty} \left( \frac{\mu \dot{S}^2}{\hat{\theta}} (\hat{\varepsilon}')^2 + \frac{\kappa}{\hat{\theta}^2} (\hat{\theta}')^2 \right) d\xi$$
$$= -\dot{S} \int_\Gamma \left( \frac{R(\varepsilon, \theta)}{\theta} d\varepsilon + \frac{H_{KV}(\varepsilon, \theta)}{\theta^2} d\theta \right) \geq 0, \quad (44)$$

where $\Gamma = \{(\hat{\varepsilon}(\xi), \hat{\theta}(\xi)) \mid \xi \in (-\infty, \infty)\}$ is the continuous piecewise smooth curve connecting $(\varepsilon^-, \theta^-)$ and $(\varepsilon^+, \theta^+)$ in the $(\varepsilon, \theta)$ plane. Let us note that the integrand is a total differential since $\frac{\partial}{\partial \theta}\left(\frac{R(\varepsilon,\theta)}{\theta}\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\mathcal{H}_{KV}(\varepsilon,\theta)}{\theta^2}\right)$ and

$$P_{trav} = -\dot{S} \int_\Gamma d\left( -\frac{H_{KV}(\varepsilon, \theta)}{\theta} + \varrho \eta_{eq}(\varepsilon, \theta) \right)$$
$$= -\dot{S}\varrho(\eta_{eq}(\varepsilon^+, \theta^+) - \eta_{eq}(\varepsilon^-, \theta^-)) \geq 0. \quad (45)$$

It follows that the total entropy production in a profile layer does not depend on viscosity or heat conductivity. It is just the entropy production (8) generated by a thermoelastic shock wave compatible with the second law. As a consequence, in a profile layer structured by Kelvin–Voigt viscosity and heat conductivity, the entropy of the Hugoniot state $(\varepsilon^-, \theta^-)$ is never less than the entropy of the initial state $(\varepsilon^+, \theta^+)$. Therefore, a shock wave which satisfies the chord criterion with respect to the Hugoniot locus $\sigma = \sigma_H(\varepsilon; \varepsilon^+, \theta^+)$ is compatible with the entropy inequality.

Concerning the variation of the entropy inside a shock layer, one can show even more. Thus, in a viscous, heat-nonconducting shock layer, or when the viscosity effect dominates the heat conductivity effect, there is a monotonous variation of the entropy. On the other side, in a nonviscous, heat-conducting profile layer, or when the heat conductivity effect is more important than the viscosity effect, the entropy variation is non-monotone inside the layer and the entropy overshoots its final value at the Hugoniot state (see, for instance, [11, 15, 16]).

## Cross-References

- ► Maxwellian Rate-Type Thermo-viscoelastic Bar Theory – An Approach to Non-monotone Thermoelasticity
- ► Thermoelastic Bar Theory

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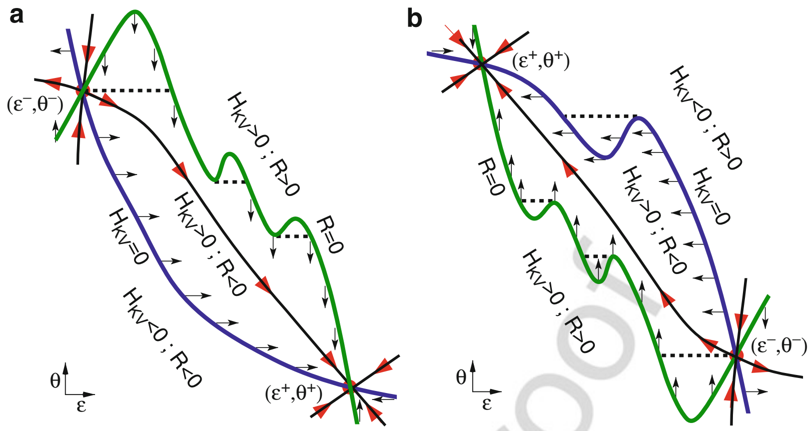


**Heat Conduction and Viscosity as Structuring Mechanisms for Shock Waves in Thermoelastic Materials, Fig. 1** Phase portrait of the system (23)–(24) and shock layer trajectory. (**a**) The compressive case $\varepsilon^- < \varepsilon^+$. (**b**) The expansive case $\varepsilon^- > \varepsilon^+$