Phase nucleation and wave propagation in phase-transforming strings. A rate-type approach

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Dedicated to Professor Nicolae Cristescu on the occasion of his 80th birthday

Abstract

This paper uses a rate-type approach of the non-monotone elasticity to describe large and rapid motions of a string made of a phase transforming material like shape memory alloy (SMA). The nucleation mechanism, the propagation of phase boundaries, of longitudinal and transverse front waves are carefully analyzed. It is studied the influence of the slope of the equilibrium curve on the growth and decay of the amplitude of the first and second order discontinuities. The nucleation and wave propagation phenomena which accompany the damping effect of SMA string submitted to a sudden transverse motion are numerically investigated.

Key words: Phase transformation; Nucleation; Strings and cables; Wave propagation; Transverse impact; SMA; Damping effect

1. Introduction

Motivated by applications, the large and rapid motions of strings have been intensively studied a few decades ago in connection with the use of synthetic strings and rubber cables. Outstanding contributions to this topic have been brought by Cristescu [1] who, by using Hadamard's theory of wave propagation has considered and investigated impact problems for elastic and viscoplastic string models in its well known monograph. From the mathematical point of view this interest has stimulated investigations on the theory of non-strictly hyperbolic conservations laws arising from non-linear elasticity with convex or non-convex stress-strain relation. There is a rich literature on these subjects related to the existence and uniqueness of weak solutions to the Riemann problem for elastic strings (see for instance Mihăilescu and Suliciu [2], Young [3] and the references therein). A different approach to investigate the finite-amplitude oscillations of strings has been applied for elastic suspended cables using nonlinear vibration analysis (see the review papers by Rega [4]).

The recent increasing interest on the mechanics of phase transforming materials, like shape memory alloys, has led Purohit and Bhattacharya [5] to consider the behavior of strings made of such materials. SMAs are crystalline solids which can abruptly change their lattice structure when subjected to mechanical or thermal loading. These phase transformation events are non-linear phenomena associated with instabilities and growth of phases. It is now unanimously accepted that from the physical point of view the basic assumption in a continuum theory description

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of phase transitions relies on the use of a non-convex Helmholtz free-energy, or equivalently a non-monotone stress-strain relation. This leads to a change of type in the governing equations and therefore to mathematical ill-posed problems. To remedy this deficiency, in [5], additional constitutive information in the form of kinetic relation and nucleation criterion has been used.

Unlike this approach, we use here a Maxwellian rate-type constitutive equation whose equilibrium relation is described by the same non-monotone tension-stretch relation as in [5]. In Section 2 we describe the general governing equations. Our constitutive thermodynamical framework is explained in Section 3. Section 4 is devoted to the analysis of the hyperbolic rate-type system. We use Hadamard's theory of propagating singular surfaces to examine the growth and decay behavior of shock waves and acceleration waves and their consequences on phase nucleation. In Section 5 we investigate numerically the transverse and longitudinal motions of a SMA string impacted by a body of mass M which moves with constant velocity. We put into evidence, on one side, the capacity of the constitutive model to describe phase nucleation phenomena accompanying the wave propagation in the string and on the other side the strong damping properties of the phase transforming string.

2. Equations for large motions of perfectly flexible strings

We consider a string as a one-dimensional continuum moving in a three dimensional space. In this section we describe the equations governing the large motions of a string of any material. A motion of a string is a continuous mapping $\overrightarrow{\chi}(X,t): I \times [0,\infty) \to \mathbb{R}^3$, $\overrightarrow{x} = \overrightarrow{\chi}(X,t)$, where I is a finite or infinite interval of the real axis. We suppose that $\overrightarrow{\chi}(\cdot,t)$ is one-to-one for any $t>0$, that is, it always represents a simple curve in $\mathbb{R}^3$. We denote by $\mathscr{S}_0 \equiv \overrightarrow{\chi}(X,0)$ the reference configuration of the string - the curve initially occupied by the string, and by $\mathscr{S}_t \equiv \overrightarrow{\chi}(X,t)$ the actual configuration of the string - the deformed configuration. X is a coordinate taken along the string starting from an arbitrary coordinate origin, while $\overrightarrow{x}$ gives the position in $\mathbb{R}^3$ at time t of the point on the string which at time $t=0$ was located at $\overrightarrow{\chi}(X,0)$ on the reference curve. Whenever $\overrightarrow{\chi}(X,t)$ is differentiable we denote by $\overrightarrow{v}(X,t) = \frac{\partial \overrightarrow{\chi}}{\partial t}$ the *particle velocity* and by $\overrightarrow{\lambda}(X,t) = \frac{\partial \overrightarrow{\chi}}{\partial X}$ the *tangent vector* to the string in the deformed configuration at material point X. $\lambda(X,t) = \left|\frac{\partial \overrightarrow{\chi}}{\partial X}\right|$ is called the *stretch*, while $\varepsilon = \frac{|d\overrightarrow{x}|-dX}{dX} = \lambda - 1$ is the *strain* at material point X. We denote by $\overrightarrow{\tau}(X,t) = \frac{\overrightarrow{\lambda}}{\lambda}$ the *tangent unit vector* to the string in the deformed configuration.

Therefore the following *compatibility condition* between $\overrightarrow{v}$ and $\overrightarrow{\lambda}$ has to be satisfied

$$\frac{\partial \overrightarrow{\lambda}}{\partial t} = \frac{\partial \overrightarrow{v}}{\partial X}. \tag{1}$$

We can relax the smoothness assumptions by allowing the particle velocity $\overrightarrow{v}$, the stretch λ or the tangent $\overrightarrow{\tau}$ to jump across a finite number of moving points along the string. According to Hadamard's lemma ([6, Sect. 173]), if $a = a(X,t)$ is one of the above functions defined and continuously differentiable in the interior of two regions $\mathscr{R}^{\pm}$ separated by a smooth curve $X = S(t)$ we have

$$\frac{d[a](t)}{dt} = \left[\frac{\partial a}{\partial t}\right] + \dot{S}(t)\left[\frac{\partial a}{\partial X}\right] \tag{2}$$

where $[a](t) = a^+(t) - a^-(t) = a(S(t)+0,t) - a(S(t)-0,t)$ and $\dot{S}$ is the speed of propagation of the discontinuity.

Therefore, the continuity condition of $\overrightarrow{\chi}$ leads to the weak form of relation (1), i.e., to the *kinematic jump condition*

$$\left[\overrightarrow{v}\right] + \dot{S}\left[\overrightarrow{\lambda}\right] = 0. \tag{3}$$

If we denote by $\overrightarrow{T} = \overrightarrow{T}(X,t)$ the *vector tension* (force per unit cross-sectional area of the string in the reference configuration) acting at a material point X of the string at time t then the *balance of linear momentum* is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} \rho \overrightarrow{v}(X,t)\mathrm{d}X = \overrightarrow{T}(X_2,t) - \overrightarrow{T}(X_1,t) + \int_{X_1}^{X_2} \rho \overrightarrow{b}(X,t)\mathrm{d}X, \tag{4}$$

for any portion of the string comprised between the coordinates X_1 and X_2 at time t. As a consequence we get its differential and weak form as

$$\rho \frac{\partial \overrightarrow{v}}{\partial t} = \frac{\partial \overrightarrow{T}}{\partial X} + \rho \overrightarrow{b}, \tag{5}$$

and, respectively

$$\rho\dot{S}[\overrightarrow{v}]+[\overrightarrow{T}]=0, \tag{6}$$

where ρ denotes the *mass density* per unit length of the string in the reference configuration and it is assumed constant; $\overrightarrow{b}=\overrightarrow{b}(X,t)$ denotes the body forces per unit mass acting on the string at the material point X and time t.

The *balance of angular momentum* for the same portion of the string, in the absence of body moments, is

$$\frac{d}{dt}\int_{X_1}^{X_2}\overrightarrow{\chi}\times\rho\overrightarrow{v}\,\mathrm{d}X=(\overrightarrow{\chi}\times\overrightarrow{T})(X_2,t)-(\overrightarrow{\chi}\times\overrightarrow{T})(X_1,t)+\int_{X_1}^{X_2}\overrightarrow{\chi}\times\rho\overrightarrow{b}\,\mathrm{d}X+\overrightarrow{m}(X_2,t)-\overrightarrow{m}(X_1,t), \tag{7}$$

where the constitutive quantity $\overrightarrow{m}(X,t)$ is the *couple* (moment) acting at the material point X and time t.

A *perfectly flexible string* is a string which can not resist to any couple forces, i.e., for which $\overrightarrow{m}(X,t)\equiv 0$ in (7). This defining constitutive property distinguishes it from a rod which can resist bending. Therefore, by using (5), the differential form of (7) leads to

$$\frac{\partial\overrightarrow{\chi}}{\partial X}\times\overrightarrow{T}=0,\quad\text{or equivalently}\quad\overrightarrow{T}=T\overrightarrow{\tau} \tag{8}$$

where $T=T(X,t)$, called tension, is the *scalar tension* at the material point X at time t. Therefore, in perfectly flexible strings, the vector tension at every point is always directed along the tangent to the string.

The *second law of thermodynamics* in this isothermal context requires

$$\frac{d}{dt}\int_{X_1}^{X_2}\rho\big(\psi+\frac{1}{2}v^2\big)\,\mathrm{d}X\le(\overrightarrow{T}\cdot\overrightarrow{v})(X_2,t)-(\overrightarrow{T}\cdot\overrightarrow{v})(X_1,t)+\int_{X_1}^{X_2}\rho\overrightarrow{b}\cdot\overrightarrow{v}\,\mathrm{d}X,\qquad\text{for any }\ X_1,X_2\in I, \tag{9}$$

where $\psi=\psi(X,t)$ denotes the free energy function.

The differential form and weak form of the *dissipation inequality* which follows from (9) are

$$\rho\frac{\partial}{\partial t}\big(\psi+\frac{1}{2}v^2\big)-\frac{\partial}{\partial X}\big(\overrightarrow{T}\cdot\overrightarrow{v}\big)\le\overrightarrow{b}\cdot\overrightarrow{v}, \tag{10}$$

and, respectively

$$D=\rho\dot{S}\Big[\psi+\frac{1}{2}v^2\Big]+[\overrightarrow{T}\cdot\overrightarrow{v}]\ge 0,\quad\text{or, equivalently}\quad D=\dot{S}\Big(\rho[\psi]-\frac{1}{2}\big(\overrightarrow{T}^{+}+\overrightarrow{T}^{-}\big)\cdot\big[\overrightarrow{\lambda}\big]\Big)\ge 0. \tag{11}$$

By using the balance of momentum (5) we get the following *reduced form of the dissipation inequality*

$$\rho\frac{\partial\psi}{\partial t}\le\overrightarrow{T}\cdot\frac{\partial\overrightarrow{\lambda}}{\partial t}\equiv T\frac{\partial\lambda}{\partial t}\equiv T\frac{\partial\varepsilon}{\partial t}. \tag{12}$$

The *conservation of mass* can be written as $\rho_a dl=\rho dX$, where ρ_a is the actual mass density, ρ the initial mass density and $dl=|\sqrt{d\overrightarrow{x}\cdot d\overrightarrow{x}}|=\lambda dX$. Therefore, we have $\rho_a\lambda=\rho$.

3. Phase transforming materials

3.1. *Constitutive assumptions*

In order to complete the system of equations (1), (5) and $(8)_2$ in the seven unknowns $\overrightarrow{v}$, $\overrightarrow{\lambda}$ and T, we must add a constitutive equation relating the scalar tension T and the stretch λ (or, equivalently the strain $\varepsilon=\lambda-1$). We consider in this paper the following rate-type constitutive equation

$$\frac{\partial T}{\partial t}-E\frac{\partial\lambda}{\partial t}=-\frac{E}{\mu}(T-T_{eq}(\lambda)), \tag{13}$$

where $T=T_{eq}(\lambda)$ describes the equilibrium states of the material. E =const.>0 is the dynamic Young modulus which characterizes the instantaneous response of the material. μ =const.>0 is a Newtonian viscosity coefficient and $\frac{\mu}{E}$ has the meaning of a relaxation time of the model, while $k=\frac{E}{\mu}$ is usually called Maxwellian viscosity coefficient. When $\mu\to 0$ this Maxwellian rate-type constitutive equation can be seen as a rate-type approach of the elastic model in a sense which will be explained below. From the physical point of view it introduces a mechanism of energy dissipation of kinetic origin modelled by a Maxwell-type viscosity. Moreover when $E \to \infty$ relation (13) reduces to the well-known Kelvin-Voigt constitutive equation $T = T_{eq}(\lambda) + \mu \frac{\partial \lambda}{\partial t}$.

Starting with the paper by Ericksen [7] for the static case, and James [8] for the dynamic case, (see also Abeyaratne and Knowles [9]) it is now unanimously accepted that, for the one-dimensional theory, the bulk behavior of solids that undergo reversible stress-induced phase transitions is described by *stress-strain (stretch) relations* $T = T_{eq}(\lambda)$ *which are non-monotone*, or equivalently by *non-convex free-energy functions* $\psi = \psi_{eq}(\lambda)$ with $\rho \frac{\mathrm{d}\psi_{eq}(\lambda)}{\mathrm{d}\lambda} = T_{eq}(\lambda)$.

As usual (see for instance [10]), we will consider sometimes here, as an explicit example, the piece-wise linear non-monotone tension-stretch relation

$$T = T_{eq}(\lambda) = \begin{cases} E_1(\lambda - 1), & \text{for } 1 \le \lambda \le \lambda_a \\ -E_2(\lambda - \lambda_a) + T_a, & \text{for } \lambda_a < \lambda < \lambda_m \\ E_3(\lambda - \lambda_m) + T_m, & \text{for } \lambda_m \le \lambda \end{cases} \tag{14}$$

where $T_a = E_1(\lambda - 1)$ and $T_m = T_a - E_2(\lambda_m - \lambda_a)$ (see Fig. 1). This assumption does not influence the strong non-linear character of phase transforming phenomena.

Relation (14) is viewed as corresponding to a material which can exist in an austenite phase $\mathscr{A}$ with the elastic modulus E_1 =const.> 0 and in a martensite phase $\mathscr{M}$ with the elastic modulus E_3 =const.> 0. $-E_2$ =const.< 0 is called the softening modulus and it corresponds to the unstable phase (spinodal region) $\mathscr{I}$ of the material.

The monotone increasing parts of such an equilibrium tension-stretch relation can be obtained in quasi-static experiments at constant temperature for shape memory alloys, like NiTi for example (see for instance Shaw and Kyriakides [11]). Indeed, during an uniaxial test of a thin bar the material is found to remain in the homogeneous austenite phase $\mathscr{A}$ for sufficiently small values of the stretch ($\lambda \le \lambda_a$), while for sufficiently large tensile stretch ($\lambda \ge \lambda_m$) it is found in the homogeneous martensite phase $\mathscr{M}$. The slope of the experimental stress-strain curve in a homogeneous phase is strictly increasing and almost linear elastic. The monotone decreasing part of the equilibrium tension-stretch relation can not be directly determined. It is chosen in an arbitrary manner and its slope affects only the instability phenomena which accompany phase transformation processes. The reversible stress-induced $\mathscr{A} \leftrightharpoons \mathscr{M}$ transformations result in inhomogeneous deformations corresponding to stress plateaus (one for loading and one for unloading) during which phase boundaries nucleate and propagate along the length of the specimen. This unusual characteristic of the material called *pseudo-elasticity* (for $T_m > 0$) refers to the material's ability to be strained significantly and to return to its unstrained configuration via a hysteresis loop. The width of the hysteresis loop is proportional with $T_a - T_m$.

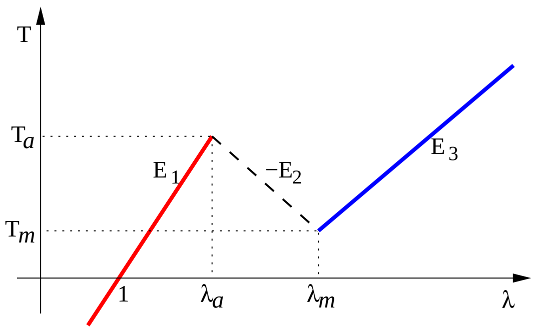


Fig. 1. The trilinear material. Tension-stretch curve. $T = T_{eq}(\lambda)$

Thus, we say that a particle of the string, labelled by X, is in the austenite phase $\mathscr{A}$, unstable phase $\mathscr{I}$, or in the martensite phase $\mathscr{M}$ according to whether the stretch $\lambda(X,t)$ is in $(1, \lambda_a]$, (λ_a, λ_m), or $[\lambda_m, \infty)$, respectively.

The same as in the case of the bar theory the non-monotone elastic constitutive equation leads to a change of type in the governing equations, and this in turn leads to mathematical ill-posed problem in the sense of Hadamard and thus to the incapacity of the model to uniquely describe processes of the body involving transitions from one phase to another, i.e. the evolution of the microstructure. Indeed, it is well known that the characteristic directions (characteristic speeds) of the elastic system

$$\rho\frac{\partial\overrightarrow{v}}{\partial t}=\frac{\partial}{\partial X}\Big(\frac{T}{\lambda}\overrightarrow{\lambda}\Big),\qquad \frac{\partial\overrightarrow{\lambda}}{\partial t}=\frac{\partial\overrightarrow{v}}{\partial X},\qquad T=T_{eq}(\lambda),\tag{15}$$

are given by

$$\frac{dX}{dt}=\pm C_L(\lambda)\equiv\pm\sqrt{\frac{1}{\rho}\frac{\mathrm{d}T_{eq}(\lambda)}{\mathrm{d}\lambda}},\quad \text{and}\quad \frac{\mathrm{d}X}{\mathrm{d}t}=\pm C_T(\lambda)\equiv\pm\sqrt{\frac{T_{eq}(\lambda)}{\rho\lambda}}\ \text{(twice)}.\tag{16}$$

and they correspond to the longitudinal and transverse acceleration waves, respectively.

Therefore, the system is of mixed type, i.e. hyperbolic for $\lambda\in(1,\lambda_a)\bigcup(\lambda_m,\infty)$, when $T=T_{eq}(\lambda)$ is monotone increasing and the material is in the austenite or martensite phase, and hyperbolic-elliptic for $\lambda\in(\lambda_a,\lambda_m)$ when the equilibrium curve is monotone decreasing and the material is in the unstable phase. The loss of the hyperbolicity of the system when the tension $T_{eq}(\lambda)$ in the string is negative, i.e when the string is submitted to a compression, is a specific limitation for perfectly flexible strings, which arises from the constitutive assumption that such ideal bodies can not resist any couple forces (shear or curvature) i.e., $\overrightarrow{m}\equiv 0$ in (7).

It is known that from the physical point of view the nucleation and propagation of phase boundaries are strongly dissipative phenomena. The elastic model does not contain a dissipative mechanism, which obviously is its main constitutive deficiency. Therefore, one needs to provide additional constitutive information. Which is the appropriate dissipative mechanism to accurately describe phase transition phenomena is still an open problem. *One widespread way* is to supplement a relation like (14) with a constitutive kinetic relation and a nucleation criterion. The kinetic relation gives the velocity of propagation of the phase boundary as a function of a driving force (see for instance the thermodynamical framework developed by Truskinovski [12], Abeyaratne and Knowles [13]). In such a constitutive context the dynamics of strings made of phase-transforming materials has been investigated by Purohit and Bhattacharya [5].

A second way to remedy this constitutive deficiency is to embed the elastic theory as a special case into a broader theory. Usually this includes viscosity of Kelvin-Voigt type and/or strain gradient effects (see for instance James [8], Slemrod [14], Ngan and Truskinovski [15]). A new and different rate-type approach of the non-monotone elasticity based on a Maxwell's rate type constitutive equation (13) has been considered for phase transformation phenomena by Suliciu [16] and investigated by Făciu and Mihăilescu-Suliciu [17], Suliciu [18], Suliciu [19], Făciu and Suliciu [20], Făciu [21], Făciu and Molinari [22]. This approach does not require a separate nucleation criterion. Moreover, it possesses its own kinetics, since the material instability phenomena incorporated in the unstable region lead automatically to the formation and evolution of phase boundaries.

In order to investigate the phase nucleation and wave propagation in phase transforming strings we adopt in this paper the Maxwellian rate-type approach (13). It has been shown, in the frame of the bar theory, that this rate-type constitutive equation has the capacity to describe the kinetics of phase transformation for quasistatic strain and stress-controlled experiments ([20], [21]) and recently, for impact experiments ([22]), too. The new parameters E and μ which intervene in the constitutive description describe in fact the kinetics of the growth of phases and should be connected with the time of growth, or time of nucleation of microscopic theories of phase transitions. This constitutive relation includes as a limiting case for $E\to\infty$ the Kelvin-Voigt viscoelastic model which has been considered by James [8], later by Pego [23] and more recently by Vainchtein and Rosakis [24] in relation with phase transformation in solids. Rate-type constitutive equations of type (13) have been used intensively by Cristescu (see [1, Chap. IV, §12] and the references therein), to describe the mechanical properties of synthetic strings.

3.2. *Energy and thermodynamic considerations.*

Let us note that according to the second law of thermodynamics (12) one gets for rate-type strings a similar result to that obtained by Făciu and Mihăilescu-Suliciu [17] for the viscoelastic bar theory. The rate-type constitutive equation (13) admits a unique free energy function depending on stretch and tension $\psi=\psi(\lambda,T)$ if and only if the slope of the straight line connecting any two points on the equilibrium curve is lower than the instantaneous Young modulus E. Therefore for the piecewise linear case (14) the following restrictions have to be satisfied by the elastic phase moduli: $E_1<E$ and $E_3<E$. Moreover, we suppose here there is a positive constant E^* such that

$$\frac{T_{eq}(\lambda_1)-T_{eq}(\lambda_2)}{\lambda_1-\lambda_2}\le E^*<E,\quad \text{for any } \lambda_1,\lambda_2\in(1,\infty),\ \lambda_1\neq\lambda_2.\tag{17}$$

One shows that the free energy function of the rate type model has to satisfy the partial differential equation

$$\frac{\partial \psi}{\partial \lambda} + E\frac{\partial \psi}{\partial T} = \frac{T}{\rho}, \quad \psi(0,0) = 0, \tag{18}$$

and the dissipation inequality

$$D_{Mxw}(\lambda, T) \equiv \frac{E}{\mu}\rho\frac{\partial \psi}{\partial T}(\lambda, T)\big(T - T_{eq}(\lambda)\big) \geq 0, \quad \text{for any } (\lambda, T). \tag{19}$$

The general form of the free energy function can be explicitly determined by the equilibrium curve $T = T_{eq}(\lambda)$ and by the dynamic Young modulus E through relation

$$\rho\psi(\lambda, T) = \frac{T^2}{2E} - \frac{T_{eq}^2(\lambda^*)}{2E} + \int_1^{\lambda^*} T_{eq}(s)\mathrm{d}s, \tag{20}$$

where λ^* is the unique solution of the equation $T - E\lambda = T_{eq}(\lambda^*) - E\lambda^*$.

Moreover, one derives the following bound on the internal Maxwellian dissipation

$$D_{Mxw}(\lambda, T) \leq \frac{E}{\mu(E - E^*)}|T - T_{eq(\lambda)}|^2, \quad \text{for any } (\lambda, T). \tag{21}$$

By using the balance equations, the constitutive relation (13) and the property (18) of the free energy function one establishes the following energy identity for the smooth solutions of the PDeqs rate-type string system

$$\frac{\partial}{\partial t}\Big(\rho\frac{\overrightarrow{v}\cdot\overrightarrow{v}}{2} + \rho\psi(\lambda, T)\Big) = \frac{\partial}{\partial X}\big(\overrightarrow{T}\cdot\overrightarrow{v}\big) - \rho\frac{E}{\mu}\frac{\partial \psi}{\partial T}(\lambda, T)\big(T - T_{eq}(\lambda)\big). \tag{22}$$

If we denote by $e_{Mxw}(t) = \int_0^L \big(\rho\frac{\overrightarrow{v}\cdot\overrightarrow{v}}{2} + \rho\psi(\lambda, T)\big)(X, t)\mathrm{d}X$ the total energy of a rate-type string of lenght L we get the energy identity

$$\frac{\mathrm{d}e_{Mxw}(t)}{dt} = (\overrightarrow{T}\cdot\overrightarrow{v})(L, t) - (\overrightarrow{T}\cdot\overrightarrow{v})(0, t) - \rho\frac{E}{\mu}\int_0^L \frac{\partial \psi}{\partial T}(\lambda, T)\big(T - T_{eq}(\lambda)\big)(X, t)\mathrm{d}X. \tag{23}$$

For an isolated body problem, i.e a boundary value problem for which $(\overrightarrow{T}\cdot\overrightarrow{v})(L, t) = 0$ and $(\overrightarrow{T}\cdot\overrightarrow{v})(0, t) = 0$ for any $t > 0$, we derive, by using estimate (21) too, the following inequalities

$$\frac{\mathrm{d}e_{Mxw}(t)}{\mathrm{d}t} \leq 0 \quad \text{and} \quad \int_0^t\int_0^L |T - T_{eq}(\lambda)|^2\mathrm{d}X\mathrm{d}t \leq \mu\frac{E - E^*}{E}(e_{Mxw}(0) - e_{Mxw}(t)). \tag{24}$$

The first one tells us that the total energy of an isolated rate type string is a non-increasing function of time. The second one shows that the rate-type model approaches the non-monotone elastic model $T = T_{eq}(\lambda)$ in an L^2 sense when $\mu \to 0$.

4. Analysis of the Maxwellian rate type system

4.1. *Characteristic directions and relations along the characteristics.*

The rate-type system for strings (1), (5) and (13) can be written as a non-conservative system with sources for the unknowns $\overrightarrow{v}$, $\overrightarrow{\lambda}$, T

$$\frac{\partial}{\partial t}\begin{pmatrix}\overrightarrow{v}\\ \overrightarrow{\lambda}\\ T\end{pmatrix} - \boldsymbol{B}(\overrightarrow{\lambda}, T)\frac{\partial}{\partial X}\begin{pmatrix}\overrightarrow{v}\\ \overrightarrow{\lambda}\\ T\end{pmatrix} = \begin{pmatrix}\rho\,\overrightarrow{b}\\ \overrightarrow{0}\\ G(\lambda, T)\end{pmatrix} \tag{25}$$

where $\boldsymbol{B}(\overrightarrow{\lambda}, T)$ is a matrix in $\mathbb{R}^{7\times 7}$ and G an over-stress function defined as follows

$$\boldsymbol{B}(\overrightarrow{\lambda},T)=\begin{pmatrix}\mathbf{0} & \dfrac{T}{\rho\lambda}\Big(\mathbf{1}-\dfrac{\overrightarrow{\lambda}\otimes\overrightarrow{\lambda}}{\lambda^2}\Big) & \dfrac{\overrightarrow{\lambda}}{\rho\lambda}\\ \mathbf{1} & \mathbf{0} & \overrightarrow{0}\\ E\dfrac{\overrightarrow{\lambda}}{\lambda} & \overrightarrow{0} & 0\end{pmatrix},\qquad G(\lambda,T)=-\frac{E}{\mu}(T-T_{eq}(\lambda)). \tag{26}$$

In writing (26) we have used relations $\frac{\partial\lambda}{\partial t}=\frac{\overrightarrow{\lambda}}{\lambda}\cdot\frac{\partial\overrightarrow{\lambda}}{\partial t}$ and $\frac{\partial\lambda}{\partial X}=\frac{\overrightarrow{\lambda}}{\lambda}\cdot\frac{\partial\overrightarrow{\lambda}}{\partial X}$.

The type of the system is given by its characteristic directions $\frac{\mathrm{d}X}{\mathrm{d}t}=r$ which are defined as the eigenvalues of the matrix $-\boldsymbol{B}(\overrightarrow{\lambda},T)$, that is solutions of the equation $r\Big(r^2-\frac{T}{\rho\lambda}\Big)^2\Big(r^2-\frac{E}{\rho}\Big)=0$.

We therefore have

$$\frac{\mathrm{d}X}{\mathrm{d}t}=\pm C_L=\pm\sqrt{\frac{E}{\rho}},\qquad \frac{\mathrm{d}X}{\mathrm{d}t}=\pm C_T(\lambda,T)=\pm\sqrt{\frac{T}{\rho\lambda}}\ \text{(twice)},\qquad \frac{\mathrm{d}X}{\mathrm{d}t}=0. \tag{27}$$

Since for strings we consider only compressive states, i.e. $T>0$, it follows that unlike the elastic system (15), the rate-type system (25) *is hyperbolic irrespective of the slope of the equilibrium curve* and the initial-boundary value problems are well-posed.

The characteristic directions are also defining the discontinuity directions for the partial derivatives of $\overrightarrow{v}$, $\overrightarrow{\lambda}$ and T, that is, the speed of propagation of the *second order discontinuities*, or equivalently, of the *acceleration waves*. If we determine the right eigenvectors corresponding to the eigenvalues (27) we get the following results. For $r=\pm C_L$ the right eigenvectors are $(\mp C_L\overrightarrow{\tau},\overrightarrow{\tau},E)\in\mathbb{R}^7$ and have the property that the velocity component is tangent to the string. Therefore, $\pm C_L$ represent the speeds of propagation of *longitudinal acceleration waves*. For $r=\pm C_T(\lambda,T)$ the two right eigenvectors are $(\mp C_T(\lambda,T)\overrightarrow{\pi}_1,\overrightarrow{\pi}_1,0)\in\mathbb{R}^7$ and $(\mp C_T(\lambda,T)\overrightarrow{\pi}_2,\overrightarrow{\pi}_2,0)\in\mathbb{R}^7$ where $\overrightarrow{\pi}_1$ and $\overrightarrow{\pi}_2$ are two unit vectors perpendicular to $\overrightarrow{\tau}$ such that $\overrightarrow{\pi}_1\perp\overrightarrow{\pi}_2$. These eigenvectors have the property that their velocity component is perpendicular to the tangent direction to the string $\overrightarrow{\tau}$. Therefore $\pm C_T(\lambda,T)$ represent the speeds of propagation of *transverse acceleration waves*. For $r=0$ the corresponding eigenvector is $(\overrightarrow{0},\overrightarrow{\lambda},0)\in\mathbb{R}^7$ and it corresponds to a *stationary* or *material discontinuity*. Obviously the above 7 eigenvectors are linearly independent.

Let us note that even if the instantaneous elastic response of the material is linear, i.e. $E=$const., the hyperbolic rate-type system is quasi-linear due to the geometric non-linearities incorporated and therefore the transverse characteristics are not straight lines and depend on the solution. On the other side, none of the characteristic field families of the rate-type system is "genuinely non-linear", all are linearly degenerate, that is, each eigenvalue $r_i=r_i(\overrightarrow{v},\overrightarrow{\lambda},T)$ and corresponding eigenvector $\overrightarrow{e}_i$, $i=1,7$ satisfy $\nabla r_i\cdot\overrightarrow{e}_i=0$. That means, the main physical non-linearities, which are related with the non-monotone character of the equilibrium curve, are now incorporated in the right term of the system (25). Therefore, we expect to "approach" ill posed problems corresponding to the non-linear PDEs hyperbolic-elliptic elastic system (15) by well posed problems for the hyperbolic linearly degenerate rate-type system (25).

In order to determine the relations along the characteristics we proceed as follows. Let $X=S(t)$ be a characteristic curve of the system (25). Let us define, along this curve the smooth functions $\hat{a}(t)=a(S(t),t)$, where a stands for $\overrightarrow{v}$, $\overrightarrow{\lambda}$ and T. Therefore, we have along this curve

$$\frac{\partial\overrightarrow{v}}{\partial t}+\dot{S}\frac{\partial\overrightarrow{v}}{\partial X}=\frac{d\,\overrightarrow{\hat{v}}}{dt},\quad \frac{\partial\overrightarrow{\lambda}}{\partial t}+\dot{S}\frac{\partial\overrightarrow{\lambda}}{\partial X}=\frac{d\,\overrightarrow{\hat{\lambda}}}{dt},\quad \frac{\partial T}{\partial t}+\dot{S}\frac{\partial T}{\partial X}=\frac{d\hat{T}}{dt}. \tag{28}$$

If one substitutes $\frac{\partial\overrightarrow{v}}{\partial t}(S(t),t)$, $\frac{\partial\overrightarrow{\lambda}}{\partial t}(S(t),t)$ and $\frac{\partial T}{\partial t}(S(t),t)$ from (28) into (25) one obtains

$$\Big(\boldsymbol{B}(\overrightarrow{\lambda},T)+\dot{S}\mathbf{1}\Big)\frac{\partial}{\partial X}\begin{pmatrix}\overrightarrow{v}\\ \overrightarrow{\lambda}\\ T\end{pmatrix}(S(t),t)=\begin{pmatrix}\overrightarrow{0}\\ \overrightarrow{0}\\ G(\lambda,T)\end{pmatrix}(S(t),t)-\frac{d}{dt}\begin{pmatrix}\overrightarrow{\hat{v}}\\ \overrightarrow{\hat{\lambda}}\\ \hat{T}\end{pmatrix}. \tag{29}$$

Let us consider the above relation as an algebraic system for the unknowns $\frac{\partial\overrightarrow{v}}{\partial X}(S(t),t)$, $\frac{\partial\overrightarrow{\lambda}}{\partial X}(S(t),t)$ and $\frac{\partial T}{\partial X}(S(t),t)$. Since $\dot{S}$ are eigenvalues of $-\boldsymbol{B}(\overrightarrow{\lambda},T)$ then the determinant of this system is zero exactly for these values i.e. (27).

Therefore, the system is undetermined and admits a solution if and only if the rank of its matrix is equal with the rank of the extended matrix. This condition leads immediately to the relations along the characteristic lines of the system (25). These are

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{T}-E\hat{\lambda}\big)=G(\hat{T},\hat{\lambda}),\quad \text{for } \frac{\mathrm{d}X}{\mathrm{d}t}=0 \tag{30}$$

$$\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\mp\sqrt{\rho E}\,\frac{\overrightarrow{\hat{\lambda}}}{\lambda}\cdot\frac{d\,\overrightarrow{\hat{v}}}{\mathrm{d}t}=G(\hat{T},\hat{\lambda}),\quad \text{for } \frac{\mathrm{d}X}{\mathrm{d}t}=\pm C_L=\pm\sqrt{\frac{E}{\rho}} \tag{31}$$

$$\hat{\lambda}_3\Big(\frac{\mathrm{d}\hat{v}_2}{\mathrm{d}t}\mp\sqrt{\frac{\hat{T}}{\rho\hat{\lambda}}}\frac{\mathrm{d}\hat{\lambda}_2}{\mathrm{d}t}\Big)=\hat{\lambda}_2\Big(\frac{\mathrm{d}\hat{v}_3}{dt}\mp\sqrt{\frac{\hat{T}}{\rho\hat{\lambda}}}\frac{\mathrm{d}\hat{\lambda}_3}{\mathrm{d}t}\Big),\ \text{for } \frac{\mathrm{d}X}{\mathrm{d}t}=\pm C_T=\pm\sqrt{\frac{\hat{T}}{\rho\hat{\lambda}}} \tag{32}$$

$$\hat{\lambda}_2\Big(\frac{\mathrm{d}\hat{v}_1}{\mathrm{d}t}\mp\sqrt{\frac{\hat{T}}{\rho\hat{\lambda}}}\frac{d\hat{\lambda}_1}{\mathrm{d}t}\Big)=\hat{\lambda}_1\Big(\frac{\mathrm{d}\hat{v}_2}{\mathrm{d}t}\mp\sqrt{\frac{\hat{T}}{\rho\hat{\lambda}}}\frac{\mathrm{d}\hat{\lambda}_2}{\mathrm{d}t}\Big),\ \text{for } \frac{\mathrm{d}X}{\mathrm{d}t}=\pm C_T=\pm\sqrt{\frac{\hat{T}}{\rho\hat{\lambda}}}. \tag{33}$$

where the lower and upper signs correspond to each other. These relations are also useful for the numerical integration of initial-boundary value problems for the system (25). Finally it is useful to note that when $E\to\infty$ and the Maxwellian constitutive equation reduces to the Kelvin-Voigt model, used in [8], [23], [24], then the longitudinal wave speeds (27) become infinite and the governing PDEs system becomes parabolic-hyperbolic. This remark shows the advantage of our approach in studying wave propagation phenomena.

4.2. *First order discontinuities in rate-type strings.*

A first order discontinuity is a smooth curve $X=S(t)$ with the property that the motion $\overrightarrow{\chi}(X,t)$ is continuous across it, but at least one of the quantities $\overrightarrow{v}$, $\overrightarrow{\lambda}$ and T has a jump across this curve, being smooth functions everywhere else. In order to determine the first order discontinuities propagating in a rate-type string we have to take into account jump relations (3), (6) and the constitutive equation (13). The rate-type relation (13) defines a *linear instantaneous response curve in tension and stretch* relative to a given state (λ^+,T^+). This curve contains all states (λ,T) which can be reached by an instantaneous process from (λ^+,T^+) and is characterized by the differential relation $\mathrm{d}T=E\mathrm{d}\lambda$. An instantaneous process is a process which takes place in $\Delta t=0$ time, the jump across a propagating discontinuity being exactly such a process. Consequently, if the tension jumps at a state (λ^+,T^+) from T^+ to T^- then the corresponding stretch λ^- has to satisfy $T^-=E(\lambda^--\lambda^+)+\lambda^+$. Therefore, for any discontinuity in tension the following jump relation has to be satisfied

$$[T]=E[\lambda] \tag{34}$$

We distinguish the following main types of discontinuities in a rate-type string:

1) *Discontinuity with continuous tangent*, i.e $[\overrightarrow{\tau}]=0$. Relations (3) and (6) can be written as $[\overrightarrow{v}]+\dot{S}\overrightarrow{\tau}[\lambda]=0$ and $\rho\dot{S}[\overrightarrow{v}]+[T]\overrightarrow{\tau}=0$ wherefrom we get $\rho\dot{S}^2=\frac{[T]}{[\lambda]}$. According to (34) we obtain the speed of propagation of the discontinuity as

$$\dot{S}(t)=C_L=\pm\sqrt{\frac{E}{\rho}}. \tag{35}$$

This propagating discontinuity is called *longitudinal shock wave* since it produces a jump in tension, stretch and velocity, but does not change the shape of the string.

2) *Discontinuity with discontinuous tangent*, i.e. $[\overrightarrow{\tau}]\neq 0$. By eliminating the jump $[\overrightarrow{v}]$ from relations (3) and (6) we get $[T\overrightarrow{\tau}]-\rho\dot{S}^2[\lambda\overrightarrow{\tau}]=0$. Taking the inner product of this relation with $<\overrightarrow{\tau}>=\frac{1}{2}(\overrightarrow{\tau}^++\overrightarrow{\tau}^-)$, $\overrightarrow{\tau}^+\neq-\overrightarrow{\tau}^-$ (i.e. when there is *no cusp*) we obtain $(1+\overrightarrow{\tau}^+\cdot\overrightarrow{\tau}^-)(\rho\dot{S}^2[\lambda]-[T])=0$, while by taking the inner product with $[\overrightarrow{\tau}]$ we obtain $(1-\tau^+\cdot\tau^-)(\rho\dot{S}^2<\lambda>-<T>)=0$, where $<a>\equiv\frac{1}{2}(a^++a^-)$. One gets the following two conditions,

$$\rho\dot{S}^2[\lambda]=[T]\quad\text{and}\quad\rho\dot{S}^2<\lambda>=<T>. \tag{36}$$

Relations (36) are satisfied for *continuous stretch* and *continuous scalar value of tension* across the discontinuity, i.e., $\lambda^+=\lambda^-=\lambda$ and $T^+=T^-=T$. Such a discontinuity propagates with the speed

$$\dot{S}(t) = C_T(\lambda, T) = \pm\sqrt{\frac{T}{\rho\lambda}}. \tag{37}$$

It is called *transverse shock wave* since it affects only the shape of the string and the velocity, but not the tension nor the strain.

A second possibility to satisfy relations (36) could appear when the stretch and the tension might suffer a jump and the speed of propagation of the discontinuity is given by $\rho\dot{S}^2 = \frac{[T]}{[\lambda]} = \frac{T^+}{\lambda^+} = \frac{T^-}{\lambda^-}$. According to inequality (17) imposed by the second law of thermodynamics on the equilibrium curve and relation (34) such a situation is not possible for a rate-type string.

3) *Material discontinuity* (or *stationary discontinuity*) $\frac{\mathrm{d}X}{\mathrm{d}t} = \dot{S} = 0$. According to jump relations (3) and (6) we have $[\overrightarrow{v}] = 0$ and $[\overrightarrow{T}] = 0$. Since $[T^2] = [\overrightarrow{T}\cdot\overrightarrow{T}] = 2 < \overrightarrow{T} > \cdot[\overrightarrow{T}] = 2 < T > [T] = 0$ it follows that $[T] = 0$. Indeed, if $< T >= 0$ then $T^+ = -T^-$ which means that one part of the string is in compression and the other one is in tension which is not allowed for strings. Therefore, across a material discontinuity only the stretch can jump, i.e.

$$[T] = 0, \quad [\overrightarrow{v}] = 0, \quad [\overrightarrow{\tau}] = 0, \quad [\lambda] \neq 0. \tag{38}$$

It is interesting to note here that for the rate-type system (25) the directions of the first order discontinuities always lie through the characteristic curves (acceleration waves) of the system. This behavior is due to the fact that all the characteristic fields of the rate-type system are linearly degenerate.

4.3. *Jump relations along first order discontinuities*

We are mainly interested to determine the variation of the jump of stretch, tension, tangent vector, velocity and of their derivatives along the discontinuity directions of the rate type system. We have to use, on one side Hadamard's lemma (2) for $\overrightarrow{v}$, $\overrightarrow{\lambda}$ and T,

$$\Big[\frac{\partial\overrightarrow{v}}{\partial t}\Big] + \dot{S}\Big[\frac{\partial\overrightarrow{v}}{\partial X}\Big] = \frac{d}{dt}[\overrightarrow{v}], \quad \Big[\frac{\partial\overrightarrow{\lambda}}{\partial t}\Big] + \dot{S}\Big[\frac{\partial\overrightarrow{\lambda}}{\partial X}\Big] = \frac{d}{dt}[\overrightarrow{\lambda}], \quad \Big[\frac{\partial T}{\partial t}\Big] + \dot{S}\Big[\frac{\partial T}{\partial X}\Big] = \frac{d}{dt}[T]. \tag{39}$$

and on the other side the jump relations which follow from (25) written on each side of the discontinuity $X = S(t)$

$$\Big[\frac{\partial\overrightarrow{\lambda}}{\partial t}\Big] = \Big[\frac{\partial\overrightarrow{v}}{\partial X}\Big], \quad \Big[\frac{\partial\overrightarrow{v}}{\partial t}\Big] = \Big[\frac{1}{\rho}\frac{\overrightarrow{\lambda}}{\lambda}\frac{\partial T}{\partial X}\Big] + \Big[\frac{T}{\rho\lambda}\Big(\mathbf{1} - \frac{\overrightarrow{\lambda}\otimes\overrightarrow{\lambda}}{\lambda^2}\Big)\frac{\partial\overrightarrow{\lambda}}{\partial X}\Big], \quad \Big[\frac{\partial T}{\partial t}\Big] - E\Big[\frac{\partial\lambda}{\partial t}\Big] = -\frac{E}{\mu}([T] - [T_{eq}(\lambda)]). \tag{40}$$

One may consider jump relations (39) and (40) as an algebraic *non-linear* system consisting of six equations for the unknowns $\Big[\frac{\partial\overrightarrow{\lambda}}{\partial t}\Big]$, $\Big[\frac{\partial\overrightarrow{\lambda}}{\partial X}\Big]$, $\Big[\frac{\partial\overrightarrow{v}}{\partial t}\Big]$, $\Big[\frac{\partial\overrightarrow{v}}{\partial X}\Big]$, $\Big[\frac{\partial T}{\partial t}\Big]$, $\Big[\frac{\partial T}{\partial X}\Big]$ where $\dot{S}$ is equal to 0, or to $\pm\sqrt{E/\rho}$, or to $\pm\sqrt{T/\rho\lambda}$. Hence, we get that the above system has solutions if and only if the following relations are satisfied:

$$\frac{\mathrm{d}[T]}{\mathrm{d}t} - E\frac{\mathrm{d}[\lambda]}{\mathrm{d}t} = -\frac{E}{\mu}([T] - [T_{eq}(\lambda)]), \quad \text{for } \dot{S} = \frac{\mathrm{d}X}{\mathrm{d}t} = 0,\ t > 0, \tag{41}$$

$$\frac{\mathrm{d}[T]}{\mathrm{d}t} \mp \sqrt{\rho E}\frac{\overrightarrow{\lambda}}{\lambda}\cdot\frac{\mathrm{d}[\overrightarrow{v}]}{\mathrm{d}t} = -\frac{E}{\mu}([T] - [T_{eq}(\lambda)]), \quad \text{for } \dot{S} = \frac{\mathrm{d}X}{\mathrm{d}t} = \pm\sqrt{\frac{E}{\rho}},\ t > 0, \tag{42}$$

$$\frac{\mathrm{d}[\overrightarrow{v}]}{\mathrm{d}t} \mp \sqrt{\frac{T}{\rho\lambda}}\frac{\mathrm{d}[\overrightarrow{\lambda}]}{\mathrm{d}t} = \Big[\Big(\frac{1}{\rho}\frac{\partial T}{\partial X} - \frac{T}{\rho\lambda}\frac{\partial\lambda}{\partial X}\Big)\overrightarrow{\tau}\Big], \text{ for } \dot{S} = \frac{\mathrm{d}X}{\mathrm{d}t} = \pm\sqrt{\frac{T}{\rho\lambda}},\ t > 0. \tag{43}$$

Let us note that relations (41)-(43) are similar to the relations along the characteristics for the rate-type system (25). Now, by using the corresponding first order jump relations in (41)-(43) one obtains more detailed results.

a) Along a *material discontinuity* $\dot{S} = \frac{\mathrm{d}X}{\mathrm{d}t} = 0$, we get by using the accompanying jump relations (38) that the evolution of the jump of stretch is governed by the differential equation

$$\frac{\mathrm{d}[\lambda](t)}{\mathrm{d}t} = -\frac{1}{\mu}\big[T_{eq}(\lambda)\big]. \tag{44}$$

The evolution of the jumps of the first order derivatives are expressed in terms of $[\lambda]$ and $[\frac{\partial\lambda}{\partial X}]$ as follows

$$\left[\frac{\partial T}{\partial t}\right]=0,\ \left[\frac{\partial T}{\partial X}\right]=0,\qquad \left[\frac{\partial \overrightarrow{\tau}}{\partial t}\right]=0,\ \left[\frac{\partial \overrightarrow{\tau}}{\partial X}\right]=0,\qquad \left[\frac{\partial \overrightarrow{v}}{\partial t}\right]=0,$$
$$\left[\frac{\partial \overrightarrow{v}}{\partial X}\right]=\left[\frac{\partial \overrightarrow{\lambda}}{\partial t}\right]=\frac{\mathrm{d}[\lambda]}{\mathrm{d}t}\overrightarrow{\tau}+[\lambda]\frac{\partial \overrightarrow{\tau}}{\partial t},\qquad \left[\frac{\partial \overrightarrow{\lambda}}{\partial X}\right]=\left[\frac{\partial \lambda}{\partial X}\right]\overrightarrow{\tau}+[\lambda]\frac{\partial \overrightarrow{\tau}}{\partial X}\ \text{ and }\ \left[\lambda\frac{\partial \lambda}{\partial X}\right]=0. \tag{45}$$

b) Along a *longitudinal shock wave* $\dot{S}=\frac{\mathrm{d}X}{\mathrm{d}t}=\pm\sqrt{\frac{E}{\rho}}$ by using the accompanying jump relations $[\overrightarrow{v}]=-\dot{S}[\lambda]\overrightarrow{\tau}$, $[T]=E[\lambda]$ and $[\overrightarrow{\lambda}]=[\lambda]\overrightarrow{\tau}$ we obtain the differential equation which governs the variation of the stretch jump

$$\frac{\mathrm{d}[\lambda](t)}{\mathrm{d}t}=-\frac{1}{2\mu}\big(E[\lambda]-[T_{eq}(\lambda)]\big). \tag{46}$$

Moreover, along this discontinuity the jumps of the first order derivatives expressed in terms of $[\lambda]$ and $[\frac{\partial\lambda}{\partial X}]$ are given by

$$\left[\frac{\partial T}{\partial t}\right]=3E\frac{\mathrm{d}[\lambda]}{\mathrm{d}t}-E\dot{S}\left[\frac{\partial \lambda}{\partial X}\right],\quad \left[\frac{\partial T}{\partial X}\right]=-2\rho\dot{S}\frac{\mathrm{d}[\lambda]}{\mathrm{d}t}+E\left[\frac{\partial \lambda}{\partial X}\right] \tag{47}$$
$$\left[\frac{\partial \overrightarrow{v}}{\partial X}\right]=\left[\frac{\partial \overrightarrow{\lambda}}{\partial t}\right]=\frac{\mathrm{d}[\overrightarrow{\lambda}]}{\mathrm{d}t}-\dot{S}\left[\frac{\partial \overrightarrow{\lambda}}{\partial X}\right],\quad \left[\frac{\partial \overrightarrow{v}}{\partial t}\right]=-2\dot{S}\frac{\mathrm{d}[\overrightarrow{\lambda}]}{\mathrm{d}t}+\frac{E}{\rho}\left[\frac{\partial \overrightarrow{\lambda}}{\partial X}\right],$$

c) Along a *transverse shock wave* $\dot{S}=\frac{\mathrm{d}X}{\mathrm{d}t}=\pm\sqrt{\frac{T}{\rho\lambda}}$, we obtain by using the accompanying jump relations, i.e. $[\lambda]=0$, $[T]=0$, $[\overrightarrow{v}]=-\dot{S}\lambda[\overrightarrow{\tau}]$ that the jump of the tangent vector satisfies the differential relation

$$2\dot{S}\frac{\mathrm{d}[\overrightarrow{\lambda}]}{\mathrm{d}t}+\frac{\mathrm{d}\dot{S}}{\mathrm{d}t}[\overrightarrow{\lambda}]+\frac{1}{\rho\lambda}\Big(\frac{\partial T}{\partial X}-E\frac{\partial\lambda}{\partial X}\Big)[\overrightarrow{\lambda}]+\Big(\frac{T}{\rho\lambda}-\frac{E}{\rho}\Big)\left[\frac{\partial\lambda}{\partial X}\overrightarrow{\lambda}\right]=0. \tag{48}$$

Moreover, across this discontinuity the following jump relations are satisfied

$$\left[\frac{\partial T}{\partial t}\right]=-E\dot{S}\left[\frac{\partial \lambda}{\partial X}\right],\quad \left[\frac{\partial T}{\partial X}\right]=E\left[\frac{\partial \lambda}{\partial X}\right],\qquad \left[\frac{\partial \lambda}{\partial t}\right]=-\dot{S}\left[\frac{\partial \lambda}{\partial X}\right] \tag{49}$$
$$\left[\frac{\partial \overrightarrow{v}}{\partial t}\right]=\frac{1}{\rho}\Big(\frac{\partial T}{\partial X}-E\frac{\partial\lambda}{\partial X}\Big)[\overrightarrow{\tau}]+\frac{E}{\rho}\left[\frac{\partial\lambda}{\partial X}\overrightarrow{\tau}\right]+\frac{T}{\rho\lambda}\left[\frac{\partial \overrightarrow{\lambda}}{\partial X}\right],\quad \left[\frac{\partial \overrightarrow{v}}{\partial X}\right]=-\frac{1}{\dot{S}}\frac{d\dot{S}}{dt}[\overrightarrow{\lambda}]-\frac{d[\overrightarrow{\lambda}]}{dt}-\frac{1}{\dot{S}}\left[\frac{\partial \overrightarrow{v}}{\partial t}\right]. \tag{50}$$

4.4. *Growth and decay of waves - consequences on phase nucleation*

In the following let us consider as equilibrium curve for the rate-type constitutive equation the trilinear material described by (14). We can obtain explicit results concerning the influence of the slope of the equilibrium curve on the growth, or the decay of the amplitude of jumps along each possible first order discontinuity.

Growth and decay behavior along a material discontinuity. Suppose that the limit values of the stretch relative to a stationary discontinuity at $X=X_0$, i.e. $\lambda(X_0+0,t)=\lambda^+(t)$ and $\lambda(X_0-0,t)=\lambda^-(t)$, lie either in the austenite phase $\mathscr{A}$, or in the martensite phase $\mathscr{M}$, or in the unstable phase $\mathscr{I}$, for $t\in[0,t_0)$, $t_0>0$. Then equation (44) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[\lambda](t)=-\frac{1}{\mu}E_0[\lambda],\quad t\in[0,t_0). \tag{51}$$

and the solution is

$$[\lambda](t)=[\lambda](0)\exp\Big(-\frac{E_0}{\mu}t\Big),\quad t\in[0,t_0). \tag{52}$$

where $E_0=E_1>0$ if the limit values for the stretch are in the $\mathscr{A}$-phase, $E_0=E_3>0$ if they are in the $\mathscr{M}$-phase and $E_0=-E_2<0$ if they are in the unstable $\mathscr{I}$-phase.

Therefore, in the austenite phase $\mathscr{A}$, or in the martensite phase $\mathscr{M}$, the initial amplitude of the jump decays exponentially along the discontinuity, while in the unstable phase $\mathscr{I}$ any initial infinitesimal jump in stretch increases exponentially until one of the limit values $\lambda^{\pm}(t)$ leaves the interval (λ_a,λ_m).

We may consider the initial jump of stretch as a stress induced inhomogeneity, or initial defect, or grain boundary in the shape memory string. Relation (52) then explains how the nucleation appears and develops when particles of the string enter in the unstable interval (λ_a, λ_m) leading finally to the formation of *stationary phase boundaries* which separate stable phases of the material. The time $\frac{\mu}{E_2}$ should be connected with the time of growth, or time of nucleation of microscopic theories of phase transitions. Let us note that the larger is the slope E_0, or the smaller is the viscosity coefficient μ, then the larger is the rate of growth or decay of the initial inhomogeneity.

The above result shows that the rate type model possesses its own kinetics due to the dissipative mechanism incorporated with no need of a separate nucleation criterion. Therefore, it has the capacity to describe the localization of stretch, that is phase transition phenomena. It is also useful to note from (45) that the jump of the velocity gradient along this stationary discontinuity also increases or decreases exponentially depending on the slope of the equilibrium curve.

Decay behavior along a longitudinal shock wave. Let us suppose again the limit values of the stretch lie either in the austenite phase $\mathscr{A}$, or in the martensite phase $\mathscr{M}$, or in the unstable phase $\mathscr{I}$. Then according to (46) the variation of the jump of stretch along the discontinuities $\dot{S} = \pm\sqrt{E/\rho}$ is governed by the differential equation

$$\frac{d}{dt}[\lambda](t) = -\frac{(E-E_0)}{2\mu}[\lambda], \quad t \in [0, t_0). \tag{53}$$

and the solution is

$$[\lambda](t) = [\lambda](0)\exp\Big(-\frac{(E-E_0)}{2\mu}t\Big), \quad t \in [0, t_0). \tag{54}$$

where $E_0 = E_1 > 0$ when the limit values for the stretch are in the $\mathscr{A}$-phase, $E_0 = E_3 > 0$ when they are in the $\mathscr{M}$-phase and $E_0 = -E_2 < 0$ when they are in the $\mathscr{I}$-phase.

Restriction (17) imposed by the second law of thermodynamics requires $E > E_0$, then the jump of stretch always decays exponentially along this propagating discontinuity irrespective of the slope of the equilibrium curve. Thus, unlike the material discontinuities the longitudinal shock wave discontinuities do not promote stretch localization.

This result also shows that near the source of a disturbance the waves propagating with the speed $\pm\sqrt{\frac{E}{\rho}}$ (also called instantaneous waves) are exponentially damped. A natural question is: what happens with the main part of the disturbance far from the source? The answer is given in Făciu and Molinari [22] for rate-type phase transforming bars by using perturbation methods. It is shown there that far from the source the disturbance moves with the slower speeds $\pm\sqrt{\frac{E_0}{\rho}}$ (called delayed waves), where E_0 is the elastic modulus of the equilibrium curve in a stable phase. The amplitude of the disturbances far from their sources (large t) decreases like $1/\sqrt{at}$ and spreads out like $\sqrt{at}$ where $a = \frac{(E-E_0)\mu}{2\rho E}$ which is typical for a diffusion process (see also Whitham [25]).

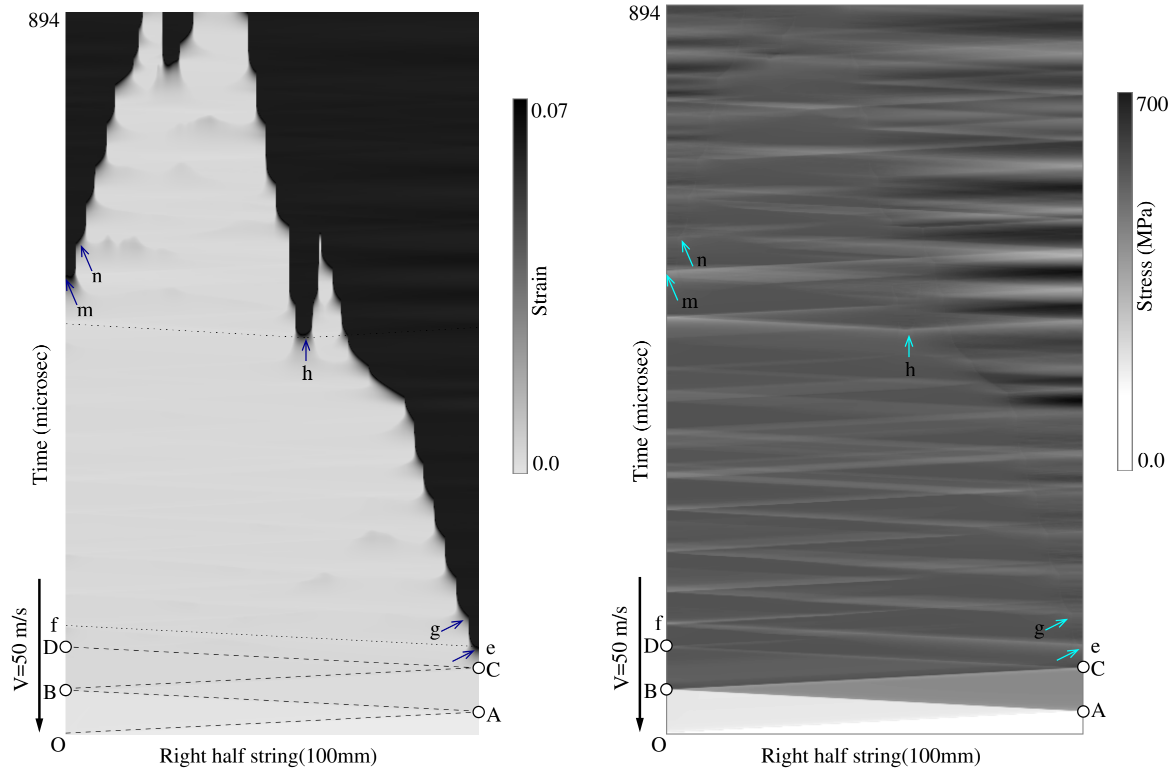
Since the *transverse shock waves* affect only the shape of the string, but do not produce elongation they do not influence the process of phase transformation.

5. Shape memory alloy string used to brake a fast moving body

In order to illustrate the phase nucleation phenomena which accompany the wave propagation in a SMA string we investigate numerically the impact of a string at its midpoint by a body of mass M which moves with a constant velocity V transverse to the string. In this case the body will be braked by the stress developed in the string. This experiment allows us to test the damping properties of a phase transforming string, too.

To this end we consider a SMA finite string of length $2L = 200.$ mm, slightly stretched, rectilinear and at rest at the initial moment, and directed along the Ox axis. Both ends of the string are fixed. Since the impact velocity is normal to the string at its mid point the motion always takes place in the plane formed by the initial string and the impact velocity $-V\overrightarrow{i}_2$. On the other hand, in the X-t plane the problem is completely symmetric with respect to $X = 0$ and the impacting body will move along the Oy axis. We only analyze the motion of the right half of the string. In order to investigate the damping effect of the model we assume the body remains attached to the string after impact and oscillates with it. According to Cristescu [1, Chap. IV], who investigated the same problem for an elastic extensible cable, the boundary condition at $X = 0$ has to be a dynamic boundary condition. Thus from the law of conservation of momentum the following condition has to be verified

$$M\frac{\mathrm{d}\overrightarrow{v}(0,t)}{\mathrm{d}t} = AT(0,t)\overrightarrow{\tau}(0,t), \quad t > 0, \tag{55}$$



a) b)

Fig. 2. Lagrangian time-space diagrams. a) Stretch distribution in the string after the impact with the projectile. Low stretch area (light grey) corresponds to the $\mathscr{A}$ phase, while high stretch area (dark grey) corresponds to the $\mathscr{M}$ phase. b) Tension distribution and stress wave propagation.

where A is the area of the transverse section of the string. By projecting this relation on the Oy axis we get the differential boundary condition

$$M\frac{\mathrm{d}v_2(0,t)}{\mathrm{d}t} = 2AT(0,t)\frac{\lambda_2(0,t)}{\lambda(0,t)}, \quad t>0. \tag{56}$$

The numerical values used for the mechanical constants characterizing the equilibrium curve (14) are: $E_1 = E_3 = 90.$ GPa, $E_2 = 3.3$ GPa, $\lambda_a = 1.005$, $\lambda_m = 1.065$, $T_a = 450.$ MPa, $T_m = 250.$ MPa, $\rho = 6500.$ kg/m^3. The kinetic parameters which enter the rate type constitutive equation are: $E = 90.1$ GPa and $\mu = 10^{-3}$ MPa$\times$s. We choose the dynamic Young modulus E very close to the elastic moduli of the stable phases such that $C_L = \sqrt{E/\rho} \cong \sqrt{E_1/\rho}$.

The predictions of the rate-type Maxwell model for an impact experiment between a string having the area of the transverse section $A = 1$ mm with a projectile having the initial velocity $V = 50$ m/s and the mass $M = 5$ g are illustrated in the following.

The numerical solution is obtained by using a second order accuracy fractional-step method. We split our hyperbolic relaxation system with stiff source terms (25) into two subproblems. In a first step we consider the hyperbolic homogeneous part of the system and we use a Godunov's type method with a second order resolution correction. In a second step we consider a simple ordinary differential equation system containing the source term which is solved by using a simple explicit second-order accurate two-stage method (see LeVeque [26, Chap. 17]). For our problem we used the time and space steps $\Delta t = 0.021$ μs and $\Delta X = 0.083$ mm (1200 nods for the half string) which satisfy the CFL condition $C_L\frac{\Delta t}{\Delta X} \leq 1$ where $C_L = \sqrt{E/\rho} = 3723$ m/s.

After impact a loading longitudinal shock wave, marked with O-A in Figs. 2 and Figs. 5, starts to propagate in the right half string followed by a transverse wave O-X moving with the velocity $C_T(1.00192, 173) \approx 163$ m/s. The longitudinal wave is reflected at the fixed end as a loading wave and interacts at X with the transverse wave being transmitted across it and generating two faster transverse waves: X-Y propagating left and X-Z propagating right with the velocity $C_T(1.0047, 427) \approx 255$ m/s. The intersection at B(53.7 μs) of the two longitudinal waves: AB and

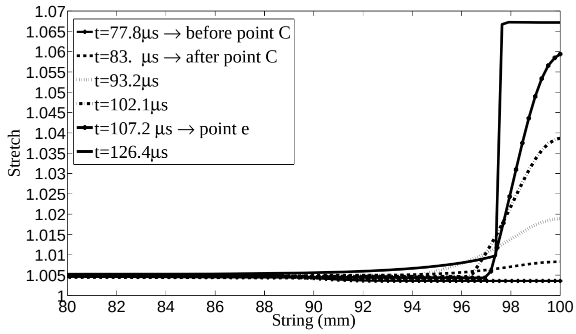
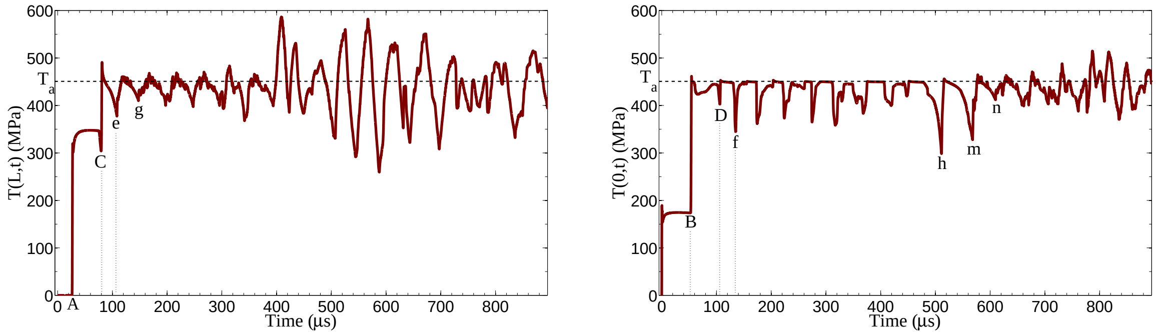


Fig. 3. First nucleation and phase transformation near the right end of the string.



a) b)

Fig. 4. a) Tension history at the right end of the string. b) Tension history at the impacted point of the string.

the symmetric one coming from the left part of the string generates a new loading longitudinal wave B-C which is reflected again at the fixed end as a loading wave C-D. After this moment, $t = 80.57\ \mu$s, the stretch in the back of this wave near point C overcomes $\lambda_a = 1.005$ ($T = 476$ MPa$> T_a$) (see Fig. 3). In other words, the values of stretch near the fixed end, due to the successive reflections of longitudinal loading shock waves enters the unstable interval (λ_a, λ_m). Consequently a nucleation process starts to develop. This evolution is illustrated in Fig. 3. The corresponding zone in the Lagrangian time-space diagrams is marked with e. Other nucleation zones, are also indicated in the figures with a small arrow and small letters g, h, m, n.

One observes, both in Fig. 2a and Fig. 3, how this nucleation process expands in the time interval $(77.8\ \mu s, 126.4\ \mu s)$, when particles of the material traverse the unstable interval, and finally leads to the formation of a stationary phase boundary. This behavior is in perfect agreement with the blow-up of stretch inhomogeneities described by relation (52). Moreover, we observe from the numerical results in Fig. 2a that, in general, each nucleation event is followed by the formation of a stationary phase boundary and the propagation of the phase transformation fronts progresses by a go-and-stop type mechanism. The transformation advances converting regions of low stretch (austenite phase) in regions of high stretch (martensite phase) until the entire string is completely transformed in phase $\mathscr{M}$.

It is well known that nucleation phenomena are dynamic local events. Figs. 2 and Figs. 5 illustrate how our constitutive approach has the capacity to describe such aspects. Indeed, let us note that at the end of the nucleation process, which mainly takes place between point C and point e, a disturbance e-f propagating in the austenite string, parallel with C-D and A-B, is emitted. It is visible in Fig. 2b and Fig. 5a. It is obvious that this wave can not be generated by the reflection at the right end of an elastic longitudinal shock wave coming from the impacted end. Indeed, the elastic pulse reflected at C has to return at the right end after a round trip in the right half of the string, i.e. after 53.7μ s while the time interval between C and e is around 26 μs. Therefore, the only source of this disturbance propagating left in the austenite string is the dynamic nucleation event. No propagating disturbance would be expected at this time if a dynamic event accompanying phase transformation would not occur.

The emission of longitudinal elastic shock waves as an accompanying phenomenon of phase nucleation can be also

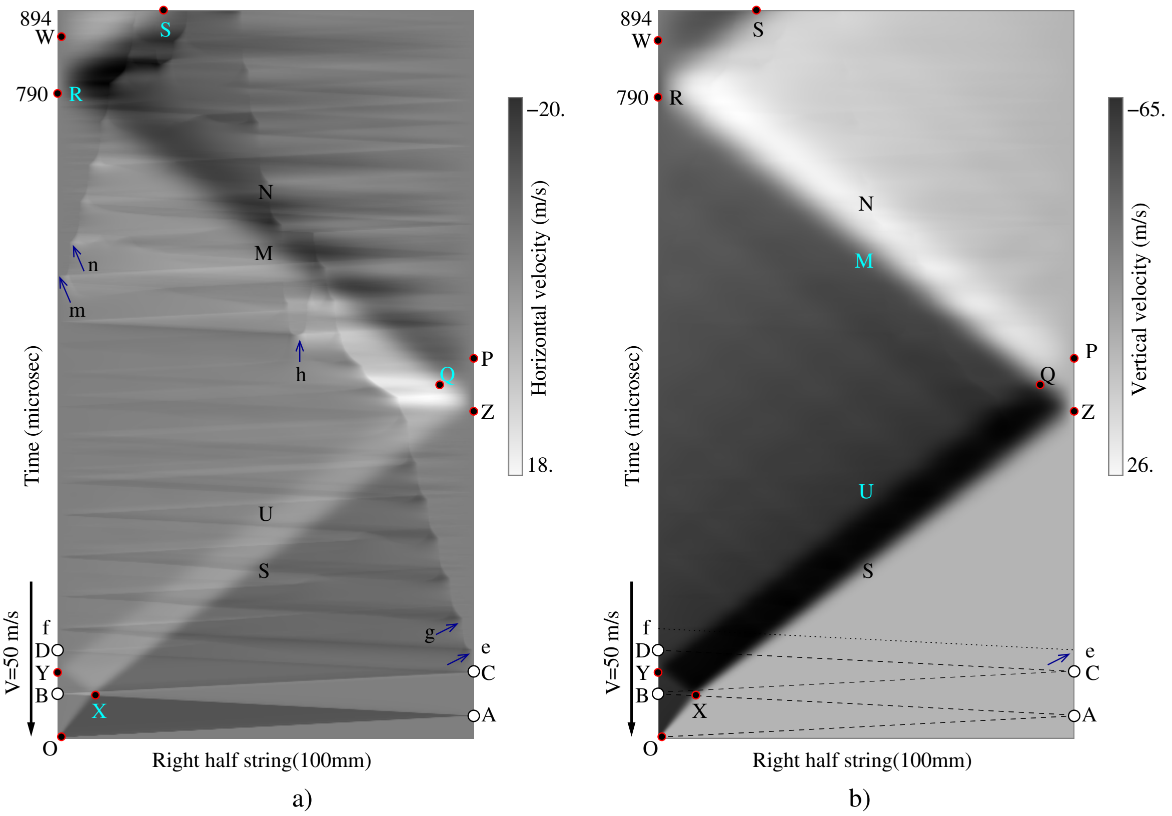


Fig. 5. Lagrangian time-space diagrams. a) Horizontal velocity and b) Vertical velocity distribution in the string after the impact with the projectile.

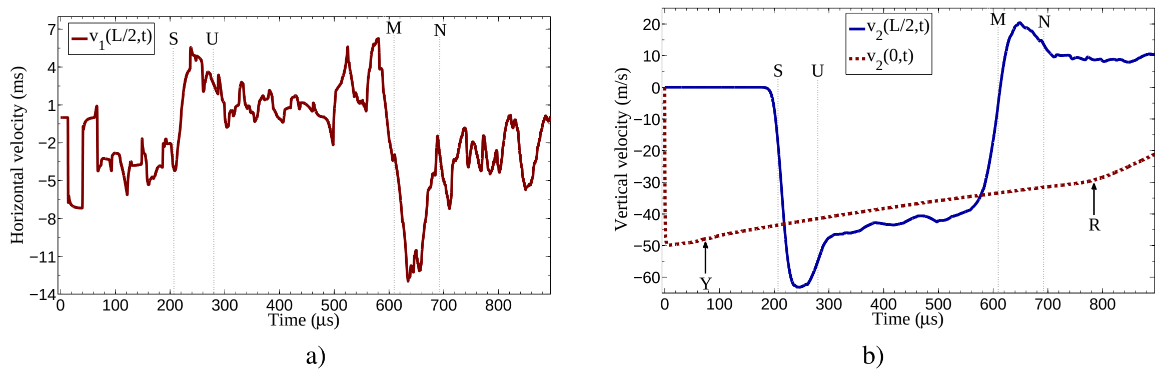


Fig. 6. a) Horizontal velocity history at 50 mm. b) Vertical velocity history at the impact point (braking) and at 50 mm.

observed in Fig. 2b and Fig. 5a at the points denoted by h, m, and n. That behavior is similar with the spontaneous emission of acoustic waves which are reported for material exhibiting instability and localization phenomena even in quasistatic loading conditions. Acoustic emissions during phase transformation in CuZnAl alloy are reported in [27].

Let us note that the nucleation of the martensite phase at point e manifests itself by a significant drop of the tension (see for instance in Fig. 4a the time interval between point C and point e). Therefore the propagating disturbance e-f will be an unloading wave. The same phenomenon happens with the disturbances emitted at the points g, h, m and n. The drops of tension induced by the corresponding nucleation events at the fixed end or at the impacted end are marked with the same letters on Fig. 4a and Fig. 4b. The stress wave propagation picture is presented in Fig. 2b. It shows how the longitudinal shock waves generated by reflection at the end points or by the nucleation phenomena propagate, being successively reflected and transmitted when crossing phase boundaries. These interactions generate large stress oscillations around the transformation stress T_a at the impacted point (Fig. 4b) and at the fixed ends of the string (Fig. 4a).

Rapid and large motions occur in the string during the braking of the high speed moving projectile. They are accompanied by the propagation of transverse wave fronts across which the shape of the string and the velocity register very large variations. The presence of these waves can be observed in Figs. 5 where the spatial-temporal behavior of the two components of the velocity is illustrated. For example, across the transverse wave X-Z the vertical velocity strongly decreases to a value lower than the impact velocity (see for instance point S in Fig. 5b and Fig. 6b). The lowest negative value of the vertical velocity is attained just above point X, i.e -65 m/s. This behavior is due to the interaction between the transverse wave OX and the longitudinal wave AX, and it can be proved theoretically in the frame of the linear elastic theory by solving corresponding Riemann problems. The transverse wave Y-P, almost parallel with X-Z, is induced by the reflection of the transverse wave X-Y at the impact point. Across it the vertical velocity increases (see for instance point U in Fig. 5b and Fig. 6b). The vertical velocity slightly decreases in absolute value between U and M due to the braking effect. Owing to the reflection of the transverse wave X-Z the vertical component changes its sign from negative to positive across Z-R (see for instance point M in Fig. 5b and Fig. 6b). The largest positive value of the vertical velocity is attained near point R, i.e 26 m/s. The reflection of the transverse wave D-P at the right end of the string induces a transverse front P-W across which the velocity slightly decreases (point N). It is useful to note that the sudden changes in curvature observed on the graph of $v_2(0,t)$ in Fig. 7 (see also Fig. 6b) correspond to the arrival and reflection at the impacted point of a transverse wave. That can be recognized at the point R on these figures and on Fig. 5.

During the whole damping process the vertical velocity suffers several times large variations alternating from negative values to positive values. The horizontal velocity also possesses a considerably large variation, much smaller however than that of the vertical one.

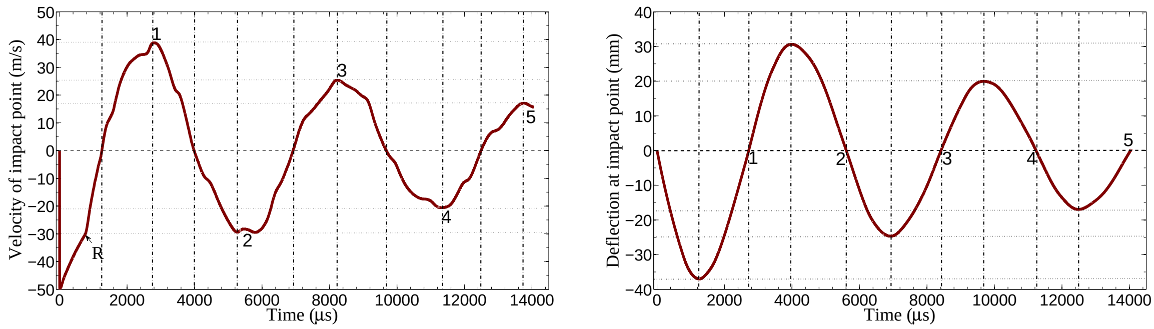


Fig. 7. Projectile braking. Long time behavior of the vertical velocity $v_2(0,t)$ and corresponding vertical displacement of the impact point.

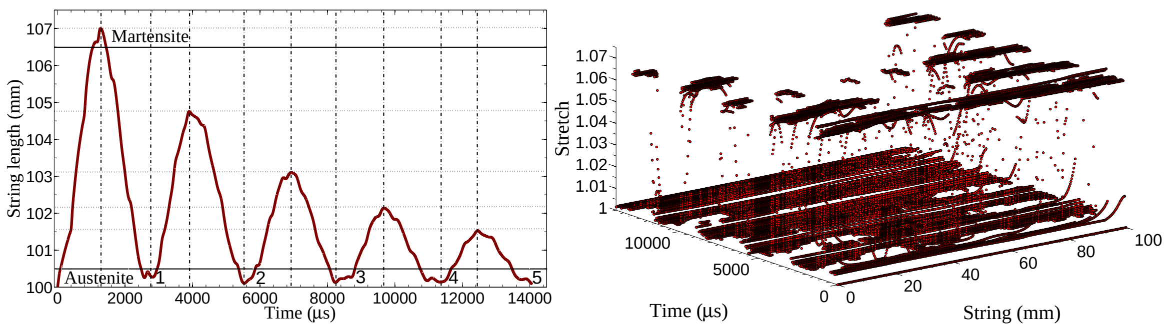


Fig. 8. Evolution of the length of the string. Long time behavior of the phase transformation process in the string.

A similar analysis can be pursued for the horizontal velocity using Fig. 5a and Fig. 6a. One observes that due to the reflection of the transverse waves X-Z and Y-P the horizontal component changes from positive values to negative ones. The largest positive value is attained near point Z, i.e 17.5 m/s and the lowest negative value is attained in the neighborhood of point R, i.e. -19.5 m/s. Let us note that the frequent oscillations recorded on the graph of the horizontal velocity in Fig. 6a are due to the longitudinal stress waves traveling in the string.

The strong damping effect of the SMA alloy string is due to the pseudo-elastic hysteretic behavior of the material, i.e. to the direct and reverse $\mathscr{A} \leftrightharpoons \mathscr{M}$ phase transformation processes which accompany the front waves propagation. It is put into evidence in Figs. 7 and Figs. 8. One can see the evolution of the velocity at the impact point, its oscillatory displacement, the variation of the length of the string as well as the evolution of the phases in the string. After impact the transformation fronts start to propagate near the string ends and later from the impact point (see Fig. 2a and Fig. 8) until the two fronts meet and the string is completely transformed. In the mean time the velocity of the body in absolute value decreases until this is completely stopped. Around this moment the string attains a maximal length of 107.2 mm, a maximal amplitude at the impact point (-40 mm), undergoing a complete $\mathscr{A} \rightarrow \mathscr{M}$ transformation. Afterwards the whole string, together with the projectile, will move in the opposite direction acting as a catapult. The positive vertical velocity increases until 39 m/s (which represents 78% from the impact velocity,) and the string completely returns in the $\mathscr{A}$ phase, nearly recovering its initial length (point 1). Further on, the damping process continues in a repetitive way. For instance, between points 1 and 2 (Fig. 7 and Fig. 8), the velocity of the projectile starts to decrease from a positive to a negative value. At the moment when the velocity of the projectile is zero, the length of the string reaches a maximum value less than 106.5 mm, which corresponds to an incomplete $\mathscr{A} \rightarrow \mathscr{M}$ transformation of the string. At point 2 the string recovers again completely the $\mathscr{A}$ phase. Similar things happen between points 2 and 3, 3 and 4, 4 and 5.

Let us note that points 1 to 5 correspond to local maxima or minima of the projectile velocity and to minimum length of the deformed string. Thus, at these time moments the string is completely in the $\mathscr{A}$ phase. The maximal values of the string length correspond to the moment when the projectile stops and the partial $\mathscr{A} \rightarrow \mathscr{M}$ transformation reaches its maximal value. The smaller is the maximal amplitude of the string length, the smaller is the amount of transformed martensite phase fraction in the string (see Fig. 8).

During the damping process the tension at the impact point exhibits significant variations due to the successive longitudinal wave fronts which cross this point (Fig. 9a). If we represent this tension value versus the length of the string (or stretch average) we can put into evidence a hysteretic behavior (Fig. 9b). The points 1 to 5 corresponds to those in Figs. 7 and 8. They also illustrate how the loop of the hysteresis diminishes as the phase transformation of the string is more and more incomplete.

The computation process ends at point 5 when in the string appears a state of compression and the tension becomes negative. In this case the PDEs system for a perfectly flexible string loses its hyperbolic character and it is no more able to describe the motion.

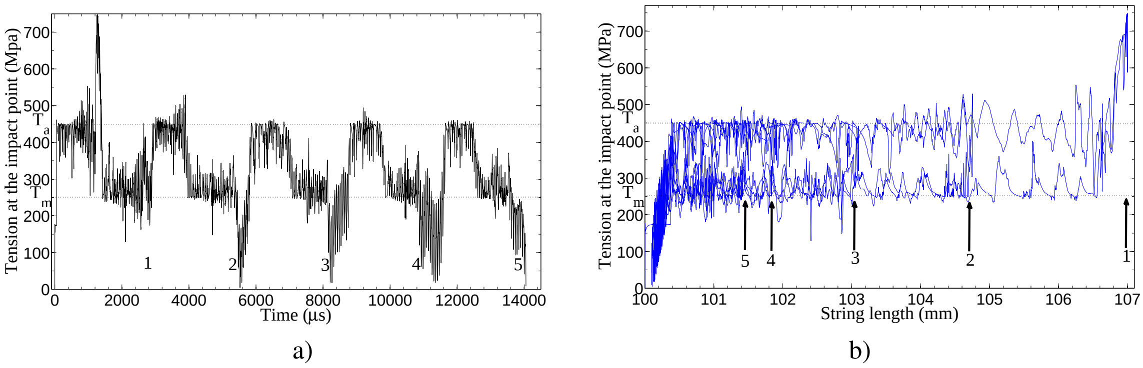


Fig. 9. Long time behavior of the tension at the impact point and its hysteretic representation with respect to the length of the string.

6. Concluding remarks

In this paper we have investigated the dynamic evolution of a phase transforming string. A Maxwellian rate-type model accounting for phase transformation phenomena has been used. The system of PDEs incorporates both geometrical nonlinearities, inherent to the description of perfectly flexible strings and physical nonlinearities corresponding to the non-monotone character of the equilibrium curve. The analysis of the first and second order discontinuities and of the growth/decay of propagating jumps put into evidence the capacity of the hyperbolic system to describe wave propagation and the ”spontaneous” nucleation of phases and their evolution. The transverse impact of a string with a projectile has been selected as an illustrative example. One captures both the propagation of phase boundaries and the change of shape produced by the longitudinal and transverse motion of the string. Numerical simulation reveals that each nucleation event is a local dynamic phenomenon which generates itself longitudinal waves. This behavior is in agreement with acoustic emissions measured in physical experiments on strain localization phenomena and their accompanying instabilities. The results presented here show that our approach has the capability to capture simultaneously local aspects related with the evolution of the microstructure in the SMA string and global facts of engineering interest as the damping of a fast moving body. These outcomes may have applications to the dynamics of smart structures.

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References

[1] N. Cristescu, Dynamic Plasticity, North-Holland Publishing Company, Amsterdam, 1967. (new version Dynamic Plasticity, World Scientific Publishing, New Jersey, 2007).

[2] M. Mihăilescu, I. Suliciu, Riemann and Goursat step data problems for extensible strings with non-convex stress-strain relation, Rev. Roumaine Math. Pures Appl. 20 (1975) 551–559,.

[3] R. Young, Nonstrictly hyperbolic waves in elasticity, J. Diff. Equations 188 (2003) 80–109.

[4] G. Rega, Nonlinear vibrations of suspended cables - Part I: Modeling and analysis. Part II: Deterministic phenomena. Appl. Mech. Rev. 57 (2004) 443–478, 479–514.

[5] K. P. Purohit, K. Bhattacharya, Dynamics of strings made of phase-transforming materials, J. Mech. Phys. Solids 51 (2003) 393–424.

[6] C. Truesdell, R. Toupin, The Classical Field Theories, Handbuch der Physik, **III/1**, Springer Verlag, Berlin-Heidelberg-New York, 1961.

[7] J. L. Ericksen, Equilibrium of bars, J. Elasticity 5 (1975) 191–201.

[8] R. D. James, The propagation of phase boundaries in elastic bars, Arch. Rational Mech. Anal. 73 (1980) 125–158.

[9] R. Abeyaratne, J. K. Knowles, On the dissipative response due to discontinuous strains in bars of unstable elastic materials, Int. J. Solids Structures 24 (1988) 1021–1044.

[10] R. Abeyaratne, S-J. Kim, J. K. Knowles, A one-dimensional continuum model for shape memory alloys, Int. J. Solids Structures 31 (1994) 2229–2249.

[11] J. A. Shaw, S. Kyriakides, On the nucleation and propagation of phase transformation fronts in a NiTi alloy 45 (1997) 683–700.

[12] L. Truskinovsky, Structure of isothermal phase jump J. Appl. Math. (PMM) 51 (1987) 777-784.

[13] R. Abeyaratne, J. K. Knowles, On the driving traction on a surface of a strain discontinuity in a continuum J. Mech. Phys. Solids 38, (1990) 345–360.

[14] M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, Arch. Rational Mech. Anal., 81 (1983) 301–315.

[15] S.-C. Ngan, L. Truskinovsky, Thermo-elastic aspects of dynamic nucleation, J. Mech. Phys. of Solids 50 (2002) 1193–1229.

[16] I. Suliciu, On the description of the dynamics of phase transitions by means of rate-type constitutive equation. A model problem, in: A. S. K. Khan, M. Tokuda (Eds.), Proceedings of Plasticity ’89, Pergamon Press, 1989 pp. 417–420.

[17] C. Făciu, M. Mihăilescu-Suliciu, The energy in one dimensional rate-type semilinear viscoelasticity, Int. J. Solids Structures 11 (1987) 1505–1520.

[18] I. Suliciu, On modeling phase transitions by means of rate-type constitutive equations. Shock wave structure, Int. J. Engng. Sci. 28 (1990) 829–841.

[19] I. Suliciu, Some stability-instability problems in phase transitions modeled by piecewise linear elastic or viscoelastic constitutive equations, Int. J. Engng. Sci. 30 (1992) 483–494.

[20] C. Făciu, I. Suliciu, A Maxwellian model for pseudoelastic materials, Scripta Metallurgica et Materialia 31 (1994) 1399–1404.

[21] C. Făciu, Initiation and growth of strain bands in rate-type viscoelastic materials. Part I: Discontinuous strain solutions. Part II: The energetics of the banding mechanism., European J. Mech. A/Solids 15, (1996) 969–988, 989–1011.

[22] C. Făciu, A. Molinari, On the longitudinal impact of two phase transforming bars. Elastic versus a rate-type approach. Part I: The elastic case. Part II: The rate-type case., Int. J. Solids Structures 43, (2006) 497–522, 523–550.

[23] R. L. Pego, Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability, Arch. Rational Mech. Anal. 97 (1987) 153–394.

[24] A. Vainchtein, P. Rosakis, Hysteresis and stick-slip motion of phase boundaries in dynamic models of phase transitions, J. Nonlinear Sci. 9 (1999) 697–719.
[25] G. B. Whitham, Some comments on wave propagation and shock wave structure with applications to magnetohydrodynamics, Commun. Pure Appl. Math. 12 (1959) 113-158.
[26] R. LeVeque, Finite Volume Methods for Hyperbolic Problems, Cambrige University Press, 2002.
[27] A. Amengual, Ll. Mañosa, F. Marco, C. Picornell, C. Segui, V. Torra, Systematic study of the martensitic transformation in a Cu-Zn-Al alloy. Reversibility versus irreversibility via acoustic emission. Thermochimica Acta 116 (1987) 195–208.