ON IMPACT-INDUCED PROPAGATING PHASE BOUNDARIES. THERMAL EFFECTS.

Cristian FĂCIU*, **Alain MOLINARI****

* Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania
** Laboratoire de Physique et Mécanique des Matériaux, Université Paul Verlaine Metz, Ile du Saulcy, F-57045 Metz, France

e-mail: Cristian.Faciu@imar.ro

Abstract: This paper investigates dynamic aspects of solid-solid phase transitions when thermal effects are taken into account. We consider an explicit one-dimensional non-monotone thermo-elastic model able to describe some aspects of the thermomechanical response of a shape memory alloy. This model is embedded in a Maxwellian rate-type constitutive equation which removes the deficiencies of the classical thermo-elastic approach. Impact problems that gives rise to both adiabatic shock waves and propagating phase boundaries are analyzed.

Keywords: non-isothermal phase transition, bar impact, SMA

1. Introduction

In a recent paper (see Făciu and Molinari, 2006) we have proposed longitudinal impact experiments of thin bars as an effective mean for understanding the kinetics of stress-induced phase transformations in shape memory alloys (SMA). This problem has been investigated in a one-dimensional and isothermal setting and has provided important insight into the wave structure. Since the influence of thermal effects on the rate of phase transformation, size and shape of pseudoelastic hysteresis at higher strain rates is very important we extend, in a forthcoming paper, this analysis to take into account their influence on the wave propagation. The present paper is a simplified version and describes only some aspects of this study.

In the next section, we set out the balance laws, the dissipation inequality, and the corresponding jump conditions governing the dynamic response of a one-dimensional bar in the adiabatic case. In Section 3, based on experimental facts for SMA, we describe the thermo-elastic constitutive assumptions for a solid which can exist in three phases. Some aspects related to the non-unicity of weak solutions for the thermo-elastic system are reminded. We consider a Maxwellian rate-type constitutive equation as a convenient alternative for the description of dynamic phase transitions. In Section 4, we formulate the problem of a semi-infinite bar in the austenite phase impacted at one end and discuss some general features of its solution. Section 5 is concerned with the numerical predictions of the rate-type model for the longitudinal impact of two phase transforming bars. One focuses on the results which can be measured in laboratory experiments like the time of separation of the bars after impact, the profile of the particle velocity at the rear end of the target and the stress-history at the contact point.

2. Adiabatic thermomechanical bar theory

We consider here the longitudinal adiabatic motions of thin cylindrical bars by taking into account simultaneously inertia and heat release. The bar is supposed to be thermally isolated and moreover, the shocks and phase boundaries are treated as adiabatic. The balance laws of momentum, mass and energy in a 1-D Lagrangian description are

$$\varrho\frac{\partial v}{\partial t}-\frac{\partial \sigma}{\partial X}=0,\quad \frac{\partial \varepsilon}{\partial t}-\frac{\partial v}{\partial X}=0,\quad \varrho\frac{\partial e}{\partial t}-\sigma\frac{\partial \varepsilon}{\partial t}=0. \tag{1}$$

where v, σ, ε, e and ϱ are the particle velocity, the stress, the strain, the specific internal energy and constant mass density at a reference point X and time t, respectively.

Across the discontinuities, the system (1) must be supplemented by the Rankine-Hugoniot jump conditions and the dissipation inequality

$$\varrho\dot{S}[v]+[\sigma]=0,\quad \dot{S}[\varepsilon]+[v]=0,\quad \varrho\dot{S}[e+\frac{1}{2}v^2]+[\sigma v]=0,\quad -\varrho\dot{S}[\eta]\geq 0, \tag{2}$$

where $\eta=\eta(X,t)$ is the specific entropy and $\dot{S}$ is the Lagrangian velocity of the discontinuity. Here for the jump we use the standard notation $[\,]=()_+-()_-$ and "+" corresponds to the state at the right of the discontinuity. From (2) we have, either a *propagating strain discontinuity* $\dot{S}\neq 0$ across which

$$[\sigma]=\varrho\dot{S}^2[\varepsilon]\quad\text{and}\quad \varrho[e]=\frac{1}{2}(\sigma^+ +\sigma^-)[\varepsilon] \tag{3}$$

or, a *contact discontinuity* $\dot{S}=0$ across which the strain and the temperature may have jump, but

$$[\sigma]=0\quad\text{and}\quad [v]=0. \tag{4}$$

3. Constitutive assumptions - three phase materials

a) The thermo-elastic approach.

It is now a common use to model materials undergoing solid-solid phase transitions by using non-monotone, up-down-up stress-strain relations. We consider here such an explicit simple stress-strain-temperature relation $\sigma=\sigma_{eq}(\varepsilon,\theta)$ derived from physical considerations on the behavior of shape memory alloys in quasi-static tests. It is inspired by the model proposed by Abeyaratne et al. (1994) and has been used by Făciu and Mihăilescu-Suliciu (2002) as an equilibrium relation for a Maxwellian rate-type approach to the thermo-elasticity. Moreover, the predictions of this rate-type approach for quasi-static strain controlled tests have shown a quite good agreement with the full-field measurements of strain and temperature fields performed by Shaw and Kyriakides (1997) and with the experimental results obtained by Pieczyska et al. (2004) (see also the references therein).

We consider a material which exists in a high-temperature phase austenite $\mathcal{A}$ and has two variants $\mathcal{M}^-$ and $\mathcal{M}^+$ of a low-temperature martensite. That means, there are two critical temperature θ_m and θ_M such as for $\theta>\theta_M$ the material only exists in its austenite form no matter what the stress level is, whereas for $\theta<\theta_m$ the material only exists in its martensitic forms. For $\theta\in[\theta_m,\theta_M]$ all three phases are available to the material. The stress response function $\sigma_{eq}(\varepsilon,\theta)$ must therefore be a monotonically increasing function of ε for $\theta>\theta_M$. At each fixed temperature $\theta\in[\theta_m,\theta_M]$ the thermo-elastic curve $\sigma_{eq}(\varepsilon,\theta)$ is a *continuous*, monotonically *increasing* function of strain for $\varepsilon \in [\varepsilon_I(\theta), \varepsilon_m^-(\theta)] \cup [\varepsilon_M^-(\theta), \varepsilon_M^+(\theta)] \cup [\varepsilon_m^+(\theta), \varepsilon_{II}(\theta)]$ and a monotonically *decreasing* function of strain over the intervals $(\varepsilon_m^-(\theta), \varepsilon_M^-(\theta))$ and $(\varepsilon_M^+(\theta), \varepsilon_m^+(\theta))$. For $\theta < \theta_m$, the isotherm $\sigma_{eq}(\varepsilon, \theta)$ is a monotonically *increasing* function of strain on two intervals, i.e. $(\varepsilon_I(\theta), \varepsilon_m^-(\theta)) \cup (\varepsilon_m^+(\theta), \varepsilon_{II}(\theta))$, while on the remaining one $(\varepsilon_m^-(\theta), \varepsilon_m^+(\theta))$ it is monotonically decreasing (see Fig. 1a).

Quasi-static experiments on SMAs have shown that we can approximate their behavior in a pure phase by a linear thermo-elastic relation. Thus, we can assume that elastic moduli of the austenite phase $\mathcal{A}$ and martensite variants $\mathcal{M}^\pm$ are constant and equal to $E_1 > 0$ and $E_3 > 0$, respectively. Moreover, for simplicity we consider that the elastic softening moduli of the unstable (spinodal) regions $\mathcal{I}^\pm$ are also constant and equal to $-E_2 < 0$. The magnitude of E_2 is proportional with the size of the hysteresis loop (see Fig. 1b).

For $\theta \in [\theta_m, \theta_M]$, we consider the following continuous and piecewise linear stress-strain-temperature relation

$$\sigma = \sigma_{eq}(\varepsilon, \theta) = \begin{cases} E_3(\varepsilon - \varepsilon_m^-(\theta)) + \sigma_m^-(\theta) & \text{for } \varepsilon_I(\theta) < \varepsilon \le \varepsilon_m^-(\theta) \\ -E_2(\varepsilon - \varepsilon_m^-(\theta)) + \sigma_m^-(\theta) & \text{for } \varepsilon_m^-(\theta) < \varepsilon < \varepsilon_M^-(\theta) \\ E_1(\varepsilon - \varepsilon_M^-(\theta)) + \sigma_M^-(\theta) & \text{for } \varepsilon_M^-(\theta) \le \varepsilon \le \varepsilon_M^+(\theta) \\ -E_2(\varepsilon - \varepsilon_m^+(\theta)) + \sigma_m^+(\theta) & \text{for } \varepsilon_M^+(\theta) < \varepsilon < \varepsilon_m^+(\theta) \\ E_3(\varepsilon - \varepsilon_m^+(\theta)) + \sigma_m^+(\theta) & \text{for } \varepsilon_m^+(\theta) \le \varepsilon < \varepsilon_I(\theta) \end{cases} \quad (5)$$

Here the functions $\sigma_M^\pm(\theta) = \sigma_{eq}(\varepsilon_M^\pm(\theta), \theta)$ and $\sigma_m^\pm(\theta) = \sigma_{eq}(\varepsilon_m^\pm(\theta), \theta)$ are the local maxima and minima with respect to ε of the isothermal stress-strain curves. In our interpretation $\sigma_M^\pm(\theta)$ $(\varepsilon_M^\pm(\theta))$ are the limit values of the stress (strain) required for the start of $\mathcal{A} \to \mathcal{M}^\pm$ transformation, whereas $\sigma_m^\pm(\theta)$ $(\varepsilon_m^\pm(\theta))$ are the limit values of the stress (strain) required for the start of the reverse transformation $\mathcal{M}^\pm \to \mathcal{A}$ and they can be determined from quasi-static experiments. The boundary curves $\varepsilon = \varepsilon_M^\pm(\theta)$ and $\varepsilon = \varepsilon_m^\pm(\theta)$ fix the limits of the regions of the (ε, θ)-plane on which the respective phases $\mathcal{A}$ and $\mathcal{M}^\pm$ exist. According to our previous assumption on linear thermoelastic behavior of the material in a single phase these constitutive functions are linear with respect to θ, i.e.

$$\varepsilon_M^\pm(\theta) = \alpha(\theta - \theta_T) \pm M(\theta - \theta_m), \quad \varepsilon_m^\pm(\theta) = \alpha(\theta - \theta_T) \pm (m - M)(\theta - \theta_M) \pm M(\theta - \theta_m) \quad (6)$$

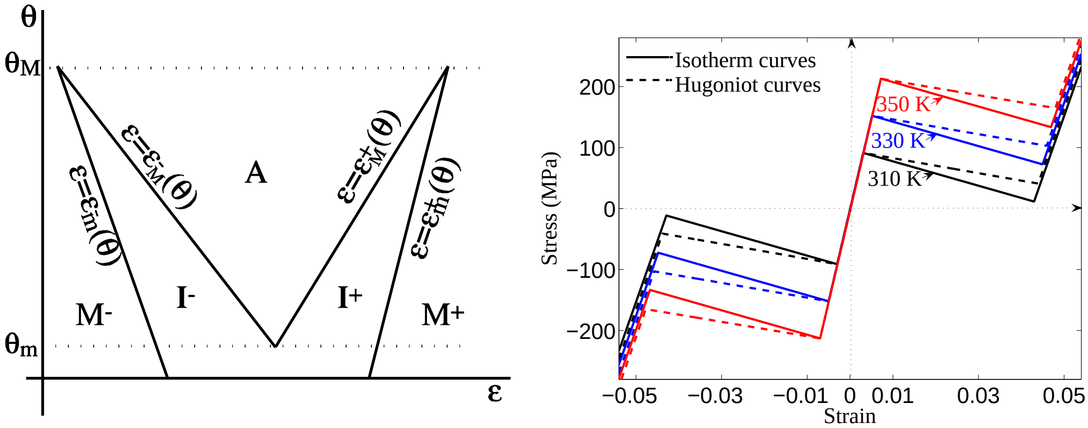
and, consequently

$$\sigma_M^+(\theta) = -\sigma_M^-(\theta) = E_1 M(\theta - \theta_m), \quad \sigma_m^+(\theta) = -\sigma_m^-(\theta) = E_1 M(\theta - \theta_m) + E_2(M - m)(\theta - \theta_M). \quad (7)$$

Here $\alpha > 0$ is the *thermal expansion coefficient* of the material in phase $\mathcal{A}$ and θ_T is a reference temperature which is precised when constructing the free energy function of the thermo-elastic model. The positive constants M and m can be determined from laboratory experiments. Indeed, $\frac{d\sigma_M^+(\theta)}{d\theta} = E_1 M$ and $\frac{d\sigma_m^+(\theta)}{d\theta} = E_1 M + E_2(M - m)$ represent the rates with respect to the temperature at which the stresses required for the direct $(\mathcal{A} \to \mathcal{M}^+)$ and inverse $(\mathcal{M}^+ \to \mathcal{A})$ transformation increase. These quantities are in general constant and determined from laboratory experiments (see for instance Shaw and Kyriakides, 1997).

Once we have established the stress-strain-temperature relation $\sigma = \sigma_{eq}(\varepsilon, \theta)$ the second law of thermodynamics determines uniquely: the free energy function $\psi = \psi_{eq}(\varepsilon, \theta)$ (modulo an additive function of temperature $\phi = \phi(\theta)$), the entropy $\eta = \eta_{eq}(\varepsilon, \theta)$ and the internal energy of the material $e = e_{eq}(\varepsilon, \theta)$ as follows

$$\varrho\psi_{eq}(\varepsilon, \theta) = \int_{\varepsilon_0}^{\varepsilon} \sigma_{eq}(s, \theta)\mathrm{d}s + \varrho\phi(\theta), \quad \eta_{eq} = -\frac{\partial \psi_{eq}}{\partial \theta}, \quad \varrho e_{eq} = \varrho\psi_{eq} + \varrho\theta\eta_{eq}. \quad (8)$$



a) b)

Figure 1: a) Phase diagram in the (ε, θ)-plane. b) Isotherm curves $\sigma = \sigma_{eq}(\varepsilon, \theta^+)$ and Hugoniot stress-strain curves based at $(\varepsilon^+ = 0, \sigma^+)$, $\sigma = \sigma_H(\varepsilon; 0, \sigma^+)$ for $\theta^+ \in \{310\,^\circ\text{K},\ 330\,^\circ\text{K},\ 350\,^\circ\text{K}\}$, $\sigma^+ = \sigma_{eq}(0, \theta^+)$.

The function $\phi = \phi(\theta)$ is determined by supposing that the specific heat at constant strain of the thermo-elastic material is constant in the austenitic phase i.e., $-\theta\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\theta^2} = C = const. > 0$, for $\varepsilon \in (\varepsilon_M^-(\theta), \varepsilon_M^+(\theta))$.

In order to investigate quantitative, as well as qualitative, the longitudinal impact of two phase transforming bars we use the following material parameters which are appropriate for a Cu-Zn-Ni shape memory alloy:

$$\begin{array}{llll} E_1 = 30.\,\text{GPa}, & E_2 = 2.\,\text{GPa}, & E_3 = 20.\,\text{GPa}, & \varrho = 8000.\,\text{kg/m}^3, \\ \alpha = 1.6\times 10^{-6}/^\circ\text{K}, & M = 10.1371\times 10^{-5}/^\circ\text{K}, & & m = 9.7253\times 10^{-5}/^\circ\text{K}, \\ \theta_m = 280\,^\circ\text{K}, & \theta_M = 10000\,^\circ\text{K}, & \theta_T = 283.3\,^\circ\text{K}, & C = 500.\,\text{J/Kg/}^\circ\text{K}. \end{array} \tag{9}$$

The PDE system composed by (1) and the constitutive relations for stress and internal energy (5) and $(8)_3$ is called the *thermo-elastic phase transforming system*.

Let $\dot{S} > 0$ and denote by $(\varepsilon^+, \theta^+, v^+)$ the state in the front of the discontinuity. For a thermo-elastic material, the Rankine-Hugoniot equation $(3)_2$ read

$$\varrho e_{eq}(\varepsilon, \theta) - \varrho e^+ - \frac{1}{2}(\sigma_{eq}(\varepsilon, \theta) + \sigma^+)(\varepsilon - \varepsilon^+) = 0, \tag{10}$$

where $e^+ = e_{eq}(\varepsilon^+, \theta^+)$ and $\sigma^+ = \sigma_{eq}(\varepsilon^+, \theta^+)$. This equation provides only restrictions on the back state (ε, θ) which can be reached in a shock process which has $(\varepsilon^+, \theta^+)$ as a front state. We can show that, for our previous constitutive assumptions, the above equation can be solved with respect to θ. Therefore, there exists a unique continuous and piecewise smooth function called the *Hugoniot temperature-strain curve based at* $(\varepsilon^+, \theta^+)$

$$\theta = \Theta_H(\varepsilon; \varepsilon^+, \theta^+) \tag{11}$$

with the property that it satisfies equation (10) and $\theta^+ = \Theta_H(\varepsilon^+; \varepsilon^+, \theta^+)$. This function describes explicitly all those states in the (ε, θ) plane that are potentially attainable as back states in a discontinuity process which has $(\varepsilon^+, \theta^+)$ as a front state. In the present setting a strain discontinuity is called a *phase boundary* if the particles separated by the discontinuity are in different phases and an *elastic shock* if they are in the same phase.

We denote by

$$\sigma = \sigma_H(\varepsilon; \varepsilon^+, \sigma^+) = \sigma_{eq}(\varepsilon, \Theta_H(\varepsilon; \varepsilon^+, \theta^+)), \tag{12}$$

the *Hugoniot stress-strain curve based at* $(\sigma^+, \varepsilon^+)$ in the space (σ, ε) where $\sigma^+ = \sigma_{eq}(\varepsilon^+, \theta^+)$. This function describes all reachable (σ, ε) back states in a wave discontinuity which has $(\sigma^+, \varepsilon^+)$ as a front state. Moreover the back states (σ, ε) have to satisfy the Rayleigh line construction

$$\sigma - \sigma^+ = \varrho \dot{S}^2(\varepsilon - \varepsilon^+). \tag{13}$$

The influence of thermal effects on the isothermal curves and on the Hugoniot stress-strain curve is illustrated in Fig. 1b.

Let us note that *not all* the back state (ε, θ) satisfying the Rankine-Hugoniot equation (11) are compatible with the second law of thermodynamics when the front state is $(\varepsilon^+, \theta^+)$. Indeed, the entropy inequality $(2)_4$, which for η given by $(8)_2$ and $\dot{S} > 0$ read as

$$\eta_{eq}(\varepsilon, \theta) \geq \eta_{eq}(\varepsilon^+, \theta^+), \tag{14}$$

will select these states. This condition expresses the fact that after the passage of the discontinuity the entropy of the particle has not to decrease. This situation is illustrated in Fig. 2.

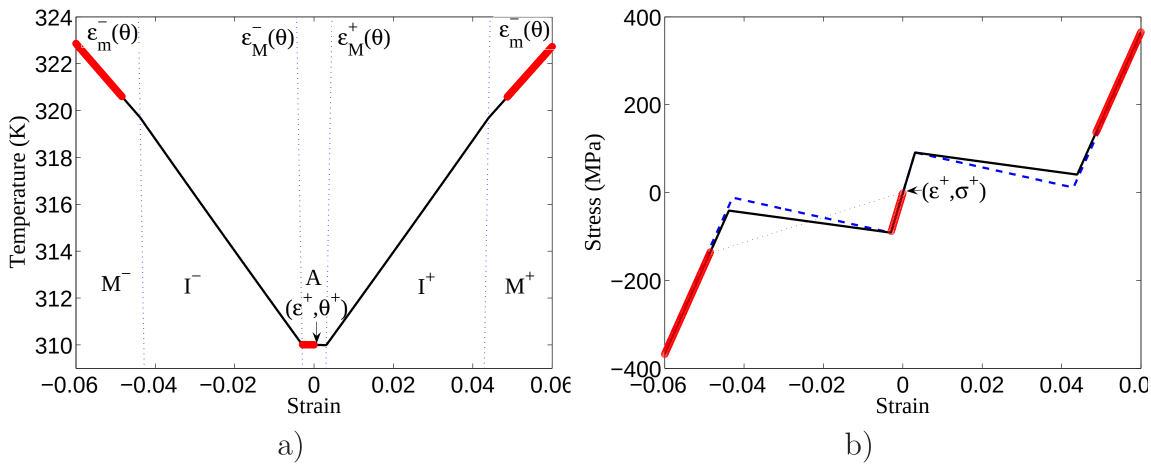


Figure 2: a) Hugoniot temperature-strain curve based at $(\varepsilon^+ = 0, \theta^+ = 310\,^\circ\mathrm{K})$. b) Solid-line is the Hugoniot stress-strain curve based at $(\varepsilon^+ = 0, \sigma^+ = -1.2\,\mathrm{MPa})$; dashed-line $(-\ -\ -)$ is the isotherm curve $\sigma = \sigma_{eq}(\varepsilon, 310\,^\circ\mathrm{K})$. In both figures the thickened lines represents the states thermodynamically admissible, according to (14). A jump from $(\varepsilon^+, \sigma^+)$ to $\mathcal{M}^+$ or to $\mathcal{M}^-$ is not physically admissible according to the viscosity criterion.

It is known that the use of non-monotone thermo-elastic constitutive equations leads to non-unique solutions for dynamic initial-boundary value problems and thus these are ill-posed. This situation reflects in fact a constitutive insufficiency in describing the behavior of phase transforming materials. To overcome this difficulty two mechanisms have been used in the literature. One approach consists in adding a *nucleation criterion* for the initiation of phase transition and a *kinetic relation* between interface velocity and the *driving force* of phase transformation. These two ingredients are sufficient to obtain unique solution of the dynamic problem (see for instance Abeyaratne and Knowles, 1991). An *alternative approach* is to incorporate in the thermo-elastic theory rate effects of Kelvin-Voigt's type, or Maxwell's type , and/or effects due to the gradient of strain (see for instance Ngan and Truskinovsky (2002) and the literature therein).

b) The rate-type approach
In this paper we adopt the second approach and we use the following *Maxwellian rate-type constitutive equation*

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = -\frac{E}{\mu}(\sigma - \sigma_{eq}(\varepsilon, \theta)), \tag{15}$$

where $E = \text{const.} > E_1 > 0$ is the *dynamic Young modulus*, $\mu = \text{const.} > 0$ is a Newtonian *viscosity coefficient* ($\frac{\mu}{E}$ is called a relaxation time) and $\sigma = \sigma_{eq}(\varepsilon, \theta)$ is the equilibrium state equation for phase transforming materials (5). When $\mu \to 0$ this constitutive equation is seen as a thermo-viscoelastic rate-type approach of the thermo-elastic model. From physical point of view it introduces a mechanism of energy dissipation of kinetic origin and an internal structure in shocks and phase boundaries replacing the sharp interfaces by transition layers of finite thickness.

The rate-type constitutive equation (15) includes as a limiting case for $E \to \infty$ the Kelvin-Voigt model

$$\sigma = \sigma_{eq}(\varepsilon, \theta) + \mu\frac{\partial \varepsilon}{\partial t}. \tag{16}$$

An additional advantage of this approach is that it does not require a separate nucleation criterion. In this model instability automatically leads to formation of phases which nucleates in the unstable regions $\mathcal{I}^{\pm}$. The new parameters E and μ influence in fact the kinetics of the growth of phases and the thickness of the transition layers.

In order to determine the wave structure of thermo-elastic solids different *admissibility criteria* designed to select physically relevant solutions have been considered (see for instance Slemrod, 1983). Since we propose the rate-type approach (15) as a way to describe impact-induced propagating phase boundaries in phase transforming materials we use it to derive a selection criterion for admissible waves. This *viscosity criterion* asserts that a discontinuity for the thermo-elastic model is admissible if and only if the strains ε^- and ε^+ and the temperature θ^- and θ^+ on either side of the discontinuity can be smoothly connected by a traveling wave constructed within the rate-type theory. In fact we identify as admissible those waves which arise in the frame of the augmented theory in the limit of vanishing viscosity.

4. Semi-infinite bar in phase $\mathcal{A}$ impacted at one end

In order to illustrate our approach we consider the case of a semi-infinite bar in the austenite phase $\mathcal{A}$ which is suddenly loaded at one end. That means we have to solve for the adiabatic thermo-elastic system the problem of wave propagation generated by the following initial-boundary value condition,

$$(\varepsilon, \theta, v)(X, 0) = (\varepsilon_R, \theta_R, v_R), \quad \text{for } X > 0, \text{ and } \quad \sigma(0, t) = \sigma^* \text{ for } t > 0, \tag{17}$$

where $(\varepsilon_R, \theta_R, v_R)$ and σ^* are given values, $\varepsilon_R \in (\varepsilon_M(\theta)^-, \varepsilon_M(\theta)^+)$. We denote by $\sigma_R = \sigma_{eq}(\varepsilon_R, \theta_R)$ the initial stress. This problem is known as the right Goursat problem in stress.

The solution is sought in the form of self-similar solutions, i.e. $(\varepsilon, \theta, v)(X, t) = (\varepsilon, \theta, v)(\xi)$, where $\xi = \frac{X}{t}$ and is build by using the jump relations (2) and the viscosity criterion. Let us consider the compressive case when the impact stress σ^* is less than the initial stress σ_R. We determine critical values θ_{ph}^-, ε_{ph}^-, σ_{ph}^- which depend on $(\varepsilon_R, \theta_R)$ such that if the applied stress $\sigma^* < \sigma_{ph}^-(\varepsilon_R, \theta_R)$ then a phase transformation is induced. These values are given by

$$\theta_{ph}^- \approx \theta_R + \frac{\varepsilon_R - \varepsilon_M^-(\theta_R)}{\alpha + M + \varrho C/(\alpha E_1 \theta_R)} > \theta_R, \quad \varepsilon_{ph}^- = \varepsilon_M^-(\theta_{ph}^{\pm}), \quad \sigma_{ph}^- = \sigma_M^-(\theta_{ph}^{\pm}). \tag{18}$$

We give a schematic representation of the solution in Fig. 3 and Fig. 4. If $\sigma^* \in [\sigma_{ph}^-, \sigma_R]$ the solution consists of a thermo-elastic adiabatic shock wave $\frac{X}{t} = \dot{S}_a \approx \sqrt{\frac{E_1}{\varrho} + \frac{E_1^2 \alpha^2}{\varrho^2 C}\theta_R}$, which separates the initial state $(\varepsilon_R, \theta_R, \sigma_R, v_R)$ and the impact state $(\varepsilon^*, \theta^*, \sigma^*, v^*)$. The unknown quantities ε^* and θ^* have to satisfy the Hugoniot temperature-strain relation based at $(\varepsilon_R, \theta_R)$ while the pair $(\varepsilon^*, \sigma^*)$ has to belong to the Hugoniot stress-strain relation based at $(\varepsilon_R, \sigma_R)$. One observes that in this case the temperature only slightly increases due to the thermo-elastic effect.

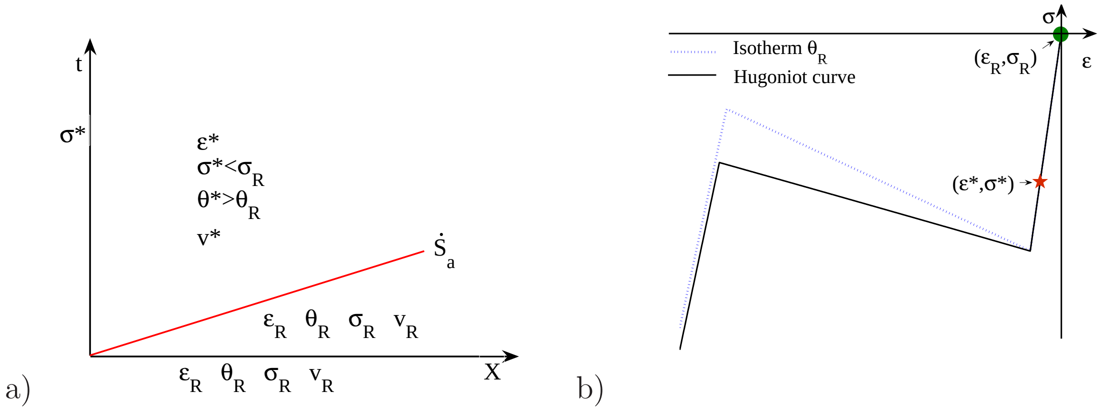


Figure 3: a) Self-similar solution: adiabatic shock wave in phase $\mathcal{A}$. b) Stress-strain states ahead and behind the discontinuity.

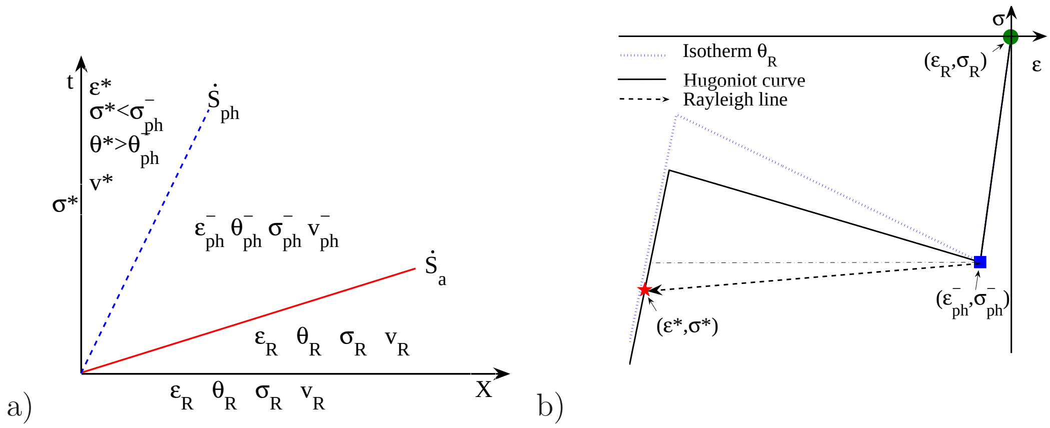


Figure 4: a) Self similar solution: adiabatic shock wave in phase $\mathcal{A}$ followed by a propagating phase boundary transforming the material from $\mathcal{A}$ to $\mathcal{M}^-$ b) Stress-strain states ahead and behind the discontinuities.

If the impact stress σ^* is less than the critical value σ_{ph}^- an inverse transformation from $\mathcal{A}$-phase to $\mathcal{M}^-$-phase is induced. The solution consists of an adiabatic compressive shock wave $\frac{X}{t} = \dot{S}_a = \sqrt{\frac{E_1}{\varrho} + \frac{E_1^2 \alpha^2}{2\varrho^2 C}(\theta_R + \theta_{ph}^-)}$, which separates the constant states $(\varepsilon_R, \theta_R, \sigma_R, v_R)$ and $(\varepsilon_{ph}^-, \theta_{ph}^-, \sigma_{ph}^-, v_{ph}^-)$, followed by a phase boundary propagating with the speed $\dot{S}_p = \sqrt{\frac{\sigma^* - \sigma_{ph}^-}{\varrho(\varepsilon^* - \varepsilon_{ph}^-)}}$, which relates the intermediate state $(\varepsilon_{ph}^-, \theta_{ph}^-, \sigma_{ph}^-, v_{ph}^-)$ and the impact state $(\varepsilon^*, \theta^*, \sigma^*, v^*)$. The solution is built by using the Hugoniot curves and the viscosity criterion. Thus, the pair $(\varepsilon_{ph}^-, \theta_{ph}^-)$, which lies on the boundary between phase $\mathcal{A}$ and $\mathcal{I}^-$, belongs to the Hugoniot temperature-strain curve based at $(\varepsilon_R, \theta_R)$ and the final state $(\varepsilon^*, \theta^*)$ belongs to the Hugoniot temperature-strain curve based at $(\varepsilon_{ph}^-, \theta_{ph}^-)$ (see Fig. 5a). The pair $(\varepsilon_{ph}^-, \sigma_{ph}^-)$ has to belong to the Hugoniot stress-strain curve based at

$(\varepsilon_R, \theta_R)$ and $(\varepsilon^*, \sigma^*)$ has to belong to the Hugoniot stress-strain relation based at $(\varepsilon^-_{ph}, \sigma^-_{ph})$ (see Fig. 5). Let us note that since the Hugoniot stress-strain curve is non-monotone the final state $(\varepsilon^*, \theta^*)$ can not lie in the unstable phase $\mathcal{I}^-$.

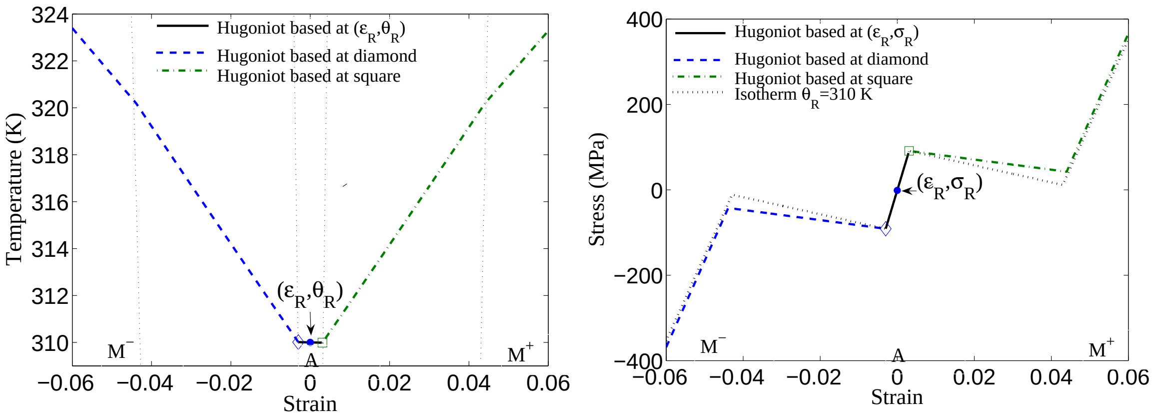


Figure 5: Hugoniot curves in the plane (θ, ε) and in the plane (σ, ε) for $\varepsilon_R = 0$, $\theta_R = 310\,°\mathrm{K}$, $\sigma_R = -1.3\mathrm{MPa}$. Here $\square$ corresponds to the coordinates $(\varepsilon^+_{ph}, \theta^+_{ph}, \sigma^+_{ph})$ and $\diamond$ to $(\varepsilon^-_{ph}, \theta^-_{ph}, \sigma^-_{ph})$.

Finally we can build three functions

$$v^* = v_R + V^A_R(\sigma^*; \varepsilon_R, \theta_R), \quad \varepsilon^* = G^A_R(\sigma^*; \varepsilon_R, \theta_R), \quad \theta^* = T^A_R(\sigma^*; \varepsilon_R, \theta_R), \tag{19}$$

with the property that for given initial data $(\varepsilon_R, \theta_R, v_R)$, where $(\varepsilon_R, \theta_R) \in \mathcal{A}$-phase, associate to the impact stress σ^* the impact velocity $v^* = v(0, t)$, the impact strain $\varepsilon^* = \varepsilon(0, t)$ and the impact temperature $\theta^* = \theta(0, t)$, solution of the Goursat problem (17) at the boundary $(X = 0, t > 0)$.

The behavior of these functions is illustrated in Fig. 6a), Fig. 6b) and Fig. 6c), respectively, for both compressive and expansive case. Let us note that function V^A_R is continuous and strictly decreasing function on σ^*, thus being inversible. Therefore, if we consider the right Goursat problem in velocity, that is when instead of giving σ^* at $X = 0$ for $t > 0$ in (17) we prescribe a constant velocity v^* we obtain immediately a unique solution due the one-to-one correspondence between v^* and σ^*. Let us note that functions G^A_R and T^A_R which prescribe the strain and temperature at the impacted end are discontinuous functions at $\sigma^* = \sigma^\pm_{ph} \approx \pm 91MPa$. This property reflects just the sudden phase transformation which can appear at the impacted end. Indeed, for impact stress $\sigma^* \in [\sigma^-_{ph}, \sigma^+_{ph}]$ the material remains in phase $\mathcal{A}$ and its behavior is thermo-elastic linear. In this case we only have a slight variation of the temperature and of the strain. The temperature increases in compression and decreases in tension. Once the critical values of the stress $\sigma^\pm_{ph} = \sigma^\pm_{ph}(\varepsilon_R, \theta_R)$ are overcome then a phase transformation appears which materializes by a strong localization of the strain and by an important increase of the temperature in the transformed zone due to the latent heat of transformation.

5. Application: longitudinal impact experiment of SMA bars

We consider at the initial moment a bar called "target " impacted at one end by another bar called "flyer" which is moving with a constant velocity V_0 (see Fig. 7). After impact the two bars remains in contact and move together until a time t_S called time of separation This time corresponds to the moment when the first tensile wave arrives at the point of

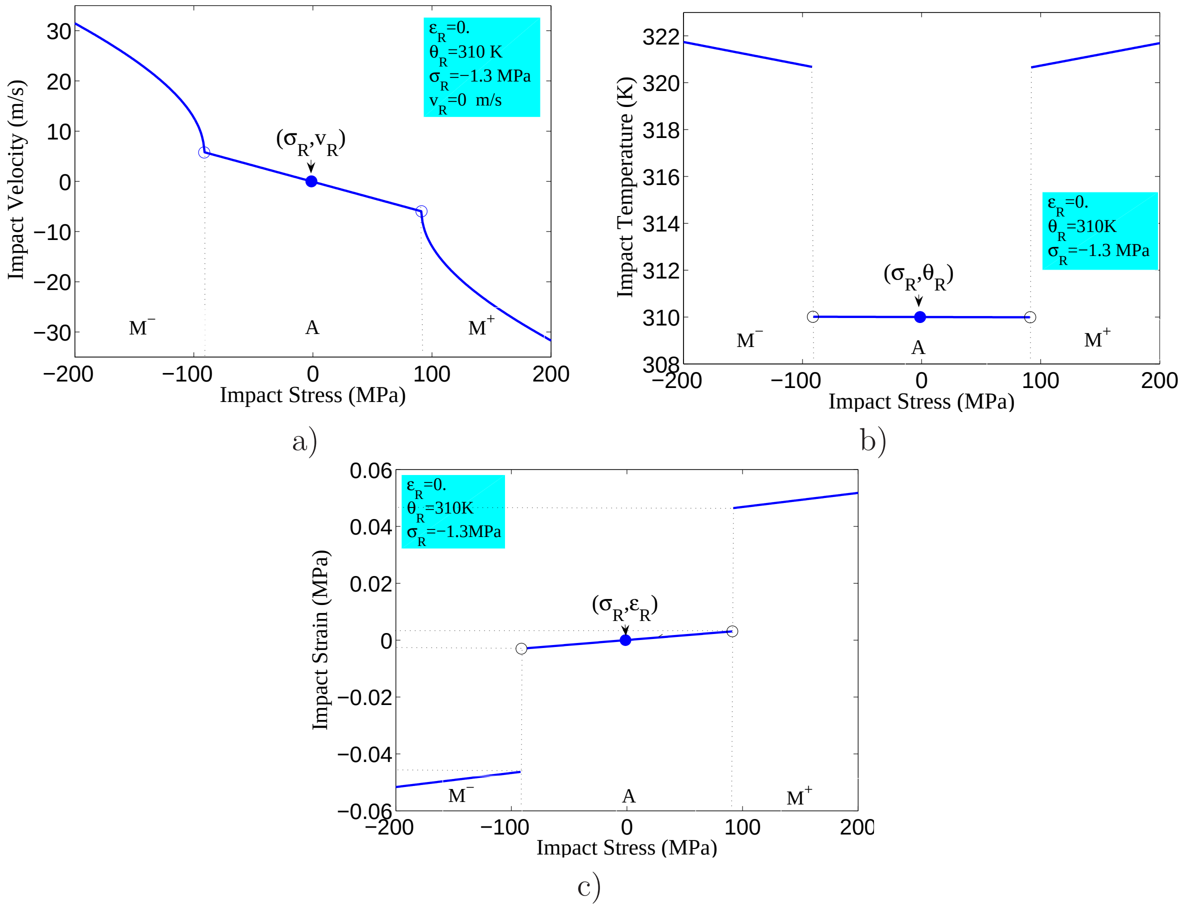


Figure 6: Velocity v^*, temperature θ^* and strain ε^* at the impacted end of a semi-infinite bar for sudden load σ^*. Predictions of the thermo-elastic three-phase material endowed with the viscosity criterion.

contact. In this simple dynamic laboratory test one can measure: the time of separation between the bars after the impact by optical methods, the particle velocity at the rear end of the target by a VISAR interferometry system, the stress history at the impacted end by piezoelectric wafers, the variation in time of the strain at various cross section by using diffraction gratings. The correlation of such measurements allow a better understanding of the kinetics of phase transformation. For instance, one can determine in an indirect way the influence of the impact velocity on the speed of propagation of the phase boundary. Such kind of experiments have been done by Escobar and Clifton (1995) on SMA foils. Although the heating of the transformed zone is very important due to the latent heat of transformation the possibility of measuring its temperature variation in a time interval of only tenth μs is unknown for us. Nevertheless, the use of an explicit temperature-dependent constitutive approach to simulate the longitudinal impact of bars can provide useful information on the dynamic of interfaces.

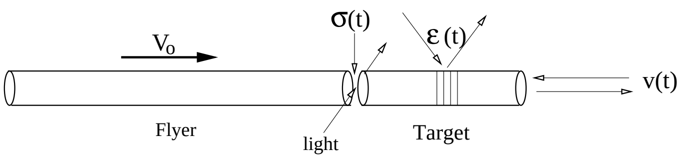


Figure 7: Longitudinal impact of thin bars

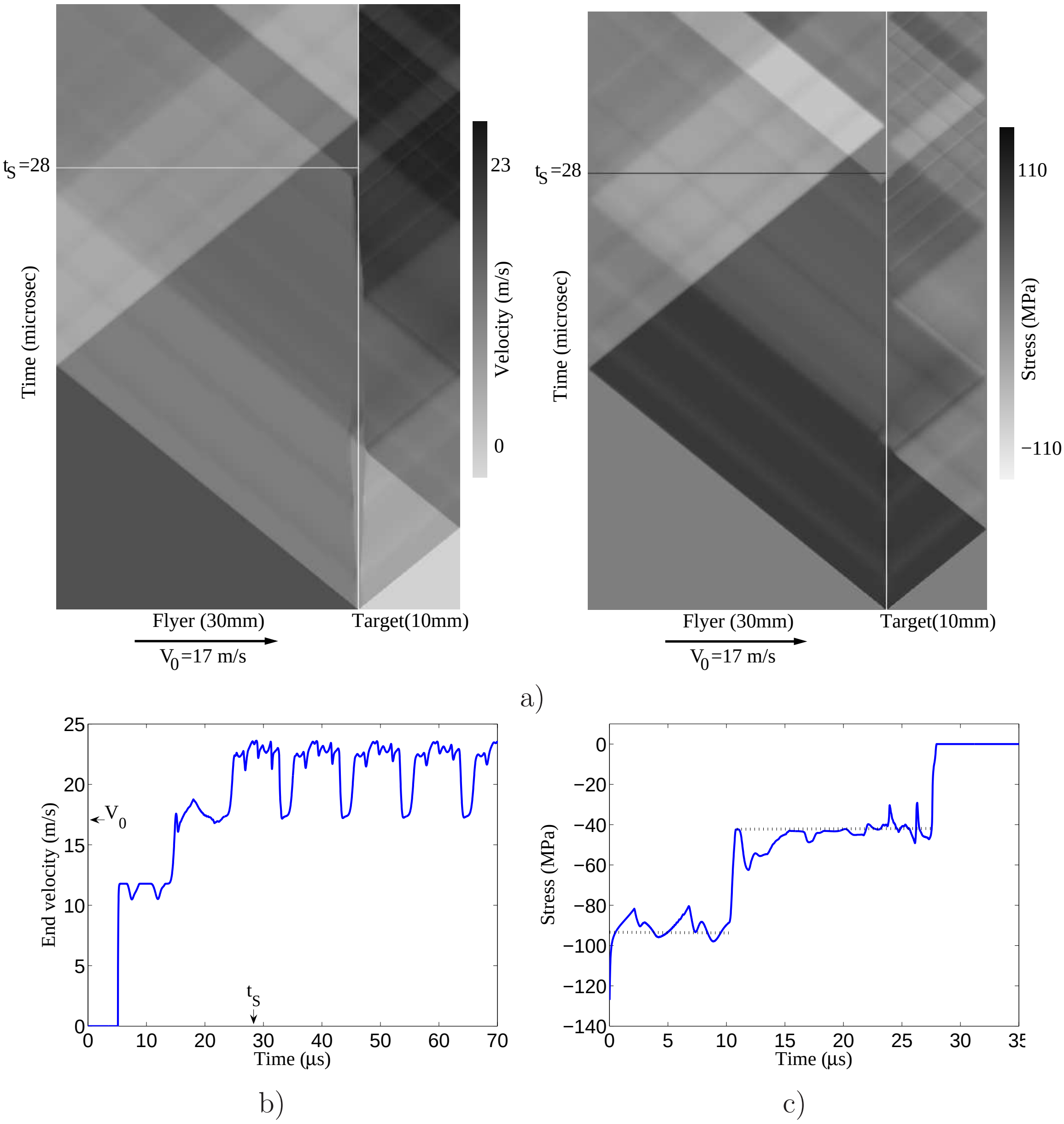


Figure 8: a) Velocity and stress distribution in the bars after a moderate impact; t_S denotes the time of separation of the bars. b) Particle velocity history at the target free-end. c) Impact-stress history. The horizontal lines denote the stress level predicted by the thermo-elastic model.

We can construct an analytical solution of the above impact problem for a short time interval which includes the first wave interactions. The building blocks in solving this problem are the simplest initial-boundary value problems for the thermo-elastic system, i.e the Goursat and Riemann problems. Thus, by using Riemann solvers we can construct solutions for a variety of impact conditions. For example, interactions of the unloading elastic wave reflected at the free end of the target with the propagating phase boundary can be explored and exact critical values of the impact velocity can be determined such that this phase boundary propagates backward, remains stationary or propagates forward.

One aim of our study is to compare the exact solution obtained using the thermo-elastic model, endowed with the viscosity criterion, with the numerical solution obtained using the Maxwellian rate-type model (15). We present in the following some numerical results which describe the evolution in time of the stress, strain, particle velocity and temperature in the flyer and target for an impact velocity $V_0 = 17$ m/s when the dynamic Young's modulus $E = 30.5\,\text{GPa}$ and the Newtonian viscosity coefficient $\mu = 1.016 \times 10^{-3}\,\text{MPa} \times \text{s}$.

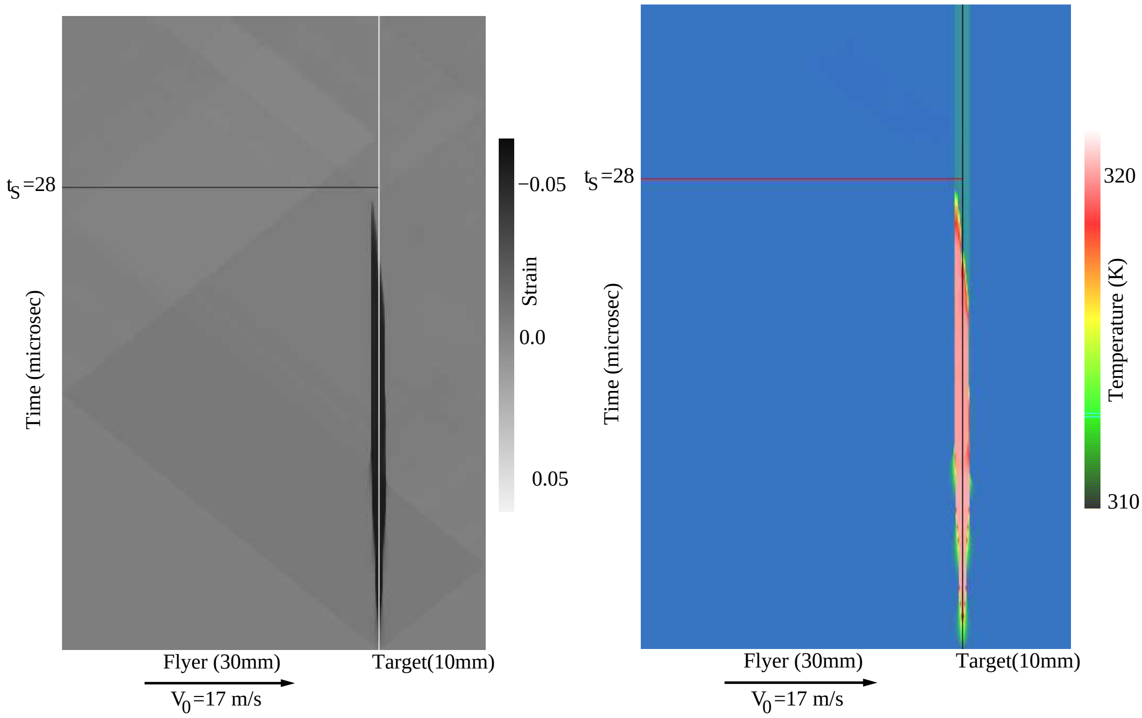


Figure 9: Strain and temperature distribution in the bars after a moderate impact

One observes (see Fig. 9) near the impact point a transformed zone (high strain) which is bounded by two phase boundaries propagating left and right with a small velocity. Since the $\mathcal{A} \to \mathcal{M}^-$ transformation is exothermic the temperature increases with about $10\,^\circ$K in this region. The adiabatic compressive shock wave propagating in the target is reflected by the free end as an unloading wave which interacts successively with the phase boundaries being reflected and transmitted across them. The propagation of both phase boundaries is stopped becoming contact discontinuities.

The presence of a transformation can be detected by recording the changes in the velocity profile at the rear end of the target (see Fig. 8b). Indeed, the adiabatic shock wave reflected by the phase boundary induces a significant step-like increase of the velocity. No increase in particle velocity would be expected at this moment if no phase transformation would occur. Moreover, as long as the transformed material exists in the target any round trip propagation of the adiabatic shock wave between the phase boundary and the free-end of the target will lead to a new significant increase of the particle velocity. One can also observe small fluctuations of the particle velocity in Fig. 8b as well as small fluctuations of the impact stress in Fig. 8c . These are due to the adiabatic shock waves propagating inside the transformed zone and which are transmitted across the phase boundaries. The interaction of the two phase boundaries and their disappearance generates two tensile waves propagating left and right. The last one leads to the separation of the two bars. A manifestation of this tensile wave can be also detected on the profile of the particle velocity (Fig. 8b) and is associated with a significant velocity drop.

These numerical experiments for the Maxwell's rate-type system illustrate the potentiality of the adopted model to describe specific phenomena accompanying stress-induced phase transformations during impact tests. Other features related with the impact-induced propagating phase boundaries will be presented elsewhere.

The results predicted by the rate-type model and by the thermo-elastic model endowed with the viscosity criterion have to be compared with experimental results which for the moment are missing from the literature. Since different selection criteria (Slemrod, 1983) and different approaches (Abeyaratne and Knowles, 1991) may furnish different solutions to the same impact problem, only a systematic experimental investigation could decide which is the physical relevant one.

Acknowledgements

C.F acknowledges support from the Romanian Ministry of Education and Research through Project 2-CEx06-11-12/2006. A.M acknowledges support of the European Network of Excellence KMM NOE project NMP3-CT-2004-502243. C.F. is also grateful for the warm hospitality of the University of Metz during his research stays at LPMM.

References

Abeyaratne, R., Knowles, J. K., 1991. Kinetic relations and the propagation of phase boundaries in solids. *Arch. Rational Mech. Anal.* 114, 119-154.

Abeyaratne, R., Kim, S-J., Knowles, J. K., 1994. A one-dimensional continuum model for shape memory alloys. *Int. J. Solids Structures* 31, 2229-2249.

Escobar, J.C., Clifton, R.J.,1995. Stress-wave induced martensitic phase transformations. *Metallurgical and Materials Applications of Shock-Wave and High-Strain-Rate Phenomena*, Eds. L.E. Murr, K.P. Staudhammer and M.A. Meyers, 451-462, Elsevier Science.

Făciu, C., Mihăilescu-Suliciu, M., 2002. On modelling phase propagation in SMAs by a Maxwellian thermo-viscoelastic approach. *Int. J. Solids Structures*, 39, 3811-3830.

Făciu, C., Molinari, A., 2006. On the longitudinal impact of two phase transforming bars. Elastic versus a rate-type approach. Part I: The elastic case. Part II: The rate-type case. *Int. J. Solids Structures*, 43, 497-522, 523-550.

Pieczyska, E. A., Gadaj, S. P., Nowacki, W. K., Tobushi, H., 2004. Thermomechanical investigations of martensitic and reverse transformations in TiNi shape memory alloy. *Bull. Pol. Ac.: Tech.* 52(3), 165-171.

Shaw, J.A., Kyriakides, S., 1997. On the nucleation and propagation of phase transformation fronts in a NiTi alloy. *Acta Materialia* 45, 683-700.

Slemrod, M., 1983. Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Rational Mech. Anal.* 81, 301-315.

Ngan, S.-C., Truskinovsky, L., 2002. Thermo-elastic aspects of dynamic nucleation. *J. Mech. Phys. of Solids* 50, 1193-1229.