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STABILITY OF INFINITESIMAL MOTIONS OF RATE-TYPE SEMILINEAR VISCOELASTIC SOLIDS. A NON-ISOLATED BODY PROBLEM

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We establish the uniqueness and continuous dependence in total energy upon initial-state, supply term and boundary data of smooth solutions of a non-isolated body problem with prescribed boundary motion for the 3-D case of small deformations in semilinear rate-type viscoelasticity with convex or non-convex free energy function.

1. INTRODUCTION

In the last years the energetic investigation of rate-type viscoelastic constitutive equations has lead to a better description of phase transition phenomena as well as of plastic flow localization phenomena in metallic materials (see for example [4] and the references therein). This energetic study was initiated by Suliciu in [7] and extended in [1-3], [6] and [8]. The starting point in these studies was the existence of a free energy function for such constitutive equations, which is a requirement of the second law of thermodynamics. Moreover, when the free energy function is non-negative the total energy may be used as a global measure in estimating the solutions of certain initial and boundary value problems. Thus bounds in energy of these solutions as well as the continuous dependence on the input data can be derived.

For the 1-D case the stability in total energy was considered in [7] when the equilibrium curve is monotone and in [2], [8] when it is non-monotone.

For the 3-D case of small deformations in rate-type semilinear viscoelasticity with smooth monotone equilibrium hypersurface (convex free energy function). Suliciu [7] has discussed *the stability in total energy* and the uniqueness of smooth solutions when the boundary conditions correspond to an isolated body (i.e., a body which does not exchange energy with the external world through its boundary). In the present

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work we extend these results for *a non-isolated body problem with prescribed boundary motion* when the equilibrium hipersurface may be only continuous and non-monotone (i.e., non-convex free energy function).

The results in this paper are obtained with minimal restrictions on the constitutive functions since the free energy properties we use were established in [3] under necessary and sufficient conditions. Therefore they may apply to materials for which the stress-strain curve presents softening. Concerning the relaxation function $\mathbf{G}(\boldsymbol{\varepsilon},\boldsymbol{\sigma})$, as in [7], the essential assumption on it is the global Lipschitz continuity condition.

We remind that in [1] we have shown that a smooth boundary data in velocity $\mathbf{V}=\overline{\mathbf{V}}(\mathbf{x},t)$, $\mathbf{x}\in\Sigma$, $t\geq 0$, where Σ is the body boundary, seems to be not sufficient to obtain a bounded total energy of the body. We have therefore to impose additional restrictions on the boundary data, namely: $\overline{\mathbf{V}}(\cdot,t)$ has to belong to the Sobolev space $W_2^{1/2}(\Sigma)$, for any time $t\geq 0$. Moreover, in [1] we have given minimal and explicit conditions on the boundary data to obtain energetic bounds of the solutions. Here we prove that these conditions ensure the stability of the infinitesimal motions too.

2. STATEMENT OF THE PROBLEM

We consider a body which is identified with a bounded domain $D\subset\mathbf{R}^3$ and we denote by $\Sigma=\partial D$ its boundary, $\overline{D}=D\cup\Sigma$ and $I=[0,t_0)$ a time interval.

In this paper we consider a *rate-type semilinear viscoelastic material* subjected to small strains and small rotations. The system of PDEqs. describing such a motion is

$$\text{(1a)}\qquad \rho\frac{\partial\mathbf{v}}{\partial t}-\operatorname{div}\boldsymbol{\sigma}=\rho\mathbf{b};\quad \rho\frac{\partial v_i}{\partial t}-\frac{\partial\sigma_{ij}}{\partial x_j}=\rho b_i,\quad i=1,2,3$$

$$\text{(1b)}\qquad \frac{\partial\boldsymbol{\sigma}}{\partial t}-\mathcal{E}\frac{\partial\boldsymbol{\varepsilon}}{\partial t}=\mathbf{G}(\boldsymbol{\varepsilon},\boldsymbol{\sigma});\quad \frac{\partial\sigma_{ij}}{\partial t}-\mathcal{E}_{ijkl}\frac{\partial\varepsilon_{kl}}{\partial t}=G_{ij}(\boldsymbol{\varepsilon},\boldsymbol{\sigma}),\quad i,j,k,l=1,2,3$$

$$\text{(1c)}\qquad \frac{\partial\mathbf{u}}{\partial t}=\mathbf{v},\ \frac{\partial\mathbf{u}}{\partial\mathbf{x}}+\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T=2\boldsymbol{\varepsilon};\quad \frac{\partial u_i}{\partial t}=v_i,\quad \frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}=2\varepsilon_{ij},\quad i,j=1,2,3$$

where $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$ is the particle velocity, $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ is the displacement vector, $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\mathbf{x},t)$ is the infinitesimal deformation tensor, $\boldsymbol{\sigma}=\boldsymbol{\sigma}(\mathbf{x},t)$ denotes the Cauchy stress tensor, $\mathbf{b}=\mathbf{b}(\mathbf{x},t)$ is the body force and $\rho=\rho(\mathbf{x})$ is the mass density, for $(\mathbf{x},t)\in D\times I$.

This partial differential equations system, containing the balance law of momentum (1a) the constitutive equation (1b) and the kinematic relations (1c), must be thought as a semilinear system in the unknowns $\mathbf{u}$, $\mathbf{v}$ and $\boldsymbol{\sigma}$.

We consider for this system as initial-boundary value problem the following *non-isolated body problem with prescribed boundary motion*, i.e.,

(2) $$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \boldsymbol{\sigma}(\mathbf{x},0) = \boldsymbol{\sigma}_0(\mathbf{x}), \quad \text{for } \mathbf{x} \in D,$$

(3) $$\mathbf{v}(\mathbf{x},t) = \overline{\mathbf{V}}(\mathbf{x},t), \quad \text{for } (\mathbf{x},t) \in \Sigma \times I,$$

where the input data $(\mathbf{u}_0, \mathbf{v}_0, \boldsymbol{\sigma}_0)$ and $\overline{\mathbf{V}}$ are prescribed functions.

Since we use energetic properties of smooth solutions of problem (1)-(3) we require that the boundary Σ be smooth and the initial data fit with the boundary data such that shock and acceleration waves are not introduced into the body. Thus we require at least $\mathbf{u}_0 \in C^2(D) \cap C^0(\overline{D})$, $\mathbf{v}_0 \in C^1(D) \cap C^0(\overline{D})$, $\boldsymbol{\sigma}_0 \in C^1(D) \cap C^0(\overline{D})$, $\mathbf{b} \in C^0(\overline{D} \times I)$, $\rho \in C^0(\overline{D})$ and

(4) $$\overline{\mathbf{V}}, \frac{\partial \overline{\mathbf{V}}}{\partial t} \in C^0(\Sigma \times I).$$

3. CONSTITUTIVE ASSUMPTIONS

In the following we denote by $\mathcal{L}$ the set of all 3×3 tensors, $\mathcal{S}$ the set of all 3×3 symmetric tensors and $\mathrm{Lin}(\mathcal{S},\mathcal{S})$ the set of all linear mapping from $\mathcal{S}$ to $\mathcal{S}$. For $\mathbf{A}$, $\mathbf{B} \in \mathcal{L}$, $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^T)$, is the scalar product in $\mathcal{L}$ of $\mathbf{A}$ and $\mathbf{B}$, $|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2}$ is the norm of $\mathbf{A} \in \mathcal{L}$ and $\mathbf{x} \cdot \mathbf{y} = x_i y_i$, $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$ is the scalar product in $\mathbf{R}^3$.

In this paper we consider materials defined by *rate-type semilinear viscoelastic constitutive equations* of the form (1b) with the properties:

($\mathbf{h_1}$) $\mathcal{E} \in \mathrm{Lin}(\mathcal{S},\mathcal{S})$, $\mathcal{E} = \text{const.}$, $\mathcal{E} = \mathcal{E}^T$, $\mathcal{E}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} > 0$ for any $\boldsymbol{\varepsilon} \in \mathcal{S}$, $\boldsymbol{\varepsilon} \neq \mathbf{0}$.

($\mathbf{h_2}$) There is a hypersurface $\boldsymbol{\sigma}_R : \mathcal{D}_{\boldsymbol{\varepsilon}} \subset \mathcal{S} \to \mathcal{S}$, $\boldsymbol{\sigma}_R \in C^0(\mathcal{D}_{\boldsymbol{\varepsilon}})$ where $\mathcal{D}_{\boldsymbol{\varepsilon}}$ is a simply connected open set, such that $\mathbf{0} \in \mathcal{D}_{\boldsymbol{\varepsilon}}$, $\boldsymbol{\sigma}_R(\mathbf{0}) = \mathbf{0}$ and

(5) $$(\boldsymbol{\sigma}_R(\boldsymbol{\varepsilon}) - \boldsymbol{\sigma}_R(\tilde{\boldsymbol{\varepsilon}})) \cdot (\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}) < \mathcal{E}(\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}) \cdot (\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}), \quad \text{for any } \boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}} \in \mathcal{D}_{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon} \neq \tilde{\boldsymbol{\varepsilon}},$$

(6) $$\int_{\mathcal{C}} \boldsymbol{\sigma}_R(\boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{\varepsilon} \text{ is path independent for any curve } \mathcal{C} \subset \mathcal{D}_{\boldsymbol{\varepsilon}}.$$

($\mathbf{h_3}$) $\mathbf{G} : \tilde{\mathcal{D}} \subset \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $\mathbf{G} \in C^0(\tilde{\mathcal{D}})$, $\tilde{\mathcal{D}} = \{(\boldsymbol{\varepsilon}, \mathcal{E}(\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}) + \boldsymbol{\sigma}_R(\tilde{\boldsymbol{\varepsilon}}) \mid \boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}} \in \mathcal{D}_{\boldsymbol{\varepsilon}}\}$

(7) $$\mathbf{G}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \mathbf{0} \text{ if and only if } \boldsymbol{\sigma} = \boldsymbol{\sigma}_R(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} \in \mathcal{D}_{\boldsymbol{\varepsilon}}.$$

(8) $$\mathbf{G}(\boldsymbol{\varepsilon}, \mathcal{E}(\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}) + \boldsymbol{\sigma}_R(\tilde{\boldsymbol{\varepsilon}})) \cdot (\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}) \leqslant 0, \quad \text{for any } \boldsymbol{\varepsilon}, \tilde{\boldsymbol{\varepsilon}} \in \mathcal{D}_{\boldsymbol{\varepsilon}}.$$

$(\mathbf{h_4})$ The hypersurface $\sigma = \sigma_R(\boldsymbol{\varepsilon})$ is stable with respect to any relaxation process in $\widetilde{\mathcal{D}}$, i.e., for a solution of the Cauchy problem $\dot{\sigma}(t) = \mathbf{G}(\boldsymbol{\varepsilon}_0, \sigma(t))$, $\sigma(0) = \sigma_0$ where $(\boldsymbol{\varepsilon}_0, \sigma_0) \in \widetilde{\mathcal{D}}$, we have $(\boldsymbol{\varepsilon}_0, \sigma(t)) \in \widetilde{\mathcal{D}}$, for any $t \in [0, \infty)$ and $\lim\limits_{t\to\infty} |\sigma(t) - \sigma_R(\boldsymbol{\varepsilon}_0)| = 0$.

These assumptions allow us to consider a very large class of models, including those with non-monotone and only continuous equilibrium hypersurfaces $\sigma_R(\boldsymbol{\varepsilon})$, (i.e., with non-convex free energy functions). A detailed discussion concerning their physical meaning can be found in [3]. Moreover, hypotheses $(\mathbf{h_1})$-$(\mathbf{h_4})$, ensure the existence and uniqueness of a *free energy function* for our mechanical model (1b) (see [3]). This means, there is a function $\psi = \psi(\boldsymbol{\varepsilon}, \sigma)$, $\psi : \widetilde{\mathcal{D}} \to \mathbf{R}$, $\psi \in C^1(\widetilde{\mathcal{D}})$, such that for any pair $(\boldsymbol{\varepsilon}(t), \sigma(t)) \in \widetilde{\mathcal{D}}$, $t \in I$, which satisfies Eq. (1b), we have $\rho\dot{\psi}(\boldsymbol{\varepsilon}(t), \sigma(t)) \leqslant \sigma(t) \cdot \dot{\boldsymbol{\varepsilon}}(t)$, for any $t \in I$. According to [3], this function is the unique solution of the following overdetermined system and inequality

$$\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} + \mathcal{E}\frac{\partial\psi}{\partial\sigma} = \frac{1}{\rho}\sigma, \quad \frac{\partial\psi}{\partial\sigma} \cdot \mathbf{G} \leqslant 0, \text{ for any } (\boldsymbol{\varepsilon}, \sigma) \in \widetilde{\mathcal{D}}, \text{ with } \psi(\mathbf{0}, \mathbf{0}) = 0. \tag{9}$$

Moreover, in [7] (see also [3]) the free energy is shown to have the form

$$\rho\psi(\boldsymbol{\varepsilon}, \sigma) = \frac{1}{2}\mathcal{E}^{-1}\sigma \cdot \sigma + \varphi(\sigma - \mathcal{E}\boldsymbol{\varepsilon}), \quad \varphi(0) = 0 \tag{10}$$

and the properties of the smooth function φ are also studied.

4. A CHANGE OF DEPENDENT VARIABLE

In order to investigate the non-isolated body problem (1)-(3) we make the following change of dependent variable

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\mathbf{x}, t) + \mathbf{w}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in D \times I, \tag{11}$$

where the function $\mathbf{V}(\mathbf{x}, t)$ has at least the properties that $\mathbf{V}$ and $\partial\mathbf{V}/\partial t \in C^0(\overline{D} \times I)$ and satisfies the boundary condition (3), i.e., $\mathbf{V}(\mathbf{x}, t) = \overline{\mathbf{V}}(\mathbf{x}, t)$ for any $(\mathbf{x}, t) \in \Sigma \times I$. Moreover, such a prolongation of the boundary data (3) should possess certain continuous dependence properties with respect to the boundary values.

Thus we reduce the initial-boundary value problem (1)-(3) to

$$\begin{gathered}
\rho\frac{\partial\mathbf{w}}{\partial t} - \operatorname{div}\sigma = \rho\overline{\mathbf{b}}, \\
\frac{\partial\sigma}{\partial t} - \mathcal{E}\frac{\partial\boldsymbol{\varepsilon}}{\partial t} = \mathbf{G}(\boldsymbol{\varepsilon}, \sigma), \\
\frac{\partial\mathbf{u}}{\partial t} = \mathbf{w} + \mathbf{V}, \quad \frac{\partial\mathbf{u}}{\partial\mathbf{x}} + \left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T = 2\boldsymbol{\varepsilon},
\end{gathered} \tag{12}$$

$$(13)\qquad \mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\quad \mathbf{v}(\mathbf{x},0)=\mathbf{v}_0(\mathbf{x})-\mathbf{V}(\mathbf{x},0),\quad \sigma(\mathbf{x},0)=\sigma_0(\mathbf{x}),\ \text{for } \mathbf{x}\in D,$$

$$(14)\qquad \mathbf{w}(\mathbf{x},t)=0,\ \text{for } (\mathbf{x},t)\in\Sigma\times I,$$

where

$$(15)\qquad \overline{\mathbf{b}}(\mathbf{x},t)=\mathbf{b}(\mathbf{x},t)-\frac{\partial\mathbf{V}}{\partial t}(\mathbf{x},t),\ \text{for } (\mathbf{x},t)\in D\times I.$$

Such a smooth prolongation of the boundary condition was proposed in [1] as the solution of the Dirichlet problem for the homogeneous Laplace equation with prescribed boundary values (3) depending on a parameter (time) $t\in I$.

The following Lemma characterizes the continuous dependence properties of this harmonic function $\mathbf{V}=\mathbf{V}(\mathbf{x},t)$, $(\mathbf{x},t)\in D\times I$ with respect to a norm of the boundary data (see for instance [5], Chap. 5, Sect. 3.5). In fact it requires that the boundary data $\overline{\mathbf{V}}(\cdot,t)$ must belong to the Sobolev space $W_2^{1/2}(\Sigma)$ for any time $t\in I$.

LEMMA 1 (The prolongation of the boundary condition). *If Σ is of C^1 class and $\overline{\mathbf{V}}(\mathbf{x},t)$, $(\mathbf{x},t)\in\Sigma\times I$ satisfies condition (4) and*

$$(16)\qquad \mathcal{I}(\overline{\mathbf{V}}(\cdot,t))=\int_\Sigma\int_\Sigma\frac{|\overline{\mathbf{V}}(\mathbf{y},t)-\overline{\mathbf{V}}(\mathbf{x},t)|^2}{|\mathbf{y}-\mathbf{x}|^{m+1}}\mathrm{d}\mathbf{a}_x\mathrm{d}\mathbf{a}_y<\infty,\ \text{for any } t\in I,$$

where $m\in\mathbf{N}$ is the dimension of Σ, then

$$\left\{\int_D|\mathbf{V}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\leqslant C_1\left\{\int_\Sigma|\overline{\mathbf{V}}(\mathbf{x},t)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2},$$

$$(17)\qquad \left\{\int_D\left|\frac{\partial\mathbf{V}}{\partial t}(\mathbf{x},t)\right|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\leqslant C_1\left\{\int_\Sigma\left|\frac{\partial\overline{\mathbf{V}}}{\partial t}(\mathbf{x},t)\right|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2},$$

$$\left\{\int_D\left|\frac{\partial\mathbf{V}}{\partial\mathbf{x}}(\mathbf{x},t)\right|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\leqslant C_2\left\{\int_\Sigma|\overline{\mathbf{V}}(\mathbf{x},t)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2}+C_3\{\mathcal{I}(\overline{\mathbf{V}}(\cdot,t))\}^{1/2},$$

for any $t\in I$, where C_1, C_2, C_3 are positive constants which depend on domain only.

5. SOME ENERGETIC PROPERTIES OF SOLUTIONS

In [7] it is observed that the free energy function $\psi=\psi(\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ can be used to derive an energy identity verified by the smooth solutions of the governing system of PDEqs. According to [1] we have the following results.

For any smooth solution of (1) on $D \times I$, the energy identity

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}+\psi(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\right)\right\}-\operatorname{div}(\boldsymbol{\sigma}\mathbf{v})-\rho\frac{\partial\psi}{\partial\boldsymbol{\sigma}}\cdot\mathbf{G}=\rho\mathbf{b}\cdot\mathbf{v}, \tag{18a}$$

holds, while for any smooth solution of (12) on $D \times I$, the energy identity is

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{\mathbf{w}\cdot\mathbf{w}}{2}+\psi(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\right)\right\}-\operatorname{div}(\boldsymbol{\sigma}\mathbf{w})-\rho\frac{\partial\psi}{\partial\boldsymbol{\sigma}}\cdot\mathbf{G}-\boldsymbol{\sigma}\cdot\frac{\partial\mathbf{V}}{\partial\mathbf{x}}=\rho\overline{\mathbf{b}}\cdot\mathbf{w}. \tag{18b}$$

Define the total energy $E_v(t)$ of body D at time t by

$$E_v(t)=E_v(\mathbf{v}(\cdot,t),\boldsymbol{\varepsilon}(\cdot,t),\boldsymbol{\sigma}(\cdot,t))=\int_D\rho(\mathbf{x})\left\{\frac{|\mathbf{v}|^2}{2}+\psi(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\right\}(\mathbf{x},t)\mathrm{d}\mathbf{x}, \tag{19a}$$

i.e., the total energy at time t corresponding to the solution $(\mathbf{v},\boldsymbol{\sigma},\boldsymbol{\varepsilon})(\mathbf{x},t)$, $(\mathbf{x},t)\in$ $\in D\times I$ of problem (1)-(3). Also define by

$$E_w(t)=E_w(\mathbf{w}(\cdot,t),\boldsymbol{\varepsilon}(\cdot,t),\boldsymbol{\sigma}(\cdot,t))=\int_D\rho(\mathbf{x})\left\{\frac{|\mathbf{w}|^2}{2}+\psi(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\right\}(\mathbf{x},t)\mathrm{d}\mathbf{x}, \tag{19b}$$

the total energy at time t corresponding to the solution $(\mathbf{w},\boldsymbol{\sigma},\boldsymbol{\varepsilon})(\mathbf{x},t)$, $(\mathbf{x},t)\in D\times I$ of problem (12)-(14).

By using Schwarz inequality we derive the following inequalities which connect the total energies $E_v(t)$ and $E_w(t)$

$$\begin{aligned}&\sqrt{E_v(t)}\leqslant\sqrt{E_w(t)}+\left\{\int_D\frac{\rho(\mathbf{x})}{2}|\mathbf{V}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}, \text{ for any } t\in I,\\&\sqrt{E_w(t)}\leqslant\sqrt{E_v(t)}+\left\{\int_D\frac{\rho(\mathbf{x})}{2}|\mathbf{V}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}, \text{ for any } t\in I.\end{aligned} \tag{20}$$

Let us note that the energy identities (18) holds for any solution of the free energy equation $(9)_1$ i.e., for any function having the general form (10). The free energy function has the supplementary property that it satisfies the residual inequality $(9)_2$ which can be exploited to derive meaningly energy estimates of the solution (see for example [1]).

6. STABILITY OF SMOOTH SOLUTIONS

In what follows we establish the continuous dependence in total energy of smooth solutions of a non-isolated body problem with prescribed boundary motion (1)-(3) with respect to the initial data $(\mathbf{u}_0,\mathbf{v}_0,\boldsymbol{\sigma}_0)(\mathbf{x})$, $\mathbf{x}\in D$, the boundary data $\overline{\mathbf{V}}(\mathbf{x},t)$,

$(\mathbf{x},t)\in\Sigma\times I$ and the body force $\mathbf{b}=\mathbf{b}(\mathbf{x},t)$, $(\mathbf{x},t)\in D\times I$. For simplicity we assume here that $\mathcal{D}_{\boldsymbol{\varepsilon}}\equiv\mathcal{S}$.

PROPOSITION 1. *Let the constitutive function $\mathbf{G}=\mathbf{G}(\boldsymbol{\varepsilon},\boldsymbol{\sigma})$ is Lipschitz continuous on D and the following conditions are satisfied:*

(a) *the boundary data Σ is of C^1 class;*

(b) *the initial data (2), the body force $\mathbf{b}$ and the mass density ρ are such that*

$$E_v(0)=\int_D\rho\left\{\frac{|v_0|^2}{2}+\psi(\boldsymbol{\varepsilon}_0,\boldsymbol{\sigma}_0)\right\}\mathrm{d}\mathbf{x}<\infty;\ \int_I\left(\int_D\rho|\mathbf{b}|^2\mathrm{d}\mathbf{x}\right)^{1/2}\mathrm{d}t<\infty; \tag{21}$$

(c) *the boundary data $\overline{\mathbf{V}}=\overline{\mathbf{V}}(\mathbf{x},t)$, $(\mathbf{x},t)\in\Sigma\times I$ satisfies condition (4) and the additional restrictions*

$$\begin{gathered}\int_I\left\{\int_\Sigma|\overline{\mathbf{V}}(\mathbf{x},t)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2}\mathrm{d}t<\infty;\ \int_I\left\{\int_\Sigma\left|\frac{\partial\overline{\mathbf{V}}}{\partial t}(\mathbf{x},t)\right|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2}\mathrm{d}t<\infty;\\ \int_I\mathcal{I}(\overline{\mathbf{V}}(\cdot,t))^{1/2}\mathrm{d}t=\int_I\left\{\int_\Sigma\int_\Sigma\frac{|\overline{\mathbf{V}}(\mathbf{y},t)-\overline{\mathbf{V}}(\mathbf{x},t)|^2}{|\mathbf{y}-\mathbf{x}|^{m+1}}\mathrm{d}\mathbf{a}_x\mathrm{d}\mathbf{a}_y\right\}^{1/2}\mathrm{d}t<\infty,\end{gathered} \tag{22}$$

where m is the dimension of Σ;

(d) *there exists $\mathcal{E}_i\in Lin(\mathcal{S},\mathcal{S})$, $\mathcal{E}_i=const.$, $\mathcal{E}_i=\mathcal{E}_i^T$, $i=1,2$*

$$0<\mathcal{E}_1\boldsymbol{\eta}\cdot\boldsymbol{\eta}<\mathcal{E}\boldsymbol{\eta}\cdot\boldsymbol{\eta}<\mathcal{E}_2\boldsymbol{\eta}\cdot\boldsymbol{\eta},\ \text{for any } \boldsymbol{\eta}\in\mathcal{S},\boldsymbol{\eta}\neq\mathbf{0}. \tag{23}$$

such that the potential $\rho\psi_R(\boldsymbol{\varepsilon})$ of the equilibrium hypersurface $\boldsymbol{\sigma}_R(\boldsymbol{\varepsilon})=\rho\partial\psi_R(\boldsymbol{\varepsilon})/\partial\boldsymbol{\varepsilon}$ satisfies the inequality

$$\frac{1}{2}\mathcal{E}_1\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\leqslant\rho\psi_R(\boldsymbol{\varepsilon})\leqslant\frac{1}{2}\mathcal{E}_2\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\ \text{for any } \boldsymbol{\varepsilon}\in\mathcal{D}_{\boldsymbol{\varepsilon}}. \tag{24}$$

Then the smooth solution of the problem (1) - (3) is unique and depends continuously in the total energy $E_v(t)$ on the input data.

Proof. Let us consider two sets of input data

$$\begin{gathered}(\mathbf{u}_0^k,\mathbf{v}_0^k,\boldsymbol{\sigma}_0^k)(\mathbf{x}),\mathbf{x}\in D,\quad \mathbf{b}^k(\mathbf{x},t),(\mathbf{x},t)\in D\times I,\\ \overline{\mathbf{V}}^k(\mathbf{x},t),(\mathbf{x},t)\in\Sigma\times I,\qquad k=1,2,\end{gathered} \tag{25}$$

which verify conditions (b) and (c). We suppose that for each set of the input data there is at least a global smooth solution $(\mathbf{u}^k,\mathbf{v}^k,\boldsymbol{\sigma}^k)(\mathbf{x},t)$, $(\mathbf{x},t)\in D\times I$, $k=1,2$ of problem (1)-(3). We denote

$$\begin{aligned}&(\mathbf{u},\mathbf{v},\boldsymbol{\sigma},\boldsymbol{\varepsilon})(\mathbf{x},t)=(\mathbf{u}^1-\mathbf{u}^2,\mathbf{v}^1-\mathbf{v}^2,\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2,\boldsymbol{\varepsilon}^1-\boldsymbol{\varepsilon}^2)(\mathbf{x},t),\quad(\mathbf{x},t)\in D\times I,\\ &(\mathbf{u}_0,\mathbf{v}_0,\boldsymbol{\sigma}_0,\boldsymbol{\varepsilon}_0)(\mathbf{x})=(\mathbf{u}_0^1-\mathbf{u}_0^2,\mathbf{v}_0^1-\mathbf{v}_0^2,\boldsymbol{\sigma}_0^1-\boldsymbol{\sigma}_0^2,\boldsymbol{\varepsilon}_0^1-\boldsymbol{\varepsilon}_0^2)(\mathbf{x}),\qquad\mathbf{x}\in D,\\ &\overline{\mathbf{V}}(\mathbf{x},t)=(\overline{\mathbf{V}}^1-\overline{\mathbf{V}}^2)(\mathbf{x},t),(\mathbf{x},t)\in\Sigma\times I,\\ &\mathbf{b}(\mathbf{x},t)=(\mathbf{b}^1-\mathbf{b}^2)(\mathbf{x},t),(\mathbf{x},t)\in D\times I,\end{aligned} \tag{26}$$

$$\mathbf{G}^*(\mathbf{x},t) = \mathbf{G}(\boldsymbol{\varepsilon}^1(\mathbf{x},t),\boldsymbol{\sigma}^1(\mathbf{x},t)) - \mathbf{G}(\boldsymbol{\varepsilon}^2(\mathbf{x},t),\boldsymbol{\sigma}^2(\mathbf{x},t)), \quad (\mathbf{x},t) \in D \times I. \tag{27}$$

By performing the change of dependent variable (11) where $\mathbf{V} = \mathbf{V}(\mathbf{x},t)$ is the harmonic function which satisfies the boundary condition $(26)_3$, we get that $(\mathbf{u},\mathbf{w},\boldsymbol{\sigma})$ verifies the initial-boundary value problem (12)-(14) with $\mathbf{G}$ substituted by $\mathbf{G}^*$ and $\overline{\mathbf{b}}$ defined by $(15)+(26)_4$.

Let us introduce the function

$$\rho\psi^*(\sigma,\boldsymbol{\varepsilon}) = \frac{1}{2}\mathcal{E}^{-1}\sigma\cdot\sigma + \mathcal{A}^*(\sigma - \mathcal{E}\boldsymbol{\varepsilon})\cdot(\sigma - \mathcal{E}\boldsymbol{\varepsilon}), \quad (\sigma,\boldsymbol{\varepsilon}) \in \widetilde{\mathcal{D}}, \tag{28}$$

where $\mathcal{A}^* \in \mathrm{Lin}(\mathcal{S},\mathcal{S})$, $\mathcal{A}^* = (\mathcal{A}^*)^T$, $\mathcal{A}^*\eta\cdot\eta > 0$ for any $\eta \in \mathcal{S}$, $\eta \neq \mathbf{0}$.

It is obvious that $\psi^* = \psi^*(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ is a solution of the free energy equation $(9)_1$ and $\sqrt{\rho\psi^*(\boldsymbol{\varepsilon},\boldsymbol{\sigma})}$ is equivalent with an Euclidean norm on $\mathcal{S}\times\mathcal{S}$.

Let us note that for the set of functions $\mathbf{w}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ defined by relations (26) and (11), there holds the energy identity (18b) with $\mathbf{G}$ substituted by $\mathbf{G}^*$ and ψ substituted by ψ^*.

Integrating that identity with respect to $\mathbf{x}$ on D and taking into account the boundary condition (14) we obtain

$$\frac{\mathrm{d}E_w^*(t)}{\mathrm{d}t} = \int_D \rho\overline{\mathbf{b}}\cdot\mathbf{w}\mathrm{d}\mathbf{x} + \int_D \sigma\cdot\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathrm{d}\mathbf{x} + \int_D \rho\frac{\partial\psi^*}{\partial\sigma}\cdot\mathbf{G}^*\mathrm{d}\mathbf{x}, \tag{29}$$

where $E_w^*(t)$ is defined according to (19b) with ψ substituted by ψ^*.

In this case $\sqrt{E_w^*(t)}$ is equivalent to a L^2-norm for the functions $(\mathbf{w}(\cdot,t)$, $\boldsymbol{\varepsilon}(\cdot,t)$, $\sigma(\cdot,t)) : D \to \mathbf{R}^3\times\mathcal{S}\times\mathcal{S}$ depending on a parameter t.

Using the norm $\sqrt{\rho\psi^*(\boldsymbol{\varepsilon},\boldsymbol{\sigma})}$ we can now write the estimates

$$|\mathbf{G}^*(\mathbf{x},t)| \leqslant M_1\sqrt{\rho\psi^*(\boldsymbol{\varepsilon},\boldsymbol{\sigma})}, \quad \rho\left|\frac{\partial\psi^*}{\partial\sigma}(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\right| \leqslant M_2\sqrt{\rho\psi^*(\boldsymbol{\varepsilon},\boldsymbol{\sigma})}, \tag{30}$$

where M_1 and M_2 are two positive constants which come out from the equivalence of two Euclidean norms on $\mathcal{S}\times\mathcal{S}$. Let us note that relation $(30)_1$ is just the Lipschitz condition on function $\mathbf{G}$.

By using Schwarz inequality and estimates (30) and (17), from Lemma 1 we get

$$\int_D \rho\frac{\partial\psi^*}{\partial\sigma}\cdot\mathbf{G}^*\mathrm{d}\mathbf{x} \leqslant \left\{\int_D \rho^2\left|\frac{\partial\psi^*}{\partial\sigma}\right|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_D |\mathbf{G}^*|^2\mathrm{d}\mathbf{x}\right\}^{1/2} \leqslant M_1M_2E_w^*(t), \tag{31}$$

$$\begin{aligned}\int_D \sigma\cdot\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathrm{d}\mathbf{x} &\leqslant \left\{\int_D \frac{1}{2}\mathcal{E}^{-1}\sigma\cdot\sigma\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_D 2\mathcal{E}\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathrm{d}\mathbf{x}\right\}^{1/2} \leqslant \\ &\leqslant \sqrt{2}\sqrt{E_w^*(t)}\left\{C_2\left\{\int_\Sigma |\overline{\mathbf{V}}(\mathbf{x},t)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2} + C_3\{\mathcal{I}(\overline{\mathbf{V}}(\cdot,t))\}^{1/2}\right\}.\end{aligned} \tag{32}$$

$$\int_D \rho\overline{\mathbf{b}}\cdot\mathbf{w}\mathrm{d}\mathbf{x} \leqslant \left\{\int_D \frac{\rho}{2}|\mathbf{w}|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_D 2\rho|\overline{\mathbf{b}}|^2\mathrm{d}\mathbf{x}\right\}^{1/2} \leqslant \tag{33}$$
$$\leqslant \max_{\mathbf{x}\in\overline{D}}\sqrt{2\rho(\mathbf{x})}\sqrt{E_w^*(t)}\left\{\left\{\int_D |\mathbf{b}|^2\mathrm{d}\mathbf{x}\right\}^{1/2} + C_1\left\{\int_\Sigma \left|\frac{\partial\overline{\mathbf{V}}}{\partial t}(\mathbf{x},t)\right|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2}\right\},$$

where C_1, C_2, C_3 are positive constants which depend on domain D only.

Therefore $E_w^*(t)$ satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{E_w^*(t)} \leqslant M\sqrt{E_w^*(t)} + N(t) \tag{34}$$

for any $t\in I$, where

$$N(t) = \max_{\mathbf{x}\in\overline{D}}\sqrt{\frac{\rho(\mathbf{x})}{2}}\left\{\left\{\int_D |\mathbf{b}|^2\mathrm{d}\mathbf{x}\right\}^{1/2} + C_1\left\{\int_\Sigma \left|\frac{\partial\overline{\mathbf{V}}}{\partial t}(\mathbf{x},t)\right|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2}\right\} + \tag{35}$$
$$+\frac{C_2}{\sqrt{2}}\left\{\int_\Sigma |\overline{\mathbf{V}}(\mathbf{x},t)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2} + \frac{C_3}{\sqrt{2}}\{\mathcal{I}(\overline{\mathbf{V}}(\cdot,t))\}^{1/2}$$

and $M = M_1M_2/2$.

Thus

$$\sqrt{E_w^*(t)} \leqslant \left\{\sqrt{E_w^*(0)} + \int_0^t N(\tau)\exp(-M\tau)\mathrm{d}\tau\right\}\exp(Mt) \tag{36}$$

for any $t\in I$.

By using inequalities (20) and estimates (17) from Lemma 1, we obtain the following bound in total energy $E_v^*(t)$ corresponding to the solution $\mathbf{v}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$

(37)
$$\sqrt{E_v^*(t)} \leqslant \left\{\sqrt{E_v^*(0)} + C_1\max_{\mathbf{x}\in\overline{D}}\sqrt{\frac{\rho(\mathbf{x})}{2}}\left\{\int_\Sigma |\overline{\mathbf{V}}(\mathbf{x},0)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2} + \right.$$
$$\left. + \int_0^t N(\tau)\exp(-M\tau)\mathrm{d}\tau\right\}\exp(Mt) + C_1\max_{\mathbf{x}\in\overline{D}}\sqrt{\tfrac{\rho(\mathbf{x})}{2}}\left\{\int_\Sigma |\overline{\mathbf{V}}(\mathbf{x},t)|^2\mathrm{d}\mathbf{a}_x\right\}^{1/2},\ t\in I.$$

The inequality (37) implies the continuous dependence in the total energy $E_v^*(t)$ of the of smooth solutions of the initial-boundary value problem with prescribed boundary motion (1)-(3) with respect to the input data on any finite time interval. It also implies the *uniqueness* of smooth solutions of problem (1)-(3).

According to Proposition 2 in [3], relations (23)+(24) are necessary and sufficient conditions for the total energy $E_v(t)$ of the body D at time t to be equivalent to $E_v^*(t)$ in the following sense: there are two positive constants a_1, a_2, $a_1 < a_2$, such that $a_1E_v^*(t) \leqslant E_v(t) \leqslant a_2E_v^*(t)$, for any $t\in I$. Then, obviously, the smooth solution of problem (1)-(3) is continuously dependent on the input data with respect to the total energy of the body. This completes the proof.

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