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CONVEX AND NON-CONVEX ENERGY IN THREE-DIMENSIONAL RATE-TYPE SEMILINEAR VISCOELASTICITY

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Abstract—Existence and uniqueness of a free energy function is studied for rate-type semilinear viscoelastic constitutive equations with monotone or non-monotone equilibrium surface, in the 3-D case. The free energy may be consequently convex or non-convex; its existence is proved by effective construction. Necessary and sufficient conditions on the constitutive functions are given such that the free energy be non-negative and "equivalent" to a Euclidean norm.

1. INTRODUCTION

Rate-type viscoelastic and viscoplastic constitutive equation with a Maxwell's type viscosity, have been extensively investigated in several recent works (see for instance [1–5]). The starting point in these studies was the existence of a free energy function for such constitutive equations, which is a requirement of the second law of thermodynamics. Consequently, such a requirement had to impose certain restrictions on the constitutive functions of the model (see for instance [2, 6]). But free energy proved to be even more important. Indeed, when the free energy function is non-negative, the total energy may be used as a measure in estimating the solutions of certain initial and boundary value problems; a bound in energy of these solutions may be easier to get as well as the continuous dependence on the input data. Moreover the same energy allows to establish an L^p-approach ($p \geq 1$) to equilibrium of the solution when the Maxwell's viscosity coefficient tends to infinity (see for instance [1, 3, 8]). This last property opened a way to obtain the solutions to elastic (and plastic) problems, which may not be always hyperbolic, through the solutions to viscoelastic (and viscoplastic) problems, which are hyperbolic. In the last years the energetic study of rate-type viscoelastic constitutive equations seems to lead to a better description of the phase transition phenomenon by means of such constitutive equations with a non-monotone equilibrium curve (see for instance [7, 8]).

An energetic investigation of such rate-type viscoelastic constitutive equations with non-monotone equilibrium surface and consequently with non-convex free energy, is done in [5] for the 1-D case only. In the present work we extend these results for the 3-D case with large deformations. The viscoelastic constitutive equation is considered semilinear (i.e. with elastic linear instantaneous response) and the equilibrium hypersurface is continuous and may be non-monotone. The main result consists in establishing necessary and sufficient conditions imposed upon the constitutive functions such that a unique free energy, compatible with the second law of thermodynamics, exists; moreover, the existence proof is given by an effective construction of the free energy function. We also establish necessary and sufficient conditions for this free energy function to be non-negative and to be "equivalent" to (or to lie between) two Euclidean norms. Another result concerns the change in free energy that results from a charge in the equilibrium hypersurface. All the conditions (restrictions) we find are expressed by means of the equilibrium hypersurface (the equilibrium constitutive equation) and the dynamic elastic moduli and may be tested in applications; what is more important, none of the restrictions imposes the equilibrium hypersurface to be monotone, while the relaxation moduli may be only continuous and therefore the relaxation time may be finite as well as infinite.

In another work the results of the present paper are used to study the solution of a non-isolated body problem with prescribed boundary motion.

2. NOTATION

$\mathscr{L}$—the set of all 3×3 tensors;
$\mathscr{S}$—the set of all symmetric 3×3 tensors;
$\mathrm{Lin}(\mathscr{S},\mathscr{S})$—the set of all linear mappings from $\mathscr{S}$ to $\mathscr{S}$;
$\mathbf{F}$—the deformation gradient, $\mathbf{F}\in\mathscr{L}$, $\mathscr{J}\in\det\mathbf{F}>0$;
$\tilde{\mathbf{T}}$—the Cauchy stress tensor, $\tilde{\mathbf{T}}\in\mathscr{S}$;
ρ_a—the actual mass density, $\rho_a>0$;
ρ—the mass density in the reference configuration, $\rho>0$, $\rho_a\mathscr{J}=\rho$;
$\mathbf{T}$—the second Piola–Kirchhoff stress tensor, $\mathbf{T}\in\mathscr{S}$,

$$\mathbf{T}=\mathscr{J}\mathbf{F}^{-1}\tilde{\mathbf{T}}(\mathbf{F}^{\mathrm{T}})^{-1};$$

$\mathbf{E}$—the strain tensor $\mathbf{E}\in\mathscr{S}$ given by

$$\mathbf{E}=\frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{1})$$

which is conjugated to $\mathbf{T}$ with respect to the stress power

$$P=\frac{1}{\rho}\mathbf{T}\cdot\dot{\mathbf{E}}\left(\equiv\frac{1}{\rho}\operatorname{tr}(\mathbf{T}\dot{\mathbf{E}}^{\mathrm{T}})\right).$$

Both $\mathbf{E}$ and $\mathbf{T}$ are considered as functions of the time t, for each particle X of the body, and the dot denotes the time derivative.

3. THE CONSTITUTIVE MODEL

Let us consider the rate-type semilinear viscoelastic constitutive equation

$$\dot{\mathbf{T}}=\mathscr{E}\dot{\mathbf{E}}+\mathbf{G}(\mathbf{E},\mathbf{T}) \tag{1}$$

together with the following constitutive assumptions:
(i_1) $\mathscr{E}\in\mathrm{Lin}(\mathscr{S},\mathscr{S})$, $\mathscr{E}=\text{const.}$, $\mathscr{E}\mathbf{E}\cdot\mathbf{E}>0$ for any $\mathbf{E}\in\mathscr{S}$, $\mathbf{E}\neq\mathbf{0}$;
(i_2) $(\mathbf{E},\mathbf{T}_R(\mathbf{E}))$ is a hypersurface in $\mathscr{S}\times\mathscr{S}$, $\mathbf{T}_R:\mathscr{D}_E\to\mathscr{S}$, $\mathbf{T}_R\in C^0(\mathscr{D}_E)$, $\mathscr{D}_E$ a simply connected open set in $\mathscr{S}$, $\mathbf{T}_R(\mathbf{0})=\mathbf{0}$;
(i_3) $\mathbf{G}:\mathscr{D}\subset\mathscr{S}\times\mathscr{S}\to\mathscr{S}$, $\mathbf{G}\in C^0(\mathscr{D})$ where $\mathscr{D}$ is a domain with

$$\mathscr{D}\supset\tilde{\mathscr{D}}=\{(\mathbf{E},\mathscr{E}(\mathbf{E}-\tilde{\mathbf{E}})+\mathbf{T}_R(\tilde{\mathbf{E}}));\mathbf{E},\tilde{\mathbf{E}}\in\mathscr{D}_E\}$$

$$\mathscr{D}_E=\mathrm{pr}_E\,\mathscr{D}=\{\mathbf{E}\in\mathscr{S};\text{ there exists }\mathbf{T}\in\mathscr{S}\text{ with }(\mathbf{E},\mathbf{T})\in\mathscr{D}\}\ni\mathbf{0} \tag{2}$$

and

$$\mathbf{G}(\mathbf{E},\mathbf{T})=\mathbf{0}\quad\text{iff}\quad\mathbf{T}=\mathbf{T}_R(\mathbf{E}),\qquad\mathbf{E}\in\mathscr{D}_E; \tag{3}$$

(i_4) The hypersurface $\mathbf{T}=\mathbf{T}_R(\mathbf{E})$ is stable with respect to any relaxation process in $\mathscr{D}$, i.e. the solution of the Cauchy problem

$$\dot{\mathbf{T}}(t)=\mathbf{G}(\mathbf{E}_0,\mathbf{T}(t)),\qquad\mathbf{T}(0)=\mathbf{T}_0,\qquad(\mathbf{E}_0,\mathbf{T}_0)\in\mathscr{D} \tag{4}$$

has the properties: $(\mathbf{E}_0,\mathbf{T}(t))\in\mathscr{D}$ for any $t>0$ and

$$\lim_{t\to\infty}|\mathbf{T}(t)-\mathbf{T}_R(\mathbf{E}_0)|=0. \tag{5}$$

The constitutive assumptions (i_1)–(i_4) reflect some experimental remarks on the behaviour of certain materials. Thus (i_1) ensures an elastic linear instantaneous response and real acceleration waves propagating over the undisturbed state ($\mathbf{E}=\mathbf{0},\mathbf{T}=\mathbf{0}$). However, unlike the 1-D case, the hyperbolic character of the governing system of equations is not granted (see [9] where the loss of hyperbolicity for a physically linear but geometrically non-linear constitutive equation (1) is discussed). According to (i_3) $\mathbf{G}$ vanishes only on a hypersurface (the equilibrium region), the constitutive equation (1) is therefore viscoelastic; moreover, by (i_4), any relaxation process starting at a state in $\mathscr{D}$ is required to remain in $\mathscr{D}$ and to end on the equilibrium hypersurface $\mathbf{T}=\mathbf{T}_R(\mathbf{E})$ after a finite or infinite time interval. Condition (2) requires $\mathscr{D}$ to contain any instantaneous process starting at a point on the equilibrium hypersurface. In the 1-D case (see [5]) $\tilde{\mathscr{D}}$ is shown to be the only proper domain for $\mathbf{G}$ in the sense that any process starting in $\tilde{\mathscr{D}}$ always remain in $\tilde{\mathscr{D}}$ and $\tilde{\mathscr{D}}$ is defined by $\mathscr{D}_E$ and $\mathbf{T}_R$ only. By (i_3) rate-type constitutive equations with a finite relaxation time are also included since $\mathbf{G}$ is a continuous function (see for instance [4, Chap. V] and [8, 15]).

In this mechanical frame hypotheses (i_1)–(i_4) are minimal from the mathematical point of view.

4. EXISTENCE AND UNIQUENESS OF THE FREE ENERGY FUNCTION

The existence of a free energy function for a constitutive equation is a requirement of the second law of thermodynamics. We therefore investigate in this section what additional conditions are necessary (and eventually sufficient) to impose upon the constitutive functions $\mathscr{E}$, $\boldsymbol{T}_R$ and $\mathbf{G}$ such that (1) together with (i_1)–(i_4) posesses a free energy depending on stress and strain.

We need first some additional notions.

A *process in* $\mathscr{D}$ is a pair $(\mathbf{E}(t), \mathbf{T}(t))$, $t \in [0, t_0)$ with the following properties: (1) $\mathbf{E}(t) \in \mathscr{D}_E$ for any $t \in [0, t_0)$, $\mathbf{E}(t) \in C^1([0, t_0))$; (2) $\mathbf{T}(t)$, $t \in [0, t_0)$ is the solution of (1) for the given $\mathbf{E}(t)$, with $\mathbf{T}(0) = \mathbf{T}_0$; (3) $(\mathbf{E}(t), \mathbf{T}(t)) \in \mathscr{D}$ for any $t \in [0, t_0)$.

The constitutive equation (1) + (i_1)–(i_4) is said to prossess a free energy function on $\mathscr{D}$, compatible with the second law of thermodynamics if there exists a function $\psi : \mathscr{D} \to R$, $\psi = \psi(\mathbf{E}, \mathbf{T})$, $\psi \in C^1(\mathscr{D})$ such that for any process in $\mathscr{D}$

$$\mathbf{T}(t) \cdot \dot{\mathbf{E}}(t) - \rho \dot{\psi}(\mathbf{E}(t), \mathbf{T}(t)) \geq 0 \qquad \text{for any } t \in [0, t_0). \tag{6}$$

The constitutive equation (1) has a free energy function depending on stress and strain if and only if there exists $\psi = \psi(\mathbf{E}, \mathbf{T}) : \mathscr{D} \to R \psi \in C^1(\mathscr{D})$ that satisfies

$$\frac{\partial \psi}{\partial \mathbf{E}} + \mathscr{E}^{\mathrm{T}} \frac{\partial \psi}{\partial \mathbf{T}} = \frac{1}{\rho} \mathbf{T} \tag{7}$$

(which is an overdetermined system) and

$$\frac{\partial \psi}{\partial \mathbf{T}} \cdot \mathbf{G}(\mathbf{E}, \mathbf{T}) \leq 0 \tag{8}$$

for any $(\mathbf{E}, \mathbf{T}) \in \mathscr{D}$ (see [3, 6]).

The answer on the existence of a free energy is then given by the following.

THEOREM (A) If the constitutive equation (1) + (i_1)–(i_4) has a free energy function $\psi = \psi(\mathbf{E}, \mathbf{T})$ on $\mathscr{D}$ then

$$\mathscr{E} = \mathscr{E}^{\mathrm{T}} \tag{9}$$

$$(\mathbf{T}_R(\mathbf{E}) - \mathbf{T}_R(\tilde{\mathbf{E}})) \cdot (\mathbf{E} - \tilde{\mathbf{E}}) < \mathscr{E}(\mathbf{E} - \tilde{\mathbf{E}}) \cdot (\mathbf{E} - \tilde{\mathbf{E}}) \qquad \text{for any } \mathbf{E}, \tilde{\mathbf{E}} \in \mathscr{D}_E, \qquad \mathbf{E} \neq \tilde{\mathbf{E}} \tag{10}$$

$$\int_{\mathscr{C}} \mathbf{T}_R(\mathbf{E}) \cdot \mathrm{d}\mathbf{E} \text{ is path independent for any curve } \mathscr{C} \subset \mathscr{D}_E \tag{11}$$

$$\mathbf{G}(\mathbf{E}, \mathscr{E}(\mathbf{E} - \tilde{\mathbf{E}}) + \mathbf{T}_R(\tilde{\mathbf{E}})) \cdot (\mathbf{E} - \tilde{\mathbf{E}}) \leq 0 \qquad \text{for any } \mathbf{E}, \tilde{\mathbf{E}} \in \mathscr{D}_E. \tag{12}$$

Moreover, this free energy is unique (up to a constant) on $\tilde{\mathscr{D}} \subset \mathscr{D}$.

(B) If conditions (9)–(12) are satisfied then (1) + (i_1)–(i_4) possesses a unique free energy function (up to a constant) on $\tilde{\mathscr{D}}$.

PROOF. (A) Let $(\mathbf{E}_0, \mathbf{T}_0) \in \tilde{\mathscr{D}}$. The instantaneous response hypersurface $\mathbf{T} = \mathbf{T}_I(\mathbf{E}; \mathbf{E}_0, \mathbf{T}_0)$ at $(\mathbf{E}_0, \mathbf{T}_0)$ is the solution of the problem

$$\frac{\partial \mathbf{T}_I}{\partial \mathbf{E}}(\mathbf{E}; \mathbf{E}_0, \mathbf{T}_0) = \mathscr{E}, \qquad \mathbf{T}_I(\mathbf{E}_0; \mathbf{E}_0, \mathbf{T}_0) = \mathbf{T}_0$$

i.e.

$$\mathbf{T}_I(\mathbf{E}; \mathbf{E}_0, \mathbf{T}_0) = \mathscr{E}(\mathbf{E} - \mathbf{E}_0) + \mathbf{T}_0, \qquad \mathbf{E} \in \mathscr{D}_E. \tag{13}$$

The free energy $\psi_I(\mathbf{E}) = \psi(\mathbf{E}, \mathscr{E}(\mathbf{E} - \mathbf{E}_0) + \mathbf{T}_0)$ along the instantaneous response hypersurface will satisfy, according to (7),

$$\frac{\partial \psi_I}{\partial \mathbf{E}} = \frac{1}{\rho}[\mathscr{E}(\mathbf{E} - \mathbf{E}_0) + \mathbf{T}_0], \qquad \mathbf{E} \in \mathscr{D}_E. \tag{14}$$

But the necessary and sufficient condition for (14) to have a solution is that $\mathscr{E} = \mathscr{E}^{\mathrm{T}}$, i.e. condition (9).

Now, the general solution of (7) is

$$\rho\psi(\mathbf{E}, \mathbf{T}) = \frac{1}{2}\mathscr{E}^{-1}\mathbf{T} \cdot \mathbf{T} + \varphi(\mathbf{T} - \mathscr{E}\mathbf{E}) \tag{15}$$

where φ is a C^1-class real function of argument $\mathbf{T} - \mathscr{E}\mathbf{E}$ and it is defined on $\{\mathbf{T} - \mathscr{E}\mathbf{E}; (\mathbf{E}, \mathbf{T}) \in \mathscr{D}\}$. The free energy (15) satisfies

$$\frac{\partial \psi}{\partial \mathbf{T}}(\mathbf{E}, \mathbf{T}_R(\mathbf{E})) = \mathbf{0} \qquad \text{for any } \mathbf{E} \in \mathscr{D}_E. \tag{16}$$

Indeed, the free energy $\tilde{\psi}(t) = \psi(\mathbf{E}_0, \mathbf{T}(t))$ along the relaxation process starting at $(\mathbf{E}_0, \mathbf{T}_0) \in \mathscr{D}$, is decreasing according to (8) and (5) to the value $\psi(\mathbf{E}_0, \mathbf{T}_R(\mathbf{E}))$, i.e.

$$\psi(\mathbf{E}_0, \mathbf{T}_0) \geqslant \psi(\mathbf{E}_0, \mathbf{T}(t)) \geqslant \psi(\mathbf{E}_0, \mathbf{T}_R(\mathbf{E}_0)) \qquad \text{for any } t \geqslant 0, \tag{17}$$

whence (16) follows. Therefore (16) and (17) lead to

$$\varphi'(\mathbf{T}_R(\mathbf{E}) - \mathscr{E}\mathbf{E}) = -\mathscr{E}^{-1}\mathbf{T}_R(\mathbf{E}), \qquad \mathbf{E} \in \mathscr{D}_E \tag{18}$$

where $\varphi'(\boldsymbol{\tau}) = \partial\varphi(\boldsymbol{\tau})/\partial\boldsymbol{\tau}$. Let us denote

$$\mathbf{H}(\mathbf{E}) = \mathbf{T}_R(\mathbf{E}) - \mathscr{E}\mathbf{E} \qquad \text{for any } \mathbf{E} \in \mathscr{D}_E. \tag{19}$$

The mapping $\mathbf{H}$ is one-to-one on $\mathscr{D}_E$ as one can easily verify by using (18) and (19) and we denote its inverse mapping by $\bar{\mathbf{H}}: \mathbf{H}(\mathscr{D}_E) \to \mathscr{S}$, $\bar{\mathbf{H}}(\mathbf{H}(\mathbf{E})) = \mathbf{E}$ for any $\mathbf{E} \in \mathscr{D}_E$. Then (18) may be written as

$$\varphi'(\boldsymbol{\tau}) = -\mathscr{E}^{-1}\boldsymbol{\tau} - \bar{\mathbf{H}}(\boldsymbol{\tau}), \qquad \text{for any } \boldsymbol{\tau} \in \mathbf{H}(\mathscr{D}_E). \tag{20}$$

Since φ is smooth, $\bar{\mathbf{H}}$ results a continuous mapping from (20) and $\mathbf{H}(\mathscr{D}_E)$ an open set; thus $\tilde{\mathscr{D}}$ is also an open set in $\mathscr{S} \times \mathscr{S}$. Moreover (20) implies

$$\bar{\mathbf{H}}(\boldsymbol{\tau}) = \frac{\partial\bar{\omega}(\boldsymbol{\tau})}{\partial\boldsymbol{\tau}} \qquad \text{for any } \boldsymbol{\tau} \in \mathbf{H}(\mathscr{D}_E) \tag{21}$$

where

$$\bar{\omega}(\boldsymbol{\tau}) = -\frac{1}{2}\mathscr{E}^{-1}\boldsymbol{\tau} \cdot \boldsymbol{\tau} - \varphi(\boldsymbol{\tau}) + \text{const.}, \qquad \boldsymbol{\tau} \in \mathbf{H}(\mathscr{D}_E)$$

which gives through (21), (20) and (15) the form of the free energy function on $\tilde{\mathscr{D}}$

$$\rho\psi(\mathbf{E}, \mathbf{T}) = \mathbf{E} \cdot \mathbf{T} - \frac{1}{2}\mathscr{E}\mathbf{E} \cdot \mathbf{E} - \bar{\omega}(\mathbf{T} - \mathscr{E}\mathbf{E}) + \text{const.}, \qquad (\mathbf{E}, \mathbf{T}) \in \tilde{\mathscr{D}}. \tag{22}$$

We need to show that the mapping $\mathbf{H}$ given by (19) is dissipative i.e. relation (10). We prove first the free energy function has a global minimum with respect to the stress, on the equilibrium set, i.e.

$$\psi(\mathbf{E}, \mathbf{T}_R(\mathbf{E})) < \psi(\mathbf{E}, \mathbf{T}) \qquad \text{for any } (\mathbf{E}, \mathbf{T}) \in \tilde{\mathscr{D}}, \qquad \mathbf{T} \neq \mathbf{T}_R(\mathbf{E}). \tag{23}$$

According to (17) it remains to prove there is no $t_1 > 0$ such that

$$\psi(\mathbf{E}_0, \mathbf{T}(t)) = \psi(\mathbf{E}_0, \mathbf{T}_R(\mathbf{E}_0)) \quad \text{and} \quad \mathbf{T}(t) \neq \mathbf{T}_R(\mathbf{E}_0) \qquad \text{for any } t > t_1. \tag{24}$$

Let us suppose such a $t_1 > 0$ exists and let $\bar{t} \in [t_1, \infty)$ be fixed, with $(\mathbf{E}_0, \mathbf{T}(\bar{t})) \in \tilde{\mathscr{D}}$; then according to (17) and (24) we have

$$\psi(\mathbf{E}_0, \mathbf{T}(\bar{t})) \leq \psi(\mathbf{E}_0, \mathbf{T}) \qquad \text{for any } (\mathbf{E}_0, \mathbf{T}) \in \tilde{\mathscr{D}}$$

that is

$$\frac{\partial \psi}{\partial \mathbf{T}}(\mathbf{E}_0, \mathbf{T}(\bar{t})) = \mathbf{0}. \tag{25}$$

Relations (25) and (7) lead to

$$\rho \frac{\partial \psi}{\partial \mathbf{E}}(\mathbf{E}_0, \mathbf{T}(\bar{t})) = \mathbf{T}(\bar{t})$$

and, on the other hand, we get from (22)

$$\rho \frac{\partial \psi}{\partial \mathbf{E}}(\mathbf{E}_0, \mathbf{T}(\bar{t})) = \mathbf{T}(\bar{t}) - \mathscr{E}\mathbf{E}_0 + \mathscr{E}\bar{\mathbf{H}}(\mathbf{T}(\bar{t}) - \mathscr{E}\mathbf{E}_0) = \mathbf{T}_R(\tilde{\mathbf{E}}(\bar{t}))$$

as $(\mathbf{E}_0, \mathbf{T}(\bar{t})) \in \tilde{\mathscr{D}}$, so there exists $\tilde{\mathbf{E}} \in \mathscr{D}_E$ with $\mathbf{T}(\bar{t}) - \mathscr{E}\mathbf{E}_0 = \mathbf{H}(\tilde{\mathbf{E}}(\bar{t}))$. Therefore $\mathbf{T}(\bar{t}) = \mathbf{T}_R(\tilde{\mathbf{E}}(\bar{t}))$ and $\tilde{\mathbf{E}}(\bar{t}) = \mathbf{E}_0$. In conclusion, as $\mathbf{T}(\bar{t}) = \mathbf{T}_R(\mathbf{E}_0)$ there is no $t_1 > 0$ such that (24) holds.

Now directly from (23) and (22) we deduce

$$(\mathbf{H}(\tilde{\mathbf{E}}) - \mathbf{H}(\mathbf{E})) \cdot \tilde{\mathbf{E}} < \bar{\omega}(\mathbf{H}(\tilde{\mathbf{E}})) - \bar{\omega}(\mathbf{H}(\mathbf{E})) < (\mathbf{H}(\tilde{\mathbf{E}}) - \mathbf{H}(\mathbf{E})) \cdot \mathbf{E} \tag{26}$$

for any $\mathbf{E}, \tilde{\mathbf{E}} \in \mathscr{D}_E$, $\mathbf{E} \neq \tilde{\mathbf{E}}$, that is $\mathbf{H}$, is given by (19), is a strictly dissipative mapping

$$(\mathbf{H}(\tilde{\mathbf{E}}) - \mathbf{H}(\mathbf{E})) \cdot (\tilde{\mathbf{E}} - \mathbf{E}) < 0 \qquad \text{for any } \mathbf{E}, \tilde{\mathbf{E}} \in \mathscr{D}_E, \qquad \mathbf{E} \neq \tilde{\mathbf{E}},$$

and relation (10) follows.

Let us denote by $\psi_R(\mathbf{E}) = \psi(\mathbf{E}, \mathbf{T}_R(\mathbf{E}))$, $\mathbf{E} \in \mathscr{D}_E$ the free energy along the equilibrium hypersurface, i.e. the equilibrium free energy. Although $\mathbf{T} = \mathbf{T}_R(\mathbf{E})$ is only continuous on $\mathscr{D}_E$, ψ_R is of C^1-class on $\mathscr{D}_E$ and satisfies the classical thermostatic relation

$$\frac{\partial \psi_R(\mathbf{E})}{\partial \mathbf{E}} = \frac{1}{\rho} \mathbf{T}_R(\mathbf{E}) \qquad \text{for any } \mathbf{E} \in \mathscr{D}_E. \tag{27}$$

Indeed, from (22) and (26) we can easily get

$$\lim_{|\mathbf{E}-\mathbf{E}_0| \to 0} \frac{|\rho\psi_R(\mathbf{E}) - \rho\psi_R(\mathbf{E}_0) - T_R(\mathbf{E}_0) \cdot (\mathbf{E} - \mathbf{E}_0)|}{|\mathbf{E} - \mathbf{E}_0|} = 0$$

for any $\mathbf{E}, \mathbf{E}_0 \in \mathscr{D}_E$, $\mathbf{E} \neq \mathbf{E}_0$. Therefore, since $\mathbf{T} = \mathbf{T}_R(\mathbf{E})$ is only continuous on the simply connected open set $\mathscr{D}_E$, relation (11) is a necessary and sufficient condition for the system (27) to have a solution.

Further, according to the definition (2) of $\tilde{\mathscr{D}}$, for any $(\mathbf{E}, \mathbf{T}) \in \tilde{\mathscr{D}}$ there exists $\tilde{\mathbf{E}} \in \mathscr{D}_E$ such that $\mathbf{T} - \mathscr{E}\mathbf{E} = \mathbf{H}(\tilde{\mathbf{E}})$; consequently we have from (22) that

$$\rho \frac{\partial \psi(\mathbf{E}, \mathbf{T})}{\partial \mathbf{T}} = \mathbf{E} - \bar{\mathbf{H}}(\mathbf{T} - \mathscr{E}\mathbf{E}) = \mathbf{E} - \tilde{\mathbf{E}} \qquad \text{for any } (\mathbf{E}, \mathbf{T}) \in \tilde{\mathscr{D}}$$

and condition (12) follows since ψ has to satisfy inequality (8).

The free energy function has been found to have the form (22); if there is another free energy $\psi_1(\mathbf{E}, \mathbf{T})$ on $\tilde{\mathscr{D}}$ then, according to (15) we have $(\psi - \psi_1)(\mathbf{E}, \mathbf{T}) = (\varphi - \varphi_1)(\mathbf{T} - \mathscr{E}\mathbf{E})$. But $\varphi'(\boldsymbol{\tau}) = \varphi_1'(\boldsymbol{\tau}) = -\mathscr{E}^{-1}\boldsymbol{\tau} - \bar{\mathbf{H}}(\boldsymbol{\tau})$ which proves that $\psi(\mathbf{E}, \mathbf{T})$ given by (22) is unique, up to a constant, on $\tilde{\mathscr{D}} \subset \mathscr{D}$. The proof of point A is complete.

(B) In order to prove the existence of a free energy ψ on $\tilde{\mathscr{D}}$ we construct a solution of the system (7), we show it also satisfies the inequality (8) and finally prove it is unique up to a constant.

We already know the general solution of (7) is given by (15); we look then for a function φ which satisfies (18), so we have to show first that (18) has a solution.

We start with the mapping $\mathbf{H}(\mathbf{E})$ defined by (19); condition (10) implies $\mathbf{H}$ is one-to-one and, according to a domain invariance result (see [10, Chap. III]), $\mathbf{H}(\mathscr{D}_E)$ is an open set and $\bar{\mathbf{H}}$, the inverse mapping of $\mathbf{H}$, is continuous $\mathbf{H}(\mathscr{D}_E)$. Then the integrability conditions (11) and (9) ensure the existence of a C^1-class function $\omega : \mathscr{D}_E \to R$, which satisfies

$$\frac{\partial \omega(\mathbf{E})}{\partial \mathbf{E}} = \mathbf{H}(\mathbf{E}) \qquad \text{on } \mathscr{D}_E. \tag{28}$$

Moreover (see [11]) since $\mathbf{H}$ is strictly dissipative, ω is strictly concave on any open convex subset of $\mathscr{D}_E$, i.e.

$$\omega(\lambda \mathbf{E}_1 + (1-\lambda)\mathbf{E}_2) > \lambda \omega(\mathbf{E}_1) + (1-\lambda)\omega(\mathbf{E}_2)$$

for any $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{D}_E$, $\mathbf{E}_1 \neq \mathbf{E}_2$, $\lambda \in (0, 1)$ with $\lambda \mathbf{E}_1 + (1-\lambda)\mathbf{E}_2 \in \mathscr{D}_E$; consequently

$$\mathbf{H}(\mathbf{E}_1) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \leqslant \omega(\mathbf{E}_1) - \omega(\mathbf{E}_2) \leqslant \mathbf{H}(\mathbf{E}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \tag{29}$$

on any open convex subset of $\mathscr{D}_E$. We now can define $\bar{\omega} : \mathbf{H}(\mathscr{D}_E) \to R$ by

$$\bar{\omega}(\boldsymbol{\tau}) = \boldsymbol{\tau} \cdot \bar{\mathbf{H}}(\boldsymbol{\tau}) - \omega(\bar{\mathbf{H}}(\boldsymbol{\tau})) \qquad \text{on } \mathbf{H}(\mathscr{D}_E). \tag{30}$$

A similar procedure to that used at point A to prove ω_R is of C^1-class and satisfies (27), leads here to the conclusion that $\bar{\omega}$ is of C^1-class on $\mathbf{H}(\mathscr{D}_E)$ and satisfies

$$\frac{\partial \bar{\omega}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}} = \bar{\mathbf{H}}(\boldsymbol{\tau}) \qquad \text{on } \mathbf{H}(\mathscr{D}_E). \tag{31}$$

Relation (31) shows that equation (20) has the solution

$$\varphi(\boldsymbol{\tau}) = -\frac{1}{2} \mathscr{E}^{-1} \boldsymbol{\tau} \cdot \boldsymbol{\tau} - \bar{\omega}(\boldsymbol{\tau}) \qquad \text{for any } \boldsymbol{\tau} \in \mathbf{H}(\mathscr{D}_E); \tag{32}$$

we therefore constructed a free energy (15) on $\tilde{\mathscr{D}}$ with φ given by (32), which satisfies $\partial\psi/\partial\mathbf{T}(\mathbf{E}, \mathbf{T}_R(\mathbf{E})) = \mathbf{0}$ for any $\mathbf{E} \in \mathscr{D}_E$. According to (12) this free energy does also verify inequality (8) since

$$\rho \frac{\partial \psi}{\partial \mathbf{T}}(\mathbf{E}, \mathbf{T}) = \mathscr{E}^{-1}\mathbf{T} + \varphi'(\mathbf{T} - \mathscr{E}\mathbf{E}) = \mathscr{E}^{-1}(\mathscr{E}\mathbf{E} + \mathbf{H}(\tilde{\mathbf{E}})) - \mathscr{E}^{-1}\mathbf{T}_R(\tilde{\mathbf{E}}) = \mathbf{E} - \tilde{\mathbf{E}}$$

where $\tilde{\mathbf{E}} \in \mathscr{D}_E$ such that $\mathbf{T} - \mathscr{E}\mathbf{E} = \mathbf{H}(\tilde{\mathbf{E}})$.

It only remains to prove that this free energy $\psi(\mathbf{E}, \mathbf{T})$ is unique on $\tilde{\mathscr{D}}$, up to a constant. We already know from point A that any free energy $\rho\tilde{\psi}(\mathbf{E}, \mathbf{T}) = (1/2)\mathscr{E}^{-1}\mathbf{T} \cdot \mathbf{T} + \tilde{\varphi}(\mathbf{T} - \mathscr{E}\mathbf{E})$ has to satisfy $\partial\tilde{\psi}/\partial\mathbf{T}\ (\mathbf{E}, \mathbf{T}_R(\mathbf{E})) = \mathbf{0}$ for any $\mathbf{E} \in \mathscr{D}_E$ therefore $\tilde{\varphi}$ is a solution of equation (20) as well as φ and $\varphi - \tilde{\varphi}$ can only be a constant.

Remarks. The above result states that conditions (9)–(12) are necessary and sufficient for the existence and uniqueness of the free energy on $\tilde{\mathscr{D}}$; however, if $\mathscr{D} \supset \tilde{\mathscr{D}}$, they are only necessary and no more sufficient for the existence of the free energy on whole $\mathscr{D}$. It would be natural that for any state $(\mathbf{E}, \mathbf{T})$ reached by a process which starts at the natural state $(\mathbf{E} = \mathbf{0}, \mathbf{T} = \mathbf{0}) \in \tilde{\mathscr{D}}$, the free energy function be uniquely defined but, as we have seen, this happens only for the states in $\tilde{\mathscr{D}}$. Therefore it would be desirable that hypotheses (i_1)–(i_4) and conditions (9)–(12) ensure the domain $\tilde{\mathscr{D}}$ an invariance property with respect to any process, i.e. any process which starts at a state in $\tilde{\mathscr{D}}$ remains in $\tilde{\mathscr{D}}$. Such an invariance property of $\tilde{\mathscr{D}}$ was established in the 1-D case (see [5]) and for some particular cases it also remains valid in the 3-D case (see [8, Section 1.4.2]) but we could not prove it in general. The mathematical argument that $\tilde{\mathscr{D}}$ seems to be the only domain, in the 3-D case, where both existence and uniqueness of the free energy are secured, is furnished by the following observation: the free energy ψ may also be determined as the solution of the Cauchy problem

$$\frac{\partial \psi}{\partial \mathbf{E}} + \mathscr{E}\frac{\partial \psi}{\partial \mathbf{T}} = \frac{1}{\rho}\mathbf{T}, \qquad \psi(\mathbf{E}, \mathbf{T}_R(\mathbf{E})) = \psi_R(\mathbf{E}) \qquad \text{on } \mathscr{D}_E, \tag{33}$$

where $\psi_R(\mathbf{E})$ is the solution of another Cauchy problem: $\partial\psi_R(\mathbf{E})/\partial\mathbf{E} = (1/\rho)\mathbf{T}_R(\mathbf{E})$, $\psi_R(\mathbf{0}) = 0$. Since the characteristics of the system (33) are the instantaneous response hyperplanes (13), the existence and uniqueness domain of the solution of (33) is just $\tilde{\mathscr{D}}$ when $\mathbf{T}_R(\mathbf{E})$ is smooth (see

[2, 6, 13, Chap. vii]). We can not use this result in our case where $\mathbf{T}_R(\mathbf{E})$ is only continuous but simple examples, in the 1-D case, show that an extension of the free energy function from $\tilde{\mathscr{D}}$ to $\mathscr{D} \supset \tilde{\mathscr{D}}$ may sometimes be impossible (i.e. we lose existence) or non-unique.

An interesting remark is that the free energy function is determined by the equilibrium surface $\mathbf{T} = \mathbf{T}_R(\mathbf{E})$ and the elastic modulus $\mathscr{E}$ only and once they are fixed we have exactly one free energy for the whole class of relaxation moduli $\mathbf{G}$ consistent with (12).

We also mention that a sufficient condition for the equilibrium hypersurface $\mathbf{T} = \mathbf{T}_R(\mathbf{E})$ to be stable with respect to any relaxation process is

$$\mathbf{G}(\mathbf{E}, \mathbf{T}) \cdot (\mathbf{T} - \mathbf{T}_R(\mathbf{E})) > 0 \qquad \text{for any } (\mathbf{E}, \mathbf{T}) \in \mathscr{D}, \qquad \mathbf{T} \neq \mathbf{T}_R(\mathbf{E}). \tag{34}$$

We end with some comments on the results obtained when $\mathbf{T}_R(\mathbf{E})$ is assumed of C^1-class. In this case, considered in [8], the stability of $\mathbf{T} = \mathbf{T}_R(\mathbf{E})$ with respect to the relaxation processes (i_4) is not assumed and $\mathscr{D}$ is taken as a convex bounded domain; under the following hypotheses

$$\frac{\partial \mathbf{T}_R(\mathbf{E})}{\partial \mathbf{E}} = \left(\frac{\partial \mathbf{T}_R(\mathbf{E})}{\partial \mathbf{E}}\right)^{\mathrm{T}} \qquad \text{on } \mathscr{D}_E$$

(which is equivalent to (11)) and

$$\left(\frac{\partial \mathbf{T}_R(\mathbf{E})}{\partial \mathbf{E}} - \mathscr{E}\right)\boldsymbol{\eta} \cdot \boldsymbol{\eta} < 0 \qquad \text{for any } \boldsymbol{\eta} \in \mathscr{S}, \qquad \boldsymbol{\eta} \neq 0$$

(which is equivalent to (10)), equation (7) is proved to have a unique solution which satisfies (16). Another set of additional conditions on the form of $\mathbf{G}$ is proved to be sufficient for this solution to satisfy (8) too (i.e. to be a free energy for equation (1)).

5. PROPERTIES OF THE FREE ENERGY FUNCTION

The free energy is important mainly because its existence is required by the second law of thermodynamics. In the same time energy has already proved to be quite useful when used as a "norm" for the estimation of the solutions of certain systems of partial differential equations (for instance energy is frequently used in numerical methods for testing stability). We therefore study in this section under what additional assumptions on the constitutive functions the free energy ψ is non-negative and when it is "equivalent" to an Euclidean norm on $\mathscr{S} \times \mathscr{S}$. On the other hand, certain thermal processes (such as annealing or quenching for instance) that leave the Young's modulus $\mathscr{E}$ unchanged, do change the equilibrium curve and consequently the free energy. We also present in this section a result related to these kind of changes in energy (also called the monotony property of the free energy with respect to the equilibrium hypersurface).

All these results are non-trivial extensions of some former ones obtained in the 1-D case (see [5]); the hypotheses may be tested in applications since they involve only the dynamic elastic moduli and the equilibrium hypersurface. The frame is sufficiently general to include non-monotone equilibrium stress–strain relations which are proper to certain materials (as iron for instance) as well as to some material bodies in phase transitions (see for instance [7]). In a different work we use some of these results in order to establish energy estimates for the solutions of certain initial and boundary value problems in 3-D rate-type semilinear viscoelasticity.

From now on we consider only the free energy ψ with $\psi(\mathbf{0}, \mathbf{0}) = 0$, i.e. $\varphi(\mathbf{0}) = 0$.

Proposition 1. The function φ is non-negative on $\mathbf{H}(\mathscr{D}_E)$ if and only if

$$\rho\psi_R(\mathbf{E}) \geq \frac{1}{2}\mathscr{E}^{-1}\mathbf{T}_R(\mathbf{E}) \cdot \mathbf{T}_R(\mathbf{E}) \qquad \text{on } \mathscr{D}_E. \tag{35}$$

When $\mathscr{D}_E \equiv \mathscr{S}$ condition (35) is equivalent to

$$\psi_R(\mathbf{E}) \geq 0 \qquad \text{on } \mathscr{D}_E. \tag{35'}$$

PROOF. We easily get from (32), (30), (28) and (27) that

$$\varphi(\mathbf{H}(\mathbf{E})) = -\frac{1}{2}\mathscr{E}^{-1}\mathbf{T}_R(\mathbf{E})\cdot\mathbf{T}_R(\mathbf{E}) + \rho\psi_R(\mathbf{E}) \qquad \text{on } \mathscr{D}_E \tag{36}$$

and therefore (35) is obviously equivalent to $\varphi(\boldsymbol{\tau}) \geqslant 0$ on $\mathbf{H}(\mathscr{D}_E)$. When $\mathscr{D}_E = \mathscr{S}$ we only have to prove that condition (35′) implies condition (35). Let $\mathbf{E}^* \in \mathscr{D}_E$ be fixed and let us denote by $\bar{\mathbf{E}}$ the inelastic strain for instantaneous unloading attached to the state $(\mathbf{E}^*, \mathbf{T}_R(\mathbf{E}^*))$, i.e.

$$\bar{\mathbf{E}} = \mathbf{E}^* - \mathscr{E}^{-1}\mathbf{T}_R(\mathbf{E}^*) \in \mathscr{D}_E. \tag{37}$$

For $\mathbf{E}(\lambda) = (1-\lambda)\bar{\mathbf{E}} + \lambda\mathbf{E}^* \in \mathscr{D}_E$, $\lambda \in [0, 1]$ we have

$$\rho\psi_R(\mathbf{E}(\lambda)) - \rho\psi_R(\bar{\mathbf{E}}) = \int_0^\lambda \mathbf{T}_R(\mathbf{E}(s))\cdot(\mathbf{E}^* - \bar{\mathbf{E}})\,\mathrm{d}s$$

and, according to inequality (10) written for $\mathbf{E} = \mathbf{E}(s)$ and $\tilde{\mathbf{E}} = \mathbf{E}^*$, we get

$$\rho\psi_R(\mathbf{E}^*) - \rho\psi_R(\bar{\mathbf{E}}) = \int_0^1 \mathbf{T}_R(\mathbf{E}(s))\cdot(\mathbf{E}^* - \bar{\mathbf{E}})\,\mathrm{d}s > \mathbf{T}_R(\mathbf{E}^*)\cdot(\mathbf{E}^* - \bar{\mathbf{E}}) - \frac{1}{2}\mathscr{E}(\mathbf{E}^* - \bar{\mathbf{E}})\cdot(\mathbf{E}^* - \bar{\mathbf{E}}).$$

We therefore obtain, together with (35′), that

$$\rho\psi_R(\mathbf{E}^*) \geqslant \rho\psi_R(\mathbf{E}^*) - \rho\psi_R(\bar{\mathbf{E}}) > \frac{1}{2}\mathscr{E}^{-1}\mathbf{T}_R(\mathbf{E}^*)\cdot\mathbf{T}_R(\mathbf{E}^*)$$

that is condition (35).

Conditions (35) and (35′) may be tested if we remember that $\rho\,\partial\psi_R(\mathbf{E})/\partial\mathbf{E} = \mathbf{T}_R(\mathbf{E})$ and $\int_{\mathscr{C}} \mathbf{T}_R(\mathbf{E})\cdot\mathrm{d}\mathbf{E}$ is path independent on any curve $\mathscr{C} \subset \mathscr{D}_E$ (condition (11)).

The proof to Proposition 1 points out that the equivalence between (35) and (35′) holds in a wider frame, i.e. when $\mathscr{D}_E$ is a convex domain such that for any $\mathbf{E}^* \in \mathscr{D}_E$ the inelastic strain $\bar{\mathbf{E}}(\mathbf{E}^*)$ defined by (37) also belongs to $\mathscr{D}_E$. Now, ψ is obviously non-negative if φ is, but the converse is not always true. However we may state the following.

COROLLARY. When $\mathscr{D}_E$ is convex and such that for any $\mathbf{E}^* \in \mathscr{D}_E$ the inelastic strain $\bar{\mathbf{E}}(\mathbf{E}^*) \in \mathscr{D}_E$ (in particular when $\mathscr{D}_E = \mathscr{S}$), then

$$\psi(\mathbf{E}, \mathbf{T}) \geqslant 0 \quad \text{on } \tilde{\mathscr{D}} \qquad \text{if and only if } \psi_R(\mathbf{E}) \geqslant 0 \quad \text{on } \mathscr{D}_E.$$

Indeed, let us suppose there is a $\boldsymbol{\tau} \in \mathbf{H}(\mathscr{D}_E)$ with $\varphi(\boldsymbol{\tau}) < 0$ while $\psi \geqslant 0$ on $\tilde{\mathscr{D}}$. Let $\mathbf{E}^* \in \mathscr{D}_E$ be such that $\boldsymbol{\tau} = \mathbf{H}(\mathbf{E}^*)$ and consider $\bar{\mathbf{E}}(\mathbf{E}^*) = \mathbf{E}^* - \mathscr{E}^{-1}\mathbf{T}_R(\mathbf{E}^*) \in \mathscr{D}_E$. Then $\rho\psi(\bar{\mathbf{E}}, \mathbf{0}) = \varphi(-\mathscr{E}\bar{\mathbf{E}}) = \varphi(\boldsymbol{\tau}) < 0$ which is a contradiction.

PROPOSITION 2. The function φ has the property

$$\mathscr{A}_1\boldsymbol{\tau}\cdot\boldsymbol{\tau} \leqslant \varphi(\boldsymbol{\tau}) \leqslant \mathscr{A}_2\boldsymbol{\tau}\cdot\boldsymbol{\tau} \qquad \text{on } \mathbf{H}(\mathscr{D}_E) \tag{38}$$

where $\mathscr{A}_i \in \mathrm{Lin}(\mathscr{S}, \mathscr{S})$, $i = 1, 2$ are symmetric and positive definite, if and only if there exist $\mathscr{E}_i \in \mathrm{Lin}(\mathscr{S}, \mathscr{S})$, $i = 1, 2$, symmetric and positive definite, with

$$0 < \mathscr{E}_1\boldsymbol{\eta}\cdot\boldsymbol{\eta} \leqslant \mathscr{E}_2\boldsymbol{\eta}\cdot\boldsymbol{\eta} < \mathscr{E}\boldsymbol{\eta}\cdot\boldsymbol{\eta} \qquad \text{on } \mathscr{S} - \{\mathbf{0}\} \tag{39}$$

such that

$$\frac{1}{2}\mathscr{E}_1\mathbf{E}\cdot\mathbf{E} + \frac{1}{2}(\mathscr{E} - \mathscr{E}_1)^{-1}(\mathbf{T}_R(\mathbf{E}) - \mathscr{E}_1\mathbf{E})\cdot(\mathbf{T}_R(\mathbf{E}) - \mathscr{E}_1\mathbf{E}) \leqslant \rho\psi_R(\mathbf{E})$$

$$\leqslant \frac{1}{2}\mathscr{E}_2\mathbf{E}\cdot\mathbf{E} + \frac{1}{2}(\mathscr{E} - \mathscr{E}_2)^{-1}(\mathbf{T}_R(\mathbf{E}) - \mathscr{E}_2\mathbf{E})\cdot(\mathbf{T}_R(\mathbf{E}) - \mathscr{E}_2\mathbf{E}) \qquad \text{on } \mathscr{D}_E. \tag{40}$$

Moreover

$$\mathscr{A}_i = \frac{1}{2}[(\mathscr{E} - \mathscr{E}_i)^{-1} - \mathscr{E}^{-1}], \qquad i = 1, 2. \tag{41}$$

When $\mathscr{D}_E \equiv \mathscr{S}$, condition (39) is equivalent to

$$\frac{1}{2}\mathscr{E}_1\mathbf{E}\cdot\mathbf{E} \leqslant \rho\psi_R(\mathbf{E}) \leqslant \frac{1}{2}\mathscr{E}_2\mathbf{E}\cdot\mathbf{E} \qquad \text{on } \mathscr{D}_E. \tag{40'}$$

PROOF. When (38) holds, a simple calculation starting from relation (36) leads to

$$\varphi(\mathbf{H}(\mathbf{E})) - \mathscr{A}_i\mathbf{H}(\mathbf{E}) \cdot \mathbf{H}(\mathbf{E}) = \rho\psi_R(\mathbf{E}) - \frac{1}{2}\mathscr{E}_i\mathbf{E} \cdot \mathbf{E} + \frac{1}{2}(\mathscr{E} - \mathscr{E}_i)^{-1}(\mathbf{T}_R(\mathbf{E}) - \mathscr{E}_i\mathbf{E}) \cdot (\mathbf{T}_R(\mathbf{E}) - \mathscr{E}_i\mathbf{E}) \tag{42}$$

$i = 1, 2$, on $\mathscr{D}_E$, where $\mathscr{E}_i$, $i = 1, 2$, denote the linear, symmetric and positive definite mappings

$$\mathscr{E}_i = \left(\frac{1}{2}\mathscr{E}^{-1} + \mathscr{A}_i\right)^{-1}\mathscr{A}_i\mathscr{E}, \qquad i = 1, 2.$$

Then (40) follows as well as (39).

Conversely, when (40) holds, we may define $\mathscr{A}_i$, $i = 1, 2$, by relations (41) and prove they are symmetric and positive definite. Then (38) follows from (42).

The equivalence of (40) and (40′) when $\mathscr{D}_E \equiv \mathscr{S}$ follows the same technique already used in the proof of Proposition 1.

COROLLARY. Conditions (39) and (40) (or (39) and (40′) when $\mathscr{D}_E \equiv \mathscr{S}$) are necessary and sufficient for the free energy ψ to satisfy

$$\psi_1(\mathbf{E}, \mathbf{T}) \leq \psi(\mathbf{E}, \mathbf{T}) \leq \psi_2(\mathbf{E}, \mathbf{T}) \qquad \text{on } \tilde{\mathscr{D}} \tag{43}$$

where

$$\rho\psi_i(\mathbf{E}, \mathbf{T}) = \frac{1}{2}\mathscr{E}^{-1}\mathbf{T} \cdot \mathbf{T} + \frac{1}{2}[(\mathscr{E} - \mathscr{E}_i)^{-1} - \mathscr{E}^{-1}](\mathbf{T} - \mathscr{E}\mathbf{E}) \cdot (\mathbf{T} - \mathscr{E}\mathbf{E}), \qquad i = 1, 2, \qquad \text{on } \mathscr{S} \times \mathscr{S}.$$

Now we can easily see that $\psi_i(\mathbf{E}, \mathbf{T})$, $i = 1, 2$ are the free energy functions corresponding to the linear equilibrium hypersurfaces $\mathbf{T}_R^{(i)} = \mathscr{E}_i\mathbf{E}$, $i = 1, 2$, respectively, and each $\sqrt{\rho\psi_i(\mathbf{E}, \mathbf{T})}$, $i = 1, 2$ is equivalent to the Euclidean norm on $\mathscr{S} \times \mathscr{S}$.

The "equivalence" of the free energy function to an Euclidean norm on $\mathscr{S} \times \mathscr{S}$ was first established in [3] but for rate-type semilinear viscoelastic constitutive equations with smooth and monotone equilibrium hypersurfaces $\mathbf{T} = \mathbf{T}_R(\mathbf{E})$ which satisfy

$$0 < \mathscr{E}_1\boldsymbol{\eta} \cdot \boldsymbol{\eta} \leq \frac{\partial\mathbf{T}_R(\mathbf{E})}{\partial\mathbf{E}}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \leq \mathscr{E}_2\boldsymbol{\eta} \cdot \boldsymbol{\eta} < \mathscr{E}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \qquad \text{for any } \boldsymbol{\eta} \in \mathscr{S} - \{\mathbf{0}\} \text{ and any } \mathbf{E} \in \mathscr{D}_E$$

where $\mathscr{E}_i \in \mathrm{Lin}(\mathscr{S}, \mathscr{S})$, $\mathscr{E}_i$ symmetric, $i = 1, 2$.

We end this section with the "monotony property" of the free energy function with respect to the equilibrium hypersurface.

PROPOSITION 3. Let $\mathbf{T}_{Ri}: \mathscr{D}_{Ei} \subset \mathscr{S} \to \mathscr{S}$ and $\mathbf{G}_i: \tilde{\mathscr{D}}_i \to \mathscr{S}$, $i = 1, 2$ (where $\tilde{\mathscr{D}}_i$ corresponds to $\mathscr{D}_{Ei}$, $i = 1, 2$), with $\mathscr{D}_{E1} \cap \mathscr{D}_{E2}$ convex, be two relaxation moduli and two equilibrum curves such that each triplet $(\mathscr{E}, \mathbf{T}_{Ri}, \mathbf{G}_i)$, $i = 1, 2$ satisfies the hypotheses (i_1)–(i_4) and the conditions (9)–(12). Then

$$\psi_1(\mathbf{E}, \mathbf{T}) \geq \psi_2(\mathbf{E}, \mathbf{T}) \qquad \text{on } \tilde{\mathscr{D}}_1 \cap \tilde{\mathscr{D}}_2 \tag{44}$$

if and only if

$$\psi_{R1}(\mathbf{E}) \geq \omega_{R2}(\mathbf{E}) \qquad \text{on } \mathscr{D}_{E1} \cap \mathscr{D}_{E2}, \tag{45}$$

where ψ_i is the free energy function on $\tilde{\mathscr{D}}_i$ and ψ_{Ri} is the equilibrium free energy corresponding to $(\mathscr{E}, T_{Ri})$, $i = 1, 2$, respectively.

PROOF. Let us denote by $\mathscr{D}_3 = \mathscr{D}_{E1} \cap \mathscr{D}_{E2}$ and let $\boldsymbol{\tau} \in \mathbf{H}_1(\mathscr{D}_3) \cap \mathbf{H}_2(\mathscr{D}_3)$, with $\mathbf{H}_i(\mathbf{E}) = \mathbf{T}_{Ri}(\mathbf{E}) - \mathscr{E}\mathbf{E}$, $i = 1, 2$. Then $\boldsymbol{\tau} = \mathbf{H}_1(\mathbf{E}_1) = \mathbf{H}_2(\mathbf{E}_2)$ and from (36) we have

$$\begin{aligned}\varphi_1(\boldsymbol{\tau}) - \varphi_2(\boldsymbol{\tau}) &= \rho\psi_{R1}(\mathbf{E}_1) - \rho\psi_{R2}(\mathbf{E}_2) - \frac{1}{2}\mathscr{E}(\mathbf{E}_1 - \mathbf{E}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) - \mathbf{T}_{R2}(\mathbf{E}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \\ &= \rho\psi_{R1}(\mathbf{E}_1) - \rho\psi_{R2}(\mathbf{E}_2) + \frac{1}{2}\mathscr{E}(\mathbf{E}_1 - \mathbf{E}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) - \mathbf{T}_{R1}(\mathbf{E}_1) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \end{aligned} \tag{46}$$

For $\tilde{\mathbf{E}}(\lambda) = \lambda\mathbf{E}_1 + (1-\lambda)\mathbf{E}_2$, $\lambda \in [0, 1]$ we have

$$\rho\psi_{Ri}(\tilde{\mathbf{E}}(\lambda)) - \rho\psi_{Ri}(\mathbf{E}_2) = \int_0^\lambda \mathbf{T}_{Ri}(\tilde{\mathbf{E}}(s)) \cdot (\mathbf{E}_1 - \mathbf{E}_2)\,\mathrm{d}s, \qquad i = 1, 2$$

so (46) may be written

$$\begin{aligned}\varphi_1(\boldsymbol{\tau}) - \varphi_2(\boldsymbol{\tau}) \\ &= \rho\psi_{R1}(\mathbf{E}_1) - \rho\psi_{R2}(\mathbf{E}_1) + \int_0^1 [\mathbf{T}_{R2}(\tilde{\mathbf{E}}(s)) - \mathbf{T}_{R2}(\mathbf{E}_2) - \mathscr{E}(\tilde{\mathbf{E}}(s) - \mathbf{E}_2)] \cdot (\mathbf{E}_1 - \mathbf{E}_2)\,\mathrm{d}s \\ &= \rho\psi_{R1}(\mathbf{E}_2) - \rho\psi_{R2}(\mathbf{E}_2) + \int_0^1 [\mathbf{T}_{R1}(\tilde{\mathbf{E}}(s)) - \mathbf{T}_{R1}(\mathbf{E}_1) - \mathscr{E}(\tilde{\mathbf{E}}(s) - \mathbf{E}_1)] \cdot (\mathbf{E}_1 - \mathbf{E}_2)\,\mathrm{d}s \end{aligned} \tag{47}$$

but, according to condition (10), the first integral in (47) is negative and the second one is positive, therefore

$$\rho\psi_{R1}(\mathbf{E}_2) - \rho\psi_{R2}(\mathbf{E}_2) < \varphi_1(\boldsymbol{\tau}) - \varphi_2(\boldsymbol{\tau}) < \rho\psi_{R1}(\mathbf{E}_1) - \rho\psi_{R2}(\mathbf{E}_1)$$

and the assertion of Proposition 3 follows.

Such a "monotony property" of the free energy function has been established in [12] for smooth and monotone equilibrium hypersurfaces $\mathbf{T} = \mathbf{T}_{Ri}(\mathbf{E})$, $i = 1, 2$ that satisfy the following condition

$$0 < \frac{\partial\mathbf{T}_{R2}(\mathbf{E})}{\partial\mathbf{E}}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \leq \frac{\partial\mathbf{T}_{R1}(\mathbf{E})}{\partial\mathbf{E}}\boldsymbol{\eta} \cdot \boldsymbol{\eta} < \mathscr{E}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \qquad \text{for any } \boldsymbol{\eta} \in \mathscr{S} - \{\mathbf{0}\} \text{ and any } \mathbf{E} \in \mathscr{D}_E.$$

One can show that these inequalities imply the following ones

$$0 < \rho\psi_{R2}(\mathbf{E}) \leq \rho\psi_{R1}(\mathbf{E}) \leq \frac{1}{2}\mathscr{E}\mathbf{E} \cdot \mathbf{E} \qquad \text{on } \mathscr{D}_{E1} \cap \mathscr{D}_{E2} \quad (\equiv \mathscr{D}_3).$$

6. CONCLUDING REMARKS

A rate-type semilinear viscoelastic constitutive equation $(1) + (\mathrm{i}_1)$–(i_4) possesses a unique energy function (up to a constant) only if conditions (9)–(12) are satisfied. Several examples in the 1-D case as well as similar results obtained under more restrictive hypotheses seem to point out that these conditions are also sufficient not only necessary even if we could not prove it yet. Several important properties of the free energy function, such as non-negativeness and "equivalence" to the Euclidean norm in $\mathscr{S} \times \mathscr{S}$, are studied under an almost minimal set of additional conditions. All these conditions, which refer to the equilibrium free energy function $\psi_R(\mathbf{E})$, are in fact restrictions imposed on the equilibrium hypersurface of the viscoelastic model and may be tested. Since the equilibrium free energy is a potential for the equilibrium hypersurface, it has a simple interpretation in the 1-D case, as the area defined by the equilibrium curve (see [5]). What is more important, the frame imposed by the above conditions remains large enough to consider a wide class of continuous and even non-monotone equilibrium hypersurfaces, when the free energy function may consequently be non-convex, and only continuous relaxation moduli which allow both infinite and finite relaxation time. We therefore expect, as in the 1-D case (see [7, 14]) that the rate-type semilinear viscoelastic model may lead to a better description of phase transitions in 3-D bodies.

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