On a Class of Rota-Baxter Operators with Geometric Origin

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Abstract. The Rota-Baxter operators have a long history, with manny applications in both pure mathematics and theoretical physics. After a short review of this subject, I will present a class of Rota-Baxter operators comming from the world of vector bundles over elliptic curves.

INTRODUCTION

The class of Rota-Baxter operators find application in many areas, from combinatorics and Hopf algebras, to shuffle algebras and renormalization in quantum field theory. In the first part we introduce the relevant definition of this type of operators and their relation with other well-known equation, namely the Yang-Baxter ones. The second part of the paper is devoted to the more recent associative Yang-Baxter equation and to their relation with operators of Rota-Baxter type. Finally, the last part is devoted to solutions of both of the above type of equations, constructed using the derived category of coherent sheaves over elliptic curves. The very last section contain two questions which could be considered for future research.

THE ROTA-BAXTER OPERATORS

In what follows, we fix a commutative field k, an associative and commutative k-algebra A, a k-linear operator R on A and a parameter $\theta \in k$. After the important papers [4] [11] [12] we consider the following:

Definition 1 *R is a Rota-Baxter operator of weight θ if for all $x, y \in A$ it satisfies the following identity:*

$$R(x)R(y) = R(R(x)y + xR(y) - \theta xy). \tag{1}$$

One could remark cf. [8], that in characteristic 0, an operator R is a Rota-Baxter one iff

$$B = \theta 1_A - 2R \tag{2}$$

satisfies for all $x, y \in A$ the following modified Rota-Baxter relation:

$$B(x)B(y) = B(B(x)y + xB(y)) - \theta^2 xy. \tag{3}$$

Examples

A first elementary example comes from real one variable analysis: by integration by parts formula, the definite Riemann integral is a Rota-Baxter operator of weight 0. As we will see later, a large class of Rota-Baxter operators comes from the classical Yang-Baxter equation and also from its more recent associative analogue.

The Rota-Baxter Operators with Spectral Parameters

Using [8] we consider a more general Rota-Baxter type equation depending on a "spectral" parameter varying in a domain $D \subseteq \mathbb{C}$. Let

$$R : D \to End(A) \tag{4}$$

an endomorphism valued function on a complex parameter.

Definition 2 *R is a Rota-Baxter operator with spectral parameter if for all $x, y \in A$ and all $z_1, z_2, z_3 \in D$ with $z' := z_1 - z_2 \in D$ and $z'' := z_2 - z_3 \in D$, it satisfies the following identity:*

$$R[z' + z''](x)R[z'](y) = R[z'](R[z''](x)y) + R[z' + z''](xR[-z''](y)). \tag{5}$$

Remark 1 *For $z' = z'' = 0$, a Rota-Baxter operator with spectral parameter is obviously an usual one, i.e. satisfy the equation 1 for the weight $\theta = 0$.*

More generally we have the following:

Definition 3 *R is a Rota-Baxter operator of weight θ with spectral parameter if for all $x, y \in A$ and all $z_1, z_2, z_3 \in D$ with $z' := z_1 - z_2 \in D$ and $z'' := z_2 - z_3 \in D$, it satisfies the following identity:*

$$R[z' + z''](x)R[z'](y) + \theta R[z' + z''](xy) = R[z'](R[z''](x)y) + R[z' + z''](xR[-z''](y)). \tag{6}$$

THE YANG-BAXTER EQUATION

The classical Yang-Baxter equation, according to [13] is defined in the following context: let g a Lie algebra, $g \otimes g \to k$ a non-degenerate pairing and $r \in g^{\otimes 2}$ an antisymmetric tensor, namely

$$\tau \circ r = -r \tag{7}$$

where $\tau : g^{\otimes 2} \to g^{\otimes 2}$ is the twist.

Remark 2 *In the case when g is semi-simple usually one take as non-degenerate pairing the Killing form.*

With the above notations, we have [8]:

Definition 4 *r is called an r-matrix if it satisfy the classical Yang-Baxter equation below:*

$$[r_{13}, r_{12}] + [r_{23}, r_{12}] + [r_{23}, r_{13}] = 0, \tag{8}$$

where r_{ij} are the defined from r using the corresponding map $g^{\otimes 2} \to g^{\otimes 3}$ and $[\cdot, \cdot]$ is the usual commutator on the enveloping algebra $U(g)$.

Rota-Baxter Operators from the classical Yang-Baxter Equation

The r-matrix defined above, have a beautiful connection (cf. [8]) with Rota-Baxter operator of weight 0. Namely, using the non-degenerate pairing, we can identify

$$g^{\otimes 2} \simeq g \otimes g^* \simeq Hom(g, g). \tag{9}$$

Consequently, if we denote by R the endomorphism of g which correspond to r under the above isomorphism, we have [8]:

Proposition 1 *r satisfy the classical Yang-Baxter equation 8 iff R is a Rota-Baxter operator of weight 0 on g, which means that for any $x, y \in g$ one has:*

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]). \tag{10}$$

More generally, if we denote by C the Casimir element in $g \otimes g$ associated with the non-degenerate pairing $g \otimes g \to k$, let's take an $r \in g^{\otimes 2}$ such that:

$$r + \tau \circ r = \theta^2 C, \tag{11}$$

where θ is a complex parameter. Then, if one denote by $B \in End(g)$ the element which correspond to the above r under the isomorphism 9, we have [8]:

Proposition 2 *If r satisfy the classical Yang-Baxter equation 8 and 11, then B satisfy the modified Rota-Baxter equation 3 of weight θ:*

$$[B(x), B(y)] = B([B(x), y] + [x, B(y)]) - \theta^2 [x, y]. \tag{12}$$

Moreover, if we denote by $R' = \frac{1}{2}(\theta 1_g - B)$, then R' is a Rota-Baxter operator of weight θ.

The Associative Yang-Baxter Equation

The associative Yang-Baxter equation appeared for the first time in a series of papers of Aguiar [1], [2], [3] and in the work of Polishchuck [9] [10]. Let's A be an associative unitary k-algebra over a field of characteristic 0 and

$$r = \Sigma a_i \otimes b_i \in A \otimes A. \tag{13}$$

We consider $A \otimes A$ as A-bimodule with multiplication

$$x(a \otimes b)y = (xa \otimes b)y = (xa) \otimes (by). \tag{14}$$

Definition 5 *r satisfy the associative classical Yang-Baxter equation if:*

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0, \tag{15}$$

where

$$r_{12} = \Sigma a_i \otimes b_i \otimes 1_A \tag{16}$$

and so on.

In the equation above, the multiplication in $A \otimes A \otimes A$ is the usual one; for example

$$r_{13}r_{12} = \Sigma a_i a_j \otimes b_j \otimes b_i. \tag{17}$$

Moreover, for $\theta \in k$ we can introduce the following [8]:

Definition 6 *r satisfy the associative classical Yang-Baxter equation with weight θ if:*

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} - \theta r_{13} = 0. \tag{18}$$

Rota-Baxter Operators from the Associative Yang-Baxter Equation

Our interest for the associative classical Yang-Baxter equation (with weight θ) comes from the following result of Aguiar:

Proposition 3 *Let $r = \Sigma a_i \otimes b_i \in A \otimes A$ a solution of the associative classical Yang-Baxter equation 15 and $B : A \to A$ defined as*

$$B(x) = \Sigma a_i x b_i, \tag{19}$$

for any $x \in A$. Then B satisfy the Rota-Baxter equation of weight 0

$$B(x)B(y) = B(B(x)y + xB(y)), \tag{20}$$

for all $x, y \in A$.

Moreover, according to [8] we have the same result for arbitrary weight θ:

Proposition 4 *Let $r = \Sigma a_i \otimes b_i \in A \otimes A$ a solution of the associative classical Yang-Baxter equation 15 of weight θ. Then B defined above, satisfy the Rota-Baxter equation of weight θ*

$$B(x)B(y) = B(B(x)y + xB(y) - \theta xy), \tag{21}$$

for all $x, y \in A$.

THE ASSOCIATIVE YANG-BAXTER EQUATION WITH SPECTRAL PARAMETER

The associative Yang-Baxter equation with spectral parameter, was introduced in [9]:

Definition 7 *A map $r : D \subseteq \mathbb{C} \to A \otimes A$ satisfy the associative Yang-Baxter equation with spectral parameter, if*

$$r_{13}(z_{13})r_{12}(z_{12}) - r_{12}(z_{12})r_{23}(z_{23}) + r_{23}(z_{23})r_{13}(z_{13}) = 0, \tag{22}$$

where $z_{ij} = z_i - z_j$. Also, r satisfy the associative Yang-Baxter equation with weight θ and spectral parameter if

$$r_{13}(z_{13})r_{12}(z_{12}) - r_{12}(z_{12})r_{23}(z_{23}) + r_{23}(z_{23})r_{13}(z_{13}) - \theta r_{13}(z_{13}) = 0, \tag{23}$$

As before we have:

Proposition 5 *Let $r : D \subseteq \mathbb{C} \to A \otimes A$ a solution of the associative classical Yang-Baxter equation with spectral parameter 22 (or of 23 with weight θ). Then the associated map $R : D \to End(A)$, satisfy the Rota-Baxter equation with spectral parameter 5 (or 6 with weight θ):*

$$R[z' + z''](x)R[z'](y) + \theta R[z' + z''](xy) = R[z'](R[z''](x)y) + R[z' + z''](xR[-z''](y)). \tag{24}$$

for all $x, y \in A$, where as before $z' := z_1 - z_2 \in D$ and $z'' := z_2 - z_3 \in D$.

Remark 3 *Following [8], one can also reverse the construction: starting with Rota-Baxter operators with spectral parameters of weight θ, satisfying certain additional symmetry properties, one can construct solution of the associative Yang-Baxter equation with spectral parameters of weight θ.*

As a very simple example of solution of the associative Yang-Baxter equation with spectral parameters, we can mention the following:

$$r(z) = \frac{1_A \otimes 1_A}{z}. \tag{25}$$

Remark 4 *The solution constructed in [9] [10] and [5] have all the above polar part; however, the ones from the next section can have more complicated polar part with higher multiplicity.*

Elliptic Curves and the Associative Yang-Baxter Equation with Spectral Parameters

The aim of this section is to use the main results from [6] and the construction above, in order to obtain new solutions of the Rota-Baxter equation with spectral parameters. As the title suggest, these comes from the world of vector bundles on elliptic curves.

In this section, A is the associative algebra of $n \times n$ matrices over $\mathbb{C}$. Let's τ be a complex parameter in the upper half space and B an invertible $n \times n$ matrix. We denote by E the elliptic curve associated to the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. Also, for

$$q = e^{\pi i \tau} \tag{26}$$

we denote by

$$\theta(z) = 2q^{\frac{1}{4}}\Sigma(-1)^n q^{n(n+1)} sin((2n+1)\pi z) \tag{27}$$

the Jacobi theta function of order one, and by

$$\sigma(u, x) = \frac{\theta'(0)\theta(u + x)}{\theta(u)\theta(x)} \tag{28}$$

the Kronecker theta function.
The main result in [6] can be summarized as follows:

Theorem 1 *For τ and B as above, there is a solution $r_{\tau,B} : D \subseteq \mathbb{C} \times \mathbb{C} \to A \times A$ of the associative Yang-Baxter equation with 2 spectral parameters:*

$$r_{12}(u,x)r_{23}(u+v,y) = r_{13}(u+v,x+y)r_{12}(-v,x) + r_{23}(v,y)r_{13}(u,x+y). \tag{29}$$

Moreover, for B a diagonal matrix, the solution $r_{\tau,B}$ can be expressed in terms of the Kronecker theta function, and for B a Jordan block $r_{\tau,B}$ can be expressed in terms of the derivatives of the Kronecker theta function.

Remark 5 *One should note that the proof is based on the study of certain Massey products in the derived category of coherent sheaves $D^b(Coh(E))$ on the elliptic curve E.*

Putting all the above together we obtain the main result of this paper:

Theorem 2 *For τ and B as above, there is a Rota-Baxter operator with 2 spectral parameters for the algebra A of $n \times n$ complex matrices.*

CONCLUSIONS AND FUTURE DIRECTIONS

We presented the interplay between the associative Yang-Baxter equation and Rota-Baxter type operators with or without spectral parameters or weights. Using some solution of the former from the theory of vector bundles over elliptic curves, we obtained Rota-Baxter operators with 2 spectral parameters but of weight $\theta = 0$. Viewing this last point we can address the following:

Question 1 *Which is the right geometric setting, involving a weight type parameter θ which could produce fully weighted Rota-Baxter type operators with spectral parameters?*

Also, taking into account the fact that the elliptic curves are the 1-dimensional Calabi-Yau's, we can also consider the following:

Question 2 *Is there a higher dimensional recipe, or better, one coming from an arbitrary Calabi-Yau category replacing the derived category of coherent sheaves, which can produce interesting solutions of the Rota-Baxter equation?*

ACKNOWLEDGMENTS

My interest on the subject of Rota-Baxter Operators was opened by a series of lectures given by F. Panaite in the Topology Seminar at IMAR.

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