Weak solutions for nonlinear antiplane problems

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The antiplane shear deformation is the deformation that we expect to appear after loading a long cylinder in the direction of its generators, such that the displacement field is parallel to the generators of the cylinder and it is independent of the axial coordinate. The following boundary value problem is an antiplane one.

Problem 1. Find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\mathrm{div}\big(\mu(\boldsymbol{x})\nabla u(\boldsymbol{x})\big) + f_0(\boldsymbol{x}) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$u(\boldsymbol{x}) = 0 \quad \text{on} \quad \Gamma_1, \tag{2}$$

$$\mu(\boldsymbol{x})\frac{\partial u}{\partial \nu}(\boldsymbol{x}) = f_2(\boldsymbol{x}) \quad \text{on} \quad \Gamma_2, \tag{3}$$

$$\left.\begin{array}{r} |\mu(\boldsymbol{x})\dfrac{\partial u}{\partial \nu}(\boldsymbol{x})| \le g(\boldsymbol{x}) \\ |\mu(\boldsymbol{x})\dfrac{\partial u}{\partial \nu}(\boldsymbol{x})| < g(\boldsymbol{x}) \Rightarrow u(\boldsymbol{x}) = 0 \\ |\mu(\boldsymbol{x})\dfrac{\partial u}{\partial \nu}(\boldsymbol{x})| = g(\boldsymbol{x}) \Rightarrow \\ \text{there exists } \alpha > 0 \text{ s.t. } \mu(\boldsymbol{x})\dfrac{\partial u}{\partial \nu}(\boldsymbol{x}) = -\alpha u(\boldsymbol{x}) \end{array}\right\} \quad \text{on} \quad \Gamma_3. \tag{4}$$

Here $\Omega \subset \mathbb{R}^2$ is an open, bounded, connected subset, with smooth boundary Γ partitioned in three measurable parts Γ_1, Γ_2, Γ_3 such that the Lebesgue measure of Γ_1 is positive. The domain $\Omega \subset \mathbb{R}^2$ represents the cross section of the cylinder, the unknown $u : \bar{\Omega} \to \mathbb{R}$ is the axial component of the displacement vector, $\mu : \bar{\Omega} \to \mathbb{R}_+$ denotes the Lamé coefficient of the material, and the functions $f_0 : \Omega \to \mathbb{R}$, $f_2 : \Gamma_2 \to \mathbb{R}$ are related to the densities of the body forces and surface traction, respectively. Problem 1 is a *frictional contact problem*. The boundary condition (4) is Tresca's law, the function $g : \Gamma_3 \to \mathbb{R}_+$ denoting the *friction bound*.

We are interested on the weak solvability of Problem 1 assuming that

$$
\begin{aligned}
&\mu \in L^\infty(\Omega),\ \mu(\boldsymbol{x}) \geq \mu^* > 0 \text{ a.e. in } \Omega,\\
&f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2),\\
&g \in L^\infty(\Gamma_3),\ g \geq 0 \text{ a.e. on } \Gamma_3.
\end{aligned}
$$

We define *weak solution* for Problem 1 in two ways. In both cases we prove the existence and the uniqueness of the weak solution. The results are based on the theory of variational inequalities. Finally, we comment on the advantages and disadvantages of each approach. Also, some extensions of the results are briefly mentioned.

Main references

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