Algebra and topology of group extensions

PART II

Alex Suciu

Northeastern University

Topology Seminar

Institute of Mathematics of the Romanian Academy

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Derived series and Alexander invariant

- The *derived series* of a group G is defined inductively by $G^{(0)} = G$, $G^{(1)} = G'$, $G^{(2)} = G''$, and $G^{(r)} = [G^{(r-1)}, G^{(r-1)}]$.
- Its terms are fully invariant (and thus, normal) subgroups.
- Successive quotients: $G^{(r-1)}/G^{(r)} = \big(G^{(r-1)}\big)_{\mathrm{ab}}$.
- $G/G^{(\ell)}$ is the maximal solvable quotient of G of length ℓ.
- *Alexander invariant*:

  $$B(G) := G'/G''\,,$$

  a $\mathbb{Z}[G_{\mathrm{ab}}]$-module via $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$.
- If X is a connected CW-complex with $\pi_1(X) = G$, then

  $$B(G) = H_1(X^{\mathrm{ab}}, \mathbb{Z}) = H_1(X, \mathbb{Z}[G_{\mathrm{ab}}])\,,$$

  where $q\colon X^{\mathrm{ab}} \to X$ is the universal abelian cover.

Example

- Let $X = \bigvee^n S^1$. Then $\pi_1(X) = F_n$, $(F_n)_{\mathrm{ab}} = \mathbb{Z}^n$, and $X^{\mathrm{ab}} = $ 1-skeleton of $(T^n)^{\mathrm{ab}}$.
- $(C_*((T^n)^{\mathrm{ab}}; \mathbb{Z}), \partial^{\mathrm{ab}})$ is the Koszul complex on $t_1 - 1, \dots, t_n - 1$ over the ring $\Lambda_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.
- Hence, $B(F_n) = \operatorname{coker}\left(\partial_3^{\mathrm{ab}} \colon \Lambda_n^{\binom{n}{3}} \to \Lambda_n^{\binom{n}{2}}\right)$.
  - $B(F_2) = \Lambda_2$.
  - $B(F_3) = \operatorname{coker}\left(\begin{pmatrix} 1 - t_3 & t_2 - 1 & 1 - t_1 \end{pmatrix} \colon \Lambda_2 \to \Lambda_2^3\right)$.

- *Alexander module*: $A(G) = \mathbb{Z}[G_{\mathrm{ab}}] \otimes_{\mathbb{Z}[G]} I(G)$, where $I(G) = \ker(\varepsilon \colon \mathbb{Z}[G] \to \mathbb{Z})$ is the augmentation ideal.
- $A(G) = H_1(X^{\mathrm{ab}}, q^{-1}(x_0)) = \operatorname{coker}(\partial_2^{\mathrm{ab}}) \colon C_2(X^{\mathrm{ab}}) \to C_1(X^{\mathrm{ab}})$.
- Crowell exact sequence:

$$0 \to B(G) \to A(G) \to I(G_{\mathrm{ab}}) \to 0\,.$$

- A homomorphism $\alpha\colon G \to H$ induces compatible homomorphisms, $\alpha_{\text{ab}}\colon G_{\text{ab}} \to H_{\text{ab}}$ and $B(\alpha)\colon B(G) \to B(H)$.
- That is, if $\tilde{\alpha}_{\text{ab}}\colon \mathbb{Z}[G_{\text{ab}}] \to \mathbb{Z}[H_{\text{ab}}]$ is the linear extension of α_{ab} to a ring map, then $B(\alpha)$ is a morphism of modules covering α_{ab}, i.e., $B(\alpha)(rm) = \tilde{\alpha}_{\text{ab}}(r) \cdot B(\alpha)(m)$ for all $r \in \mathbb{Z}[G_{\text{ab}}]$ and $m \in B(G)$.
- $B(\alpha)$ factors as $B(G) \to B(H)_{\alpha} \to B(H)$, where $B(H)_{\alpha}$ is the $\mathbb{Z}[G_{\text{ab}}]$-module obtained from $B(H)$ by restriction of scalars via $\tilde{\alpha}$.
- Alternatively, if $f\colon (X, x_0) \to (Y, y_0)$ is a cellular map and $f_{\sharp}\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is the induced morphism, then $B(f_{\sharp})\colon B(\pi_1(X)) \to B(\pi_1(Y))$ is equal to $f_*\colon H_1(X^{\text{ab}}) \to H_1(Y^{\text{ab}})$.

THEOREM (MASSEY 1980)

Let $I = I(G_{\text{ab}})$. Then $I^n B(G) = \gamma_{n+2}(G/G'')$, and thus

$$\operatorname{gr}_n(B) \cong \operatorname{gr}_{n+2}(G/G''), \quad \textit{for all } n \geqslant 0.$$

LCS and Chen ranks

- Let $\mathrm{gr}(G) = \bigoplus_{n\geqslant 1} \gamma_n(G)/\gamma_{n+1}(G)$ be the associated graded Lie algebra of G.
- Recall that $\mathrm{gr}(G)$ is generated by $\mathrm{gr}_1(G) = G_{\mathrm{ab}}$, and so $\mathrm{gr}(G) \otimes \mathbb{Q}$ is generated by $H_1(G, \mathbb{Q}) = G_{\mathrm{ab}} \otimes \mathbb{Q}$.
- Thus, if $b_1(G) := \dim_{\mathbb{Q}} H_1(G, \mathbb{Q})$ is finite, all graded pieces of $\mathrm{gr}(G) \otimes \mathbb{Q}$ are finite-dimensional. Define:
  - *LCS ranks*: $\phi_n(G) := \dim_{\mathbb{Q}} \mathrm{gr}_n(G) \otimes \mathbb{Q}$.
  - *Chen ranks*: $\theta_n(G) := \phi_n(G/G'')$.
- $\theta_n(G) \leqslant \phi_n(G)$, with equality for $n \leqslant 3$.
- By Massey's theorem, the Chen ranks are computed by the Hilbert series of $\mathrm{gr}(B)$:

$$\mathrm{Hilb}(\mathrm{gr}(B(G) \otimes \mathbb{Q}), t) = \sum_{n\geqslant 0} \theta_{n+2}(G) t^n.$$

Rational derived series and Alexander invariant

- The *rational derived series* of G is defined by $G_{\mathbb{Q}}^{(0)} = G$ and $G_{\mathbb{Q}}^{(r)} = \sqrt{[G_{\mathbb{Q}}^{(r-1)}, G_{\mathbb{Q}}^{(r-1)}]}$. [Stallings, Harvey, Cochran]
- $G_{\mathbb{Q}}^{(r)}/G_{\mathbb{Q}}^{(r+1)} \cong \big(G_{\mathbb{Q}}^{(r)}\big)_{\mathrm{abf}}$.
- $G'_{\mathbb{Q}}$ is also known as the *Johnson kernel*. We have: $G/G'_{\mathbb{Q}} = G_{\mathrm{abf}}$ and $G'_{\mathbb{Q}}/G' \cong \mathrm{Tors}(G_{\mathrm{ab}})$.
- *Rational Alexander invariant*: $B_{\mathbb{Q}}(G) := G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$, viewed as a $\mathbb{Z}G_{\mathrm{abf}}$-module via $gG'_{\mathbb{Q}} \cdot xG''_{\mathbb{Q}} = gxg^{-1}G''_{\mathbb{Q}}$ for $g \in G$ and $x \in G'_{\mathbb{Q}}$.

LEMMA

Let X be a connected CW-complex with $\pi_1(X) = G$. Then

- $B_{\mathbb{Q}}(G) \cong H_1(X^{\mathrm{abf}}, \mathbb{Z})/\mathbb{Z}\text{-}\mathrm{Tors}$ *as* $\mathbb{Z}[G_{\mathrm{abf}}]$*-modules.*
- $B_{\mathbb{Q}}(G) \otimes \mathbb{Q} \cong H_1(X^{\mathrm{abf}}, \mathbb{Q})$ *as* $\mathbb{Q}[G_{\mathrm{abf}}]$*-modules.*
- *If* G_{ab} *is torsion-free, then* $B_{\mathbb{Q}}(G) \cong B(G)/\mathbb{Z}\text{-}\mathrm{Tors}$ *as* $\mathbb{Z}[G_{\mathrm{ab}}]$*-mods.*

- *Rational Alexander module*: $A_{\mathbb{Q}}(G) = \mathbb{Z}[G_{\mathrm{abf}}] \otimes_{\mathbb{Z}[G]} I(G)$.
- $A_{\mathbb{Q}}(G) \otimes \mathbb{Q} = H_1(X^{\mathrm{abf}}, q_0^{-1}(x_0), \mathbb{Q}) = \operatorname{coker}(\partial_2^{\mathrm{abf}}) \otimes \mathbb{Q} \colon C_2(X^{\mathrm{abf}}, \mathbb{Q}) \to C_1(X^{\mathrm{abf}}, \mathbb{Q})$.
- $\mathbb{Q}$-Crowell: $0 \to B_{\mathbb{Q}}(G) \otimes \mathbb{Q} \to A_{\mathbb{Q}}(G) \otimes \mathbb{Q} \to I(G_{\mathrm{abf}}) \otimes \mathbb{Q} \to 0$.

Proposition

Extend $\nu \colon G_{\mathrm{ab}} \twoheadrightarrow G_{\mathrm{abf}}$ *to a ring map,* $\tilde{\nu} \colon \mathbb{Z}[G_{\mathrm{ab}}] \twoheadrightarrow \mathbb{Z}[G_{\mathrm{abf}}]$*. Then:*

- $G' \hookrightarrow G'_{\mathbb{Q}}$ *induces a functorial* $\tilde{\nu}$*-morphism,* $\kappa \colon B(G) \to B_{\mathbb{Q}}(G)$.
- *If* $\operatorname{Tors}(G_{\mathrm{ab}})$ *is finite, then* $\kappa \otimes \mathbb{Q} \colon B(G) \otimes \mathbb{Q} \to B_{\mathbb{Q}}(G) \otimes \mathbb{Q}$ *is surjective.*
- *If* G_{ab} *is torsion-free, then* $\kappa \otimes \mathbb{Q} \colon B(G) \otimes \mathbb{Q} \to B_{\mathbb{Q}}(G) \otimes \mathbb{Q}$ *is an isomorphism.*

Theorem

Let $I = I(G_{\mathrm{abf}}) \otimes \mathbb{Q}$*. Then* $I^n(B_{\mathbb{Q}}(G) \otimes \mathbb{Q}) = \gamma_{n+2}^{\mathbb{Q}}(G/G''_{\mathbb{Q}}) \otimes \mathbb{Q}$ *for all* n.

THEOREM (DIMCA–PAPADIMA–HAIN 2014, S. 2021)

Let G be a group with $b_1(G) < \infty$. Then $\kappa\colon B(G) \to B_{\mathbb{Q}}(G)$ yields

- *An isomorphism $\hat{\kappa} \otimes \mathbb{Q}\colon \widehat{B(G) \otimes \mathbb{Q}} \xrightarrow{\simeq} \widehat{B_{\mathbb{Q}}(G) \otimes \mathbb{Q}}$ of filtered modules.*
- *An isomorphism $\operatorname{gr}(\kappa) \otimes \mathbb{Q}\colon \operatorname{gr}(B(G) \otimes \mathbb{Q}) \to \operatorname{gr}(B_{\mathbb{Q}}(G) \otimes \mathbb{Q})$ of graded modules.*

- We may define $\phi_n^{\mathbb{Q}}(G)$ and $\theta_n^{\mathbb{Q}}(G)$ as before.
- By Bass–Lubotzky (or the above theorem for second equality):

$$\phi_n^{\mathbb{Q}}(G) = \phi_n(G) \text{ and } \theta_n^{\mathbb{Q}}(G) = \theta_n(G).$$

- By $\mathbb{Q}$-Massey:

$$\theta_n(G) = \dim_{\mathbb{Q}} \operatorname{gr}_{n-2}(B_{\mathbb{Q}}(G) \otimes \mathbb{Q}).$$

Mod-p derived series

- The *mod-p derived series* of G is defined by $G_p^{(0)} = G$ and

$$G_p^{(r)} = \Big\langle \big(G_p^{(r-1)}\big)^p, \big[G_p^{(r-1)}, G_p^{(r-1)}\big] \Big\rangle.$$

  [Stallings, Harvey, Cochran, Lackenby]

- Its terms are fully invariant (thus, normal) subgroups.
- $G_p^{(r-1)}/G_p^{(r)} \cong H_1(G_p^{(r-1)}, \mathbb{Z}_p)$. In particular, $G/G'_p = H_1(G, \mathbb{Z}_p)$ is the maximal elementary p-abelian quotient of G.
- This is the fastest descending normal (and even subnormal) series for which the successive quotients are $\mathbb{Z}_p$-vector spaces.
- If G is finitely generated, then $G/G_p^{(r)}$ is a finite p-group, with all elements having order dividing p^r.

Mod-p Alexander invariant

- *Mod-p Alexander invariant*: $B_p(G) := G'_p/G''_p$, viewed as a $\mathbb{Z}_p[H_1(G,\mathbb{Z}_p)]$-module via $gG'_p \cdot xG''_p = gxg^{-1}G''_p$ for $g \in G, x \in G'_p$.
- Let $G = \pi_1(X)$ and let $X^{(p)} \to X$ be the p-congruence cover of X, classified by epimorphism $\pi_1(X) \xrightarrow{\text{ab}} H_1(X,\mathbb{Z}) \xrightarrow{\nu_p} H_1(X,\mathbb{Z}_p)$. Then $\pi_1(X^{(p)}) = G'_p$ and $B_p(G) = H_1(X^{(p)},\mathbb{Z}_p)$.
- $G' \hookrightarrow G'_p$ induces a functorial $\tilde{\nu}_p$-morphism, $\kappa_p\colon B(G) \to B_p(G)$, which coincides with $H_1(X^{\text{ab}},\mathbb{Z}_p) \to H_1(X^{(p)},\mathbb{Z}_p)$.

EXAMPLE

If G is abelian, then $B(G) = B_{\mathbb{Q}}(G) = 0$, yet $B_p(G) = G^p/G^{p^2}$. E.g., $B_p(\mathbb{Z}^n) = \mathbb{Z}_p^n$. Also, $B_p(\mathbb{Z}_p \oplus \mathbb{Z}_p) = 0$ yet $B_p(\mathbb{Z}_{p^2}) = \mathbb{Z}_p$.

THEOREM

Let $I = \ker\left(\mathbb{Z}_p[H_1(G,\mathbb{Z}_p)] \xrightarrow{\varepsilon} \mathbb{Z}_p\right)$. *Then* $I^n B_p(G) = \gamma^p_{n+2}(G/G''_p)\ \forall n$.

Ab-exact sequences

- Let $1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$ be an exact sequence.
- There is then a 5-term exact sequence in homology,

$$H_2(G,\mathbb{Z}) \xrightarrow{\pi_*} H_2(Q,\mathbb{Z}) \xrightarrow{\delta} H_1(K,\mathbb{Z})_Q \xrightarrow{\iota_*} H_1(G,\mathbb{Z}) \xrightarrow{\pi_*} H_1(Q,\mathbb{Z}) \to 0$$

- If Q acts trivially on K_{ab}, then $H_1(K,\mathbb{Z})_Q = H_1(K,\mathbb{Z})$.

LEMMA

The following conditions are equivalent.

- *The group Q acts trivially on K_{ab} and $\delta\colon H_2(Q,\mathbb{Z}) \to H_1(K,\mathbb{Z})$ is the zero map.*
- *The sequence* $0 \longrightarrow K_{ab} \xrightarrow{\iota_{ab}} G_{ab} \xrightarrow{\pi_{ab}} Q_{ab} \longrightarrow 0$ *is exact.*

- A split exact sequence $1 \to K \to G \to Q \to 1$ is ab-exact iff Q acts trivially on K_{ab}; that is, $G = K \rtimes Q$ is an almost direct product.
- If $1 \to K \to G \to Q \to 1$ is an ab-exact sequence, then its restriction, $1 \to K' \to G' \to Q' \to 1$ is an exact sequence.

Example

- Let $G_\Gamma = \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E \rangle$ be the RAAG associated to a finite (simple) graph $\Gamma = (V, E)$
- Exact sequence $1 \to N_\Gamma \to G_\Gamma \xrightarrow{\pi} \mathbb{Z} \to 1$, where $\pi(v) = 1, \forall v \in V$.
- Bestvina–Brady 1997: N_Γ is finitely generated iff Γ is connected; N_Γ is finitely presented iff flag complex Δ_Γ is simply connected.
- (Papadima–S. 2007): if Γ is connected, then $\mathbb{Z}$ acts trivially on $H_1(N_\Gamma, \mathbb{Z})$, and so this sequence is $\mathtt{ab}$-exact.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be an ab-exact sequence, and assume Q is abelian. Then:

- $\iota\colon K \hookrightarrow G$ *restricts to an equality,* $K' = G'$.
- $B(\iota)$ *factors through a* $\mathbb{Z}[K_{\mathrm{ab}}]$*-linear isomorphism* $B(K) \xrightarrow{\simeq} B(G)_\iota$.
- *If* G_{ab} *is finitely generated, then* $\theta_n(K) \leqslant \theta_n(G)$ *for all* $n \geqslant 1$.

Assume now that the sequence is also split-exact. Then:

- *The map* ι *induces isomorphisms of graded Lie algebras,* $\mathrm{gr}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G)$ *and* $\mathrm{gr}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G/G'')$.
- *If* $b_1(G) < \infty$, *then* $\phi_n(K) = \phi_n(G)$ *and* $\theta_n(K) = \theta_n(G)$ *for all* $n \geqslant 2$.

Abf-exact sequences

LEMMA / DEFINITION

A sequence $1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$ *is called* abf*-exact if one of the following equivalent conditions is satisfied.*

- *The group* Q *acts trivially on* K_{abf} *and the composite* $H_2(Q, \mathbb{Z}) \xrightarrow{\delta} H_1(K, \mathbb{Z}) \to H_1(K, \mathbb{Z})/\operatorname{Tors}$ *is zero.*
- *The sequence* $0 \longrightarrow K_{\text{abf}} \xrightarrow{\iota_{\text{abf}}} G_{\text{abf}} \xrightarrow{\pi_{\text{abf}}} Q_{\text{abf}} \longrightarrow 0$ *is exact.*

- Suppose K_{abf} is finitely generated. Then the extension is abf-exact iff Q acts trivially on $H_1(K, \mathbb{Q})$ and $\delta \otimes \mathbb{Q} = 0$.
- Reverse implication may not hold if K_{abf} not f.g.

LEMMA

Let $1 \to K \to G \to Q \to 1$ *be a split exact sequence.*

- *The sequence is* abf*-exact iff* Q *acts trivially on* K_{abf}.
- *If* K_{abf} *is finitely generated, then the sequence is* abf*-exact iff* Q *acts trivially on* $H_1(K, \mathbb{Q})$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be an* abf*-exact sequence, and assume* Q *is torsion-free abelian. Then:*

- $\iota\colon K \hookrightarrow G$ *restricts to an equality,* $K'_{\mathbb{Q}} = G'_{\mathbb{Q}}$*.*
- $B_{\mathbb{Q}}(\iota)$ *factors through a* $\mathbb{Z}[K_{\mathrm{abf}}]$*-isomorphism* $B_{\mathbb{Q}}(K) \xrightarrow{\simeq} B_{\mathbb{Q}}(G)_{\iota}$*.*
- *If* G_{abf} *is finitely generated, then* $\theta_n(K) \leqslant \theta_n(G)$ *for all* $n \geqslant 1$*.*

Assume now that the sequence is also split-exact. Then:

- *The map* ι *induces isomorphisms of graded Lie algebras,* $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G)$ *and* $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G/G'')$*.*
- *If* $b_1(G) < \infty$*, then* $\phi_n(K) = \phi_n(G)$ *and* $\theta_n(K) = \theta_n(G)$ *for all* $n \geqslant 2$*.*

Corollary

Let X *be a connected CW-complex with* $b_1(X) < \infty$*, let* $f\colon X \to X$ *be a map inducing the identity on* $H_1(X, \mathbb{Q})$*, and let* T_f *be its mapping torus. Then* $\phi_n(\pi_1(X)) = \phi_n(\pi_1(T_f))$ *and* $\theta_n(\pi_1(X)) = \theta_n(\pi_1(T_f))$ *for all* $n \geqslant 2$*.*

References

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