Algebra and topology of group extensions

PART I

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June 25, 2021

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N-series

- Following Lazard (1954), we define an N-*series* for a group G to be a descending filtration $G = K_1 \geqslant K_2 \geqslant \cdots \geqslant K_n \geqslant \cdots$ such that

$$[K_m, K_n] \subseteq K_{m+n} \text{ for all } m, n \geqslant 1.$$

- In particular, $K = \{K_n\}_{n\geqslant 1}$ is a *central series*, i.e., $[G, K_n] \subseteq K_{n+1}$.
- Thus, it is also a *normal series*, that is, $K_n \lhd G$ for all $n \geqslant 1$.
- Consequently, each quotient K_n/K_{n+1} lies in the center of G/K_{n+1}, and thus is an abelian group.
- If all those quotients are torsion-free, K is called an N_0-*series*.
- *Associated graded Lie algebra*:

$$\operatorname{gr}^K(G) = \bigoplus_{n\geqslant 1} K_n/K_{n+1},$$

with addition induced by $\cdot\colon G \times G \to G$, and Lie bracket $[\ ,\ ]\colon \operatorname{gr}_m \times \operatorname{gr}_n \to \operatorname{gr}_{m+n}$ induced by $[x, y] := xyx^{-1}y^{-1}$.

- The *isolator* in G of a subset $S \subseteq G$ is the subset

$$\sqrt{S} := \sqrt[G]{S} = \{g \in G \mid g^m \in S \text{ for some } m \in \mathbb{N}\}$$

- Clearly, $S \subseteq \sqrt{S}$ and $\sqrt{\sqrt{S}} = \sqrt{S}$. Also, if $\varphi\colon G \to H$ is a homomorphism, and $\varphi(S) \subseteq T$, then $\varphi(\sqrt[G]{S}) \subseteq \sqrt[H]{T}$.

- The isolator of a subgroup of G need not be a subgroup; for instance, $\sqrt[G]{\{1\}} = \operatorname{Tors}(G)$, which is not a subgroup in general (although it is if G is nilpotent).

- If $N \lhd G$ is a normal subgroup, then $\sqrt[G]{N} = \pi^{-1}(\operatorname{Tors}(G/N))$, where $\pi\colon G \twoheadrightarrow G/N$, and so $\sqrt[G]{N}/N \cong \operatorname{Tors}(G/N)$.

PROPOSITION (MASSUYEAU 2007)

Suppose $K = \{K_n\}_{n\geqslant 1}$ is an N-series for G. Then $\sqrt{K} := \{\sqrt{K_n}\}_{n\geqslant 1}$ is an N_0-series for G.

Lower central series

- The *lower central series*, $\gamma(G) = \{\gamma_n(G)\}_{n\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{n+1}(G) = [G, \gamma_n(G)]$.
- It is an N-series (P. Hall, 1934).
- If K is a descending central series for G, then $\gamma_n(G) \leqslant K_n$ for all n.
- The γ_n's are fully invariant subgroups, i.e., $\varphi\colon G \to H$ morphism $\Rightarrow$ $\varphi(\gamma_n(G)) \subseteq \gamma_n(H)$.
- $\operatorname{gr}(G) = \bigoplus_{n\geqslant 1} \gamma_n(G)/\gamma_{n+1}(G)$ is generated by $\operatorname{gr}_1(G) = G_{\text{ab}}$.
- For any N-series K, there is a canonical map $\operatorname{gr}(G) \to \operatorname{gr}^K(G)$.
- $\Gamma_n := G/\gamma_n(G)$ is the maximal $(n-1)$-step nilpotent quotient of G.
- $G/\gamma_2(F) = G_{\text{ab}}$, while $G/\gamma_3(G) \leftrightarrow H^{\leqslant 2}(G, \mathbb{Z})$.
- G is residually nilpotent if and only if $\gamma_\omega(G) := \bigcap_{n\geqslant 1} \gamma_n(G)$ is the trivial subgroup.

The rational lower central series

- The *rational lower central series*, $\gamma^{\mathbb{Q}}(G)$, is defined by $\gamma_1^{\mathbb{Q}}(G) = G$ and $\gamma_{n+1}^{\mathbb{Q}}(G) = \sqrt{[G, \gamma_n^{\mathbb{Q}}(G)]}$. (Stallings, 1965)

LEMMA

$\gamma_n^{\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}$, *for all* $n \geqslant 1$.

- Hence, $\gamma^{\mathbb{Q}}(G)$ is an N_0-series (since $\gamma(G)$ is an N-series).
- $G/\gamma_n^{\mathbb{Q}}(G) = \Gamma_n / \operatorname{Tors}(\Gamma_n)$ is the maximal torsion-free $(n-1)$-step nilpotent quotient of G; in particular, $G/\gamma_2^{\mathbb{Q}} G = G_{\mathrm{abf}}$.
- Associated graded Lie algebra: $\operatorname{gr}^{\mathbb{Q}}(G) = \bigoplus_{n\geq 1} \gamma_n^{\mathbb{Q}} G / \gamma_{n+1}^{\mathbb{Q}} G$.
- G is residually torsion-free nilpotent (RTFN) iff $\gamma_\omega^{\mathbb{Q}}(G) = \{1\}$.

PROPOSITION (BASS & LUBOTZKY, 1994)

- $\operatorname{gr}(G) \to \operatorname{gr}^{\mathbb{Q}}(G)$ *has torsion kernel and cokernel in each degree.*
- $\operatorname{gr}(G) \otimes \mathbb{Q} \to \operatorname{gr}^{\mathbb{Q}}(G) \otimes \mathbb{Q}$ *is an isomorphism.*

Mod-p lower central series

- Fix a prime p. The (Stallings) *mod-p lower central series*, $\gamma^p(G)$, is defined by $\gamma_1^p(G) = G$ and $\gamma_{n+1}^p(G) = \big\langle \big(\gamma_n^p(G)\big)^p, \big[G, \gamma_n^p(G)\big] \big\rangle$.
- $\big(\gamma_n^p(G)\big)^p \subseteq \gamma_{n+1}^p(G)$; thus, $\gamma^p(G)$ is a p-torsion series.
- $\gamma_2^p(G) = \langle G^p, G' \rangle$, and so $G/\gamma_2^p(G) = G_{\mathrm{ab}} \otimes \mathbb{Z}_p = H_1(G, \mathbb{Z}_p)$.
- (Paris 2009) $\gamma^p(G)$ is an N-series. Moreover, G is residually p iff $\gamma_\omega^p(G) = \{1\}$.
- $\gamma^p(G)$ is the fastest descending central series among all p-torsion series for G.
- The quotients $\gamma_n^p(G)/\gamma_{n+1}^p(G)$ are elementary abelian p-groups. Thus, $\mathrm{gr}^p(G)$ is a Lie algebra over $\mathbb{Z}_p$.
- The map $G \to G$, $x \mapsto x^p$ defines maps $\mathrm{gr}_n^p(G) \to \mathrm{gr}_{n+1}^p(G)$. The $\mathbb{Z}_p$-Lie algebra $\mathrm{gr}^p(G)$ is generated—through Lie brackets and these power operations—by $\mathrm{gr}_1^p(G) = H_1(G, \mathbb{Z}_p)$.

Split extensions of groups

- Consider a split exact sequence

$$1 \longrightarrow A \xrightarrow{\alpha} B \underset{\curvearrowleft\sigma}{\xrightarrow{\beta}} C \longrightarrow 1. \qquad (*)$$

- The splitting homomorphism σ satisfies $\beta \circ \sigma = \mathrm{id}_C$; it defines an action of C on A via the homomorphism $\varphi\colon C \to \mathrm{Aut}(A)$ given by

$$\alpha(\varphi(c)(a)) = \sigma(c)\alpha(a)\sigma(c)^{-1}.$$

- This realizes B as a split extension, $B = A \rtimes_{\varphi} C$; that is, the set $A \times C$ with operation $(a_1, c_1) \cdot (a_2, c_2) = (a_1\varphi(c_1)(a_2), c_1c_2)$.
- Conversely, every split extension $B = A \rtimes_{\varphi} C$ gives rise to (*).
- We identify C with its image under σ, and thus view it as $C \leqslant B$, and identify A with its image under α and view it as $A \lhd B$.
- The action of C on A is then the restriction of the conjugation action in B, that is, $\varphi(c)(a) = cac^{-1}$. Also, every $b \in B$ can be written uniquely as $b = ac$, for some $a \in A$, $c \in C$.

The lower central series of a split extension

- Goal: Describe the lower central series $\gamma(B) = \{\gamma_n(B)\}_{n\geqslant 1}$ of a split extension, $B = A \rtimes_\varphi C$, in terms of $\gamma(A)$, $\gamma(C)$, and φ.

- Following Guaschi and Pereiro (2020), we define a sequence $L = \{L_n\}_{n\geqslant 1}$ of subgroups of A by setting $L_1 = A$ and letting
$$L_{n+1} = \langle [A, L_n], [A, \gamma_n(C)], [L_n, C] \rangle .$$

- Guaschi–Pereiro showed that L is a descending normal series. We strengthen their result with the next lemma, which we then use to give a quicker proof of the next theorem.

LEMMA

L is an N-series for A

THEOREM (GUASCHI–PEREIRO 2020, S. 2021)

- $\varphi\colon C \to \mathrm{Aut}(A)$ *restricts to* $\varphi\colon \gamma_n(C) \to \mathrm{Aut}(L_n)$.
- $\gamma_n(B) = L_n \rtimes_\varphi \gamma_n(C)$.

Theorem

Let $1 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 1$ *be a split exact sequence of groups, with monodromy* $\varphi\colon C \to \operatorname{Aut}(A)$. *There is then an induced split exact sequence of graded Lie algebras,*

$$0 \longrightarrow \operatorname{gr}^L(A) \xrightarrow{\operatorname{gr}^L(\alpha)} \operatorname{gr}(B) \xrightarrow{\operatorname{gr}(\beta)} \operatorname{gr}(C) \longrightarrow 0 .$$

Consequently, $\operatorname{gr}(B) \cong \operatorname{gr}^L(A) \rtimes_{\bar\varphi} \operatorname{gr}(C)$, *where the monodromy* $\bar\varphi\colon \operatorname{gr}(C) \to \operatorname{Der}(\operatorname{gr}^L(A))$ *is the map of Lie algebras induced by* φ.

Example

- Let $K = \langle a, t \mid tat^{-1} = a^{-1} \rangle$ be the Klein bottle group.
- $K = A \rtimes_\varphi C$, where $C = \langle t \rangle$ acts by inversion on $A = \langle a \rangle$.
- $L_n = \langle a^{2^{n-1}} \rangle$ and $\operatorname{gr}^L_n(A) = \mathbb{Z}_2$ for $n \geqslant 1$.
- $\gamma_n(A) = \{1\}$ and $\operatorname{gr}_n(A) = 0$ for $n > 1$.
- By the theorem, $\gamma_n(K) = L_n$ for $n > 1$, and thus $\gamma_\omega(K) = \{1\}$.

Almost direct products

- A split extension $B = A \rtimes_{\varphi} C$ is called an *almost direct product* if C acts trivially on the abelianization $A_{\mathrm{ab}} = H_1(A, \mathbb{Z})$.
- That is, the monodromy factors through a map $\varphi\colon C \to \mathcal{T}(A)$, where $\mathcal{T}(A) \coloneqq \ker\big(\mathrm{Aut}(A) \to \mathrm{Aut}(A_{\mathrm{ab}})\big)$ is the Torelli group of A.
- Equivalently, $\varphi(c)(a) \cdot a^{-1} \in A'$, for all $c \in C$ and $a \in A$.
- If we view C as a subgroup of G via the splitting $\sigma\colon C \to B$, so that $\varphi(c)(a) \cdot a^{-1} = [c, a]$, the condition most succinctly reads as
$$[A, C] \subseteq \gamma_2(A)\,.$$
- Ex: $P_n = F_{n-1} \rtimes \varphi P_{n-1}$, where $\varphi\colon P_{n-1} \hookrightarrow \mathcal{T}(F_{n-1}) \subset \mathrm{Aut}(F_{n-1})$ is the Artin embedding of the pure braid group.

THEOREM

If $B = A \rtimes_{\varphi} C$ is an almost direct product, then $L = \gamma(A)$.

- As a corollary, we recover well-known results of Falk and Randell.

Corollary (Falk–Randell 1985)

Let $B = A \rtimes_{\varphi} C$ be an almost direct product. Then

- $\gamma_n(B) = \gamma_n(A) \rtimes_{\varphi} \gamma_n(C)$ *for all* $n \geqslant 1$.
- *The corresponding split exact sequence restricts to split exact sequences* $1 \longrightarrow \gamma_n(A) \xrightarrow{\alpha} \gamma_n(B) \xrightarrow{\beta} \gamma_n(C) \longrightarrow 1$ *for all* n.

Corollary (Falk–Randell 1988)

Suppose $B = A \rtimes_{\varphi} C$ is an almost direct product of two residually nilpotent groups. Then B is also residually nilpotent.

Corollary (Falk–Randell 1985)

If $B = A \rtimes_{\varphi} C$ is an almost direct product, then $\operatorname{gr}(B) \cong \operatorname{gr}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}(C)$, where $\bar{\varphi}\colon \operatorname{gr}(C) \to \operatorname{Der}(\operatorname{gr}(A))$ is the map of Lie algebras induced by φ.

The rational lower central series of a split extension

- To describe $\gamma^{\mathbb{Q}}(B) = \sqrt{\gamma(B)}$ we use the sequence $\sqrt{L} = \{\sqrt{L_n}\}_{n\geqslant 1}$.
- Recall we showed that L is an N-series for A.
- Thus, $\sqrt{L}$ is an N_0-series for A.

THEOREM

Let $B = A \rtimes_{\varphi} C$. Then:

- $\varphi\colon C \to \operatorname{Aut}(A)$ *restricts to* $\varphi\colon \sqrt[C]{\gamma_n(C)} \to \operatorname{Aut}\big(\sqrt[A]{L_n}\big)$.
- $\sqrt[B]{\gamma_n(B)} = \sqrt[A]{L_n} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)}$.
- $\operatorname{gr}^{\mathbb{Q}}(B) \cong \operatorname{gr}^{\sqrt{L}}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^{\mathbb{Q}}(C)$.

Rational almost direct products

- $B = A \rtimes_{\varphi} C$ is called a *rational almost direct product* if C acts trivially on the torsion-free abelianization $A_{\mathrm{abf}} = H_1(A, \mathbb{Z}) / \operatorname{Tors}$.
- Equivalently, $\varphi(c)(a) \cdot a^{-1} \in \sqrt{A'}$, for all $c \in C$ and $a \in A$, or, $[A, C] \subseteq \sqrt{\gamma_2(A)}$.
- If C acts trivially on A_{abf}, then it acts trivially on $A_{\mathrm{abf}} \otimes \mathbb{Q}$. The converse holds if A_{abf} is finitely generated, but not in general.

THEOREM

Let $B = A \rtimes_{\varphi} C$ a rational almost direct product. Then

- $\sqrt[A]{L_n} = \sqrt[A]{\gamma_n(A)}$ *for all* n.
- $\sqrt[B]{\gamma_n(B)} = \sqrt[A]{\gamma_n(A)} \rtimes_{\varphi} \sqrt[C]{\gamma_n(C)}$.
- $\operatorname{gr}^{\mathbb{Q}}(B) \cong \operatorname{gr}^{\mathbb{Q}}(A) \rtimes_{\bar{\varphi}} \operatorname{gr}^{\mathbb{Q}}(C)$.

COROLLARY

Let $B = A \rtimes C$ be a split extension of RTFN groups. If C acts trivially on A_{abf}, then B is also RTFN.

The mod-p lower central series of a split extension

- Let $B = A \rtimes_{\varphi} C$. Define a sequence of subgroups $L^p = \{L^p_n\}_{n\geqslant 1}$ by setting $L^p_1 = A$ and letting

$$L^p_{n+1} = \big\langle (L^p_n)^p, [A, L^p_n], [A, \gamma^p_n(C)], [L^p_n, C] \big\rangle .$$

Theorem

- *L^p is a p-torsion N-series for A.*
- *$\varphi\colon C \to \operatorname{Aut}(A)$ restricts to $\varphi\colon \gamma^p_n(C) \to \operatorname{Aut}\big(L^p_n\big)$.*
- *$\gamma^p_n(B) = L^p_n \rtimes_{\varphi} \gamma^p_n(C)$.*
- *$\operatorname{gr}^p(B) \cong \operatorname{gr}^{L^p}(A) \rtimes_{\bar\varphi} \operatorname{gr}^p(C)$*

Mod-p almost direct products

- $B = A \rtimes_\varphi C$ is called a *mod-p almost direct product* if C acts trivially on $A_{\mathrm{ab}} \otimes \mathbb{Z}_p = H_1(A, \mathbb{Z}_p)$.
- Equivalently, $[A, C] \subseteq \gamma_2^p(A)$.

THEOREM

If $B = A \rtimes_\varphi C$ is a mod-p almost direct product, then $L^p = \gamma^p(A)$.

Combining the previous two theorems recovers the following result.

COROLLARY (BELLINGERI–GERVAIS, 2016)

Let $B = A \rtimes_\varphi C$ be a mod-p almost direct product. Then,

- $\gamma_n^p(B) = \gamma_n^p(A) \rtimes_\varphi \gamma_n^p(C)$, *for all $n \geqslant 1$.*
- *If A and C are residually p-finite, then B is also residually p-finite.*

COROLLARY

$\mathrm{gr}^p(B) \cong \mathrm{gr}^p(A) \rtimes_{\bar\varphi} \mathrm{gr}^p(C)$.

REFERENCES

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