Topology of complex projective hypersurfaces

Laurentiu Maxim

UW-Madison & IMAR

(based on joint work with M. Tibăr and L. Păunescu)

06/18/2021

Setup

- $\mathbb{C}P^{n+1} = \{[x_0 : x_1 : \ldots : x_{n+1}]\}$ complex projective space

Setup

- $\mathbb{C}P^{n+1} = \{[x_0 : x_1 : \ldots : x_{n+1}]\}$ complex projective space
- A homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ defines a complex projective hypersurface

$$V(f) = \{x \in \mathbb{C}P^{n+1} \mid f(x) = 0\}.$$

Setup

- $\mathbb{C}P^{n+1} = \{[x_0 : x_1 : \ldots : x_{n+1}]\}$ complex projective space
- A homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ defines a complex projective hypersurface

$$V(f) = \{x \in \mathbb{C}P^{n+1} \mid f(x) = 0\}.$$

- The singular locus of $V(f)$ is:

$$\mathrm{Sing}\, V(f) = \{x \in V(f) \mid \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_{n+1}}(x) = 0\}$$

Setup

- $\mathbb{C}P^{n+1} = \{[x_0 : x_1 : \ldots : x_{n+1}]\}$ complex projective space
- A homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ defines a complex projective hypersurface

$$V(f) = \{x \in \mathbb{C}P^{n+1} \mid f(x) = 0\}.$$

- The singular locus of $V(f)$ is:

$$\mathrm{Sing}\, V(f) = \{x \in V(f) \mid \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_{n+1}}(x) = 0\}$$

- We are interested in the topology of $V = V(f)$, i.e., its shape, reflected in the computation of topological invariants like fundamental group, Betti numbers or Euler characteristic.

Setup

- $\mathbb{C}P^{n+1} = \{[x_0 : x_1 : \ldots : x_{n+1}]\}$ complex projective space
- A homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ defines a complex projective hypersurface

$$V(f) = \{x \in \mathbb{C}P^{n+1} \mid f(x) = 0\}.$$

- The singular locus of $V(f)$ is:

$$\operatorname{Sing} V(f) = \{x \in V(f) \mid \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_{n+1}}(x) = 0\}$$

- We are interested in the topology of $V = V(f)$, i.e., its shape, reflected in the computation of topological invariants like fundamental group, Betti numbers or Euler characteristic.
- The shape of V is intimately connected to the topology of $\mathbb{C}P^{n+1} \setminus V$, i.e., the view from the outside of V.

Computational tools: Milnor fibration

- Let $V = V(f)$ be defined by a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$. We say $\deg V = d$.

- Let $V = V(f)$ be defined by a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$. We say $\deg V = d$.
- Let $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ be the affine cone on V.

Computational tools: Milnor fibration

- Let $V = V(f)$ be defined by a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_{n+1}]$. We say $\deg V = d$.
- Let $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ be the affine cone on V.
- There is a global (affine) Milnor fibration

$$F = \{f = 1\} \hookrightarrow \mathbb{C}^{n+2} \setminus \widehat{V} \xrightarrow{f} \mathbb{C}^*.$$

with monodromy homeomorphism $h : F \to F$.

Computational tools: Milnor fibration

- Let $V = V(f)$ be defined by a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_{n+1}]$. We say $\deg V = d$.
- Let $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ be the affine cone on V.
- There is a global (affine) Milnor fibration

$$F = \{f = 1\} \hookrightarrow \mathbb{C}^{n+2} \setminus \widehat{V} \xrightarrow{f} \mathbb{C}^*.$$

  with monodromy homeomorphism $h : F \to F$.
- Milnor-Kato-Matsumoto: If $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$, the Milnor fiber F is $(n - s - 1)$-connected (set $s = -1$ if V is nonsingular).

Computational tools: Milnor fibration

- Let $V = V(f)$ be defined by a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$. We say $\deg V = d$.
- Let $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ be the affine cone on V.
- There is a global (affine) Milnor fibration

$$F = \{f = 1\} \hookrightarrow \mathbb{C}^{n+2} \setminus \widehat{V} \xrightarrow{f} \mathbb{C}^*.$$

  with monodromy homeomorphism $h : F \to F$.
- Milnor-Kato-Matsumoto: If $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$, the Milnor fiber F is $(n-s-1)$-connected (set $s = -1$ if V is nonsingular).
- The map $F \to \mathbb{C}P^{n+1} \setminus V$ given by

$$(x_0, \ldots, x_{n+1}) \mapsto [x_0 : \ldots : x_{n+1}]$$

  is an unbranched d-fold cover.

Computational tools: Hopf bundle

- Let $K_V = S^{2n+3} \cap \widehat{V}$ be the link of f at $0 \in \mathbb{C}^{n+2}$.

Computational tools: Hopf bundle

- Let $K_V = S^{2n+3} \cap \widehat{V}$ be the link of f at $0 \in \mathbb{C}^{n+2}$.
- Milnor: K_V is $(n-1)$-connected.

Computational tools: Hopf bundle

- Let $K_V = S^{2n+3} \cap \widehat{V}$ be the link of f at $0 \in \mathbb{C}^{n+2}$.
- Milnor: K_V is $(n-1)$-connected.
- Restricting the Hopf bundle $S^1 \hookrightarrow S^{2n+3} \to \mathbb{C}P^{n+1}$ to V yields the Hopf bundle of the hypersurface V:

$$S^1 \hookrightarrow K_V \to V.$$

Computational tools: Hopf bundle

- Let $K_V = S^{2n+3} \cap \widehat{V}$ be the link of f at $0 \in \mathbb{C}^{n+2}$.
- Milnor: K_V is $(n-1)$-connected.
- Restricting the Hopf bundle $S^1 \hookrightarrow S^{2n+3} \to \mathbb{C}P^{n+1}$ to V yields the Hopf bundle of the hypersurface V:

$$S^1 \hookrightarrow K_V \to V.$$

Computational tools: Hopf bundle

- Let $K_V = S^{2n+3} \cap \widehat{V}$ be the link of f at $0 \in \mathbb{C}^{n+2}$.
- Milnor: K_V is $(n-1)$-connected.
- Restricting the Hopf bundle $S^1 \hookrightarrow S^{2n+3} \to \mathbb{C}P^{n+1}$ to V yields the Hopf bundle of the hypersurface V:

$$S^1 \hookrightarrow K_V \to V.$$

Milnor-Lê: One has a *local* Milnor fibration, Milnor fiber and link associated to any complex hypersurface singularity *germ* $(V, x) \subset (\mathbb{C}^{n+1}, 0)$.

Preliminary results

The homotopy sequence of the Hopf bundle of V yields:

Proposition

The projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is simply-connected for $n \geq 2$ and connected for $n = 1$.

Preliminary results

The homotopy sequence of the Hopf bundle of V yields:

Proposition

The projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is simply-connected for $n \geq 2$ and connected for $n = 1$.

Using Alexander duality and the covering $F \to \mathbb{C}P^{n+1} \setminus V$, yields:

Proposition

$V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$ iff

$$h_* \colon \widetilde{H}_*(F;\mathbb{C}) \longrightarrow \widetilde{H}_*(F;\mathbb{C})$$

has no eigenvalue 1, with $h : F \to F$ the Milnor monodromy.

Preliminary results

The homotopy sequence of the Hopf bundle of V yields:

Proposition

The projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is simply-connected for $n \geq 2$ and connected for $n = 1$.

Using Alexander duality and the covering $F \to \mathbb{C}P^{n+1} \setminus V$, yields:

Proposition

$V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$ iff

$$h_* : \widetilde{H}_*(F;\mathbb{C}) \longrightarrow \widetilde{H}_*(F;\mathbb{C})$$

has no eigenvalue 1, with $h : F \to F$ the Milnor monodromy.

Example (Dimca)

$V_n = \{x_0 x_1 \cdots x_n + x_{n+1}^{n+1} = 0\}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$. However, $H^3(V_2;\mathbb{Z})$ contains 3-torsion.

Lefschetz Theorem

No matter how singular V is, one has:

Theorem (Lefschetz)

Let $V \subset \mathbb{C}P^{n+1}$ be a projective hypersurface. The inclusion $j\colon V \hookrightarrow \mathbb{C}P^{n+1}$ induces cohomology isomorphisms

$$j^k\colon H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \xrightarrow{\cong} H^k(V;\mathbb{Z}) \quad \text{for} \quad k < n,$$

and a monomorphism if $k = n$.

Nonsingular complex projective hypersurfaces

The diffeomorphism type of a nonsingular hypersurface depends only on the degree:

Theorem

Let $f, g \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ be two homogeneous polynomials of the same degree d, such that the corresponding projective hypersurfaces $V(f)$ and $V(g)$ are nonsingular. Then:

(i) *The hypersurfaces $V(f)$ and $V(g)$ are diffeomorphic.*

(ii) *Their complements in $\mathbb{C}P^{n+1}$ are diffeomorphic.*

Nonsingular projective hypersurfaces

Together with the Milnor fibration, this yields:

Proposition

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d nonsingular projective hypersurface. The Euler characteristic of V is given by the formula:

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\}.$$

Nonsingular projective hypersurfaces

Together with the Milnor fibration, this yields:

Proposition

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d nonsingular projective hypersurface. The Euler characteristic of V is given by the formula:

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\}.$$

Example

Assume $n = 1$, i.e., V is a Riemann surface. Topologically, V is characterized by its *genus* $g(V)$, with $\chi(V) = 2 - 2g(V)$. The above formula for $\chi(V)$ yields the genus-degree formula:

$$g(V) = \frac{(d-1)(d-2)}{2}.$$

Cohomology of nonsingular projective hypersurfaces

Lefschetz theorem, Poincaré duality and formula for χ yield:

Theorem

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d nonsingular projective hypersurface. Then $H^*(V;\mathbb{Z})$ is torsion free, and the corresponding Betti numbers are given by:

Cohomology of nonsingular projective hypersurfaces

Lefschetz theorem, Poincaré duality and formula for χ yield:

Theorem

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d nonsingular projective hypersurface. Then $H^*(V;\mathbb{Z})$ is torsion free, and the corresponding Betti numbers are given by:

1. $b_i(V) = 0$ *for* $i \neq n$ *odd or* $i \notin [0, 2n]$.
2. $b_i(V) = 1$ *for* $i \neq n$ *even and* $i \in [0, 2n]$.
3. $b_n(V) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2}$.

Cohomology of a singular projective hypersurface

It is much more difficult to understand the $\mathbb{Z}$-(co)homology of a singular projective hypersurface $V \subset \mathbb{C}P^{n+1}$ in degrees $\geq n$.

Cohomology of a singular projective hypersurface

It is much more difficult to understand the $\mathbb{Z}$-(co)homology of a singular projective hypersurface $V \subset \mathbb{C}P^{n+1}$ in degrees $\geq n$.

Theorem (Kato)

Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. (Set $s = -1$ if V is nonsingular.)

Cohomology of a singular projective hypersurface

It is much more difficult to understand the $\mathbb{Z}$-(co)homology of a singular projective hypersurface $V \subset \mathbb{C}P^{n+1}$ in degrees $\geq n$.

Theorem (Kato)

Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. (Set $s = -1$ if V is nonsingular.) Then

$$H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \ \ \text{for} \ \ n+s+2 \leq k \leq 2n.$$

Cohomology of a singular projective hypersurface

It is much more difficult to understand the $\mathbb{Z}$-(co)homology of a singular projective hypersurface $V \subset \mathbb{C}P^{n+1}$ in degrees $\geq n$.

Theorem (Kato)

Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. (Set $s = -1$ if V is nonsingular.) Then

$$H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \ \ \textit{for} \ \ n+s+2 \leq k \leq 2n.$$

Moreover, if $j : V \hookrightarrow \mathbb{C}P^{n+1}$ is the inclusion, the induced homomorphisms

$$j^k : H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \longrightarrow H^k(V;\mathbb{Z}), \quad n+s+2 \leq k \leq 2n,$$

are given by multiplication by d if k is even.

Cohomology of a singular projective hypersurface

It is much more difficult to understand the $\mathbb{Z}$-(co)homology of a singular projective hypersurface $V \subset \mathbb{C}P^{n+1}$ in degrees $\geq n$.

Theorem (Kato)

Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. (Set $s = -1$ if V is nonsingular.) Then

$$H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \quad \text{for} \quad n+s+2 \leq k \leq 2n.$$

Moreover, if $j : V \hookrightarrow \mathbb{C}P^{n+1}$ is the inclusion, the induced homomorphisms

$$j^k : H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \longrightarrow H^k(V;\mathbb{Z}), \quad n+s+2 \leq k \leq 2n,$$

are given by multiplication by d if k is even.

A proof (by Dimca) uses the connectivity of the Milnor fiber F and the Gysin sequences for the Hopf bundle of V and of $\mathbb{C}P^{n+1}$.

Corollary

Let $V \subset \mathbb{C}P^{n+1}$ be a projective hypersurface which has the same $\mathbb{Z}$-cohomology algebra as $\mathbb{C}P^n$. If $n \geq 2$, then $V \cong \mathbb{C}P^n$ as varieties.

Corollary

Let $V \subset \mathbb{C}P^{n+1}$ be a projective hypersurface which has the same $\mathbb{Z}$-cohomology algebra as $\mathbb{C}P^n$. If $n \geq 2$, then $V \cong \mathbb{C}P^n$ as varieties.

Remark

The cuspidal curve $C = x^2y - z^3 = 0$ in $\mathbb{C}P^2$ is homeomorphic to $\mathbb{C}P^1$, but C is not isomorphic as a variety to $\mathbb{C}P^1$. So the assumption $n \geq 2$ in the above corollary is essential.

Zariski's example

The structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \dots, n+s+1$, can be very different from that of $\mathbb{C}P^n$.

Zariski's example

The structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \dots, n+s+1$, can be very different from that of $\mathbb{C}P^n$. Furthermore, as observed by Zariski in 1930s, the Betti numbers of $V = V(f)$ depend on the position of singularities.

Zariski's example

The structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \dots, n+s+1$, can be very different from that of $\mathbb{C}P^n$. Furthermore, as observed by Zariski in 1930s, the Betti numbers of $V = V(f)$ depend on the position of singularities.

Example (Zariski)

Let

$$V_6 = \{f(x,y,z) + w^6 = 0\} \subset \mathbb{C}P^3$$

be a sextic surface, so that f defines a plane sextic $C_6 \subset \mathbb{C}P^2$ with only six cusp singular points.

Zariski's example

The structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \ldots, n+s+1$, can be very different from that of $\mathbb{C}P^n$. Furthermore, as observed by Zariski in 1930s, the Betti numbers of $V = V(f)$ depend on the position of singularities.

Example (Zariski)

Let

$$V_6 = \{f(x,y,z) + w^6 = 0\} \subset \mathbb{C}P^3$$

be a sextic surface, so that f defines a plane sextic $C_6 \subset \mathbb{C}P^2$ with only six cusp singular points.
If the six cusps of C_6 are situated on a conic in $\mathbb{C}P^2$, e.g., $f(x,y,z) = (x^2+y^2)^3 + (y^3+z^3)^2$, then $b_2(V_6) = 2$.

Zariski's example

The structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \ldots, n+s+1$, can be very different from that of $\mathbb{C}P^n$. Furthermore, as observed by Zariski in 1930s, the Betti numbers of $V = V(f)$ depend on the position of singularities.

Example (Zariski)

Let

$$V_6 = \{f(x,y,z) + w^6 = 0\} \subset \mathbb{C}P^3$$

be a sextic surface, so that f defines a plane sextic $C_6 \subset \mathbb{C}P^2$ with only six cusp singular points.
If the six cusps of C_6 are situated on a conic in $\mathbb{C}P^2$, e.g., $f(x,y,z) = (x^2+y^2)^3 + (y^3+z^3)^2$, then $b_2(V_6) = 2$.
Otherwise, $b_2(V_6) = 0$.

Zariski's example

The structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \ldots, n+s+1$, can be very different from that of $\mathbb{C}P^n$. Furthermore, as observed by Zariski in 1930s, the Betti numbers of $V = V(f)$ depend on the position of singularities.

Example (Zariski)

Let

$$V_6 = \{f(x,y,z) + w^6 = 0\} \subset \mathbb{C}P^3$$

be a sextic surface, so that f defines a plane sextic $C_6 \subset \mathbb{C}P^2$ with only six cusp singular points.
If the six cusps of C_6 are situated on a conic in $\mathbb{C}P^2$, e.g., $f(x,y,z) = (x^2+y^2)^3 + (y^3+z^3)^2$, then $b_2(V_6) = 2$.
Otherwise, $b_2(V_6) = 0$.
This phenomenon is explained by the fact that, while the two types of sextic curves are homeomorphic, they cannot be deformed one into the other.

Smoothing of a singular projective hypersurface

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d.

Smoothing of a singular projective hypersurface

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d. Consider a one-parameter smoothing of degree d:

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \quad (t \in \mathbb{C}),$$

for g a *general* polynomial of degree d.

Smoothing of a singular projective hypersurface

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d. Consider a one-parameter smoothing of degree d:

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \quad (t \in \mathbb{C}),$$

for g a *general* polynomial of degree d. For $t \neq 0$ small enough, V_t is *nonsingular* and *"transversal"* to V.

Smoothing of a singular projective hypersurface

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d. Consider a one-parameter smoothing of degree d:

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \ \ (t \in \mathbb{C}),$$

for g a *general* polynomial of degree d. For $t \neq 0$ small enough, V_t is *nonsingular* and *"transversal"* to V. Let

$$B = \{f = g = 0\}$$

be the base locus of the pencil.

Smoothing of a singular projective hypersurface

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d. Consider a one-parameter smoothing of degree d:

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \;\; (t \in \mathbb{C}),$$

for g a *general* polynomial of degree d. For $t \neq 0$ small enough, V_t is *nonsingular* and *"transversal"* to V. Let

$$B = \{f = g = 0\}$$

be the base locus of the pencil. Consider the incidence variety

$$V_D := \{(x, t) \in \mathbb{C}P^{n+1} \times D \mid x \in V_t\},$$

with D a small disc centered at $0 \in \mathbb{C}$ so that V_t is smooth for all $t \in D^* := D \setminus \{0\}$.

Smoothing of a singular projective hypersurface

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d. Consider a one-parameter smoothing of degree d:

$$V_t := \{f_t = f - tg = 0\} \subset \mathbb{C}P^{n+1} \ \ (t \in \mathbb{C}),$$

for g a *general* polynomial of degree d. For $t \neq 0$ small enough, V_t is *nonsingular* and *"transversal"* to V. Let

$$B = \{f = g = 0\}$$

be the base locus of the pencil. Consider the incidence variety

$$V_D := \{(x, t) \in \mathbb{C}P^{n+1} \times D \mid x \in V_t\},$$

with D a small disc centered at $0 \in \mathbb{C}$ so that V_t is smooth for all $t \in D^* := D \setminus \{0\}$. Let

$$\pi \colon V_D \to D$$

be the proper projection map, with $V = V_0 = \pi^{-1}(0)$ and $V_t = \pi^{-1}(t)$ for all $t \in D^*$ a smoothing of V.

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$.

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$.
V_D is nonsingular if $V = V_0$ has only isolated singularities,

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$. V_D is nonsingular if $V = V_0$ has only isolated singularities, but otherwise V_D has singularities where the base locus B intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$. V_D is nonsingular if $V = V_0$ has only isolated singularities, but otherwise V_D has singularities where the base locus B intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Deligne: associated to $\pi : V_D \to D$ there is the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$,

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$. V_D is nonsingular if $V = V_0$ has only isolated singularities, but otherwise V_D has singularities where the base locus B intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Deligne: associated to $\pi : V_D \to D$ there is the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$, where A is a commutative ring (e.g., $\mathbb{Z}$ or a field), and $\underline{A}_{V_D}$ is the constant sheaf with stalk A at every point.

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$. V_D is nonsingular if $V = V_0$ has only isolated singularities, but otherwise V_D has singularities where the base locus B intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Deligne: associated to $\pi : V_D \to D$ there is the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$, where A is a commutative ring (e.g., $\mathbb{Z}$ or a field), and $\underline{A}_{V_D}$ is the constant sheaf with stalk A at every point.

$\varphi_\pi \underline{A}_{V_D}$ is supported on $\Sigma = \mathrm{Sing}(V)$ and it encodes the reduced Milnor fiber cohomology at points in $V = V_0$.

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$. V_D is nonsingular if $V = V_0$ has only isolated singularities, but otherwise V_D has singularities where the base locus B intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Deligne: associated to $\pi : V_D \to D$ there is the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$, where A is a commutative ring (e.g., $\mathbb{Z}$ or a field), and $\underline{A}_{V_D}$ is the constant sheaf with stalk A at every point.

$\varphi_\pi \underline{A}_{V_D}$ is supported on $\Sigma = \mathrm{Sing}(V)$ and it encodes the reduced Milnor fiber cohomology at points in $V = V_0$.

$\varphi_\pi \underline{A}_{V_D}$ fits into the specialization sequence:

$$\to H^k(V;A) \to H^k(V_t;A) \to H^k(V;\varphi_\pi \underline{A}_{V_D}) \to H^{k+1}(V;A) \to$$

Vanishing cycles. Specialization

The incidence variety V_D is of pure complex dimension $n+1$.
V_D is nonsingular if $V = V_0$ has only isolated singularities, but otherwise V_D has singularities where the base locus B intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

Deligne: associated to $\pi : V_D \to D$ there is the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$, where A is a commutative ring (e.g., $\mathbb{Z}$ or a field), and $\underline{A}_{V_D}$ is the constant sheaf with stalk A at every point.

$\varphi_\pi \underline{A}_{V_D}$ is supported on $\Sigma = \mathrm{Sing}(V)$ and it encodes the reduced Milnor fiber cohomology at points in $V = V_0$.

$\varphi_\pi \underline{A}_{V_D}$ fits into the specialization sequence:

$$\to H^k(V;A) \to H^k(V_t;A) \to H^k(V;\varphi_\pi \underline{A}_{V_D}) \to H^{k+1}(V;A) \to$$

Parusinski-Pragacz, M.-Saito-Schürmann, Tibăr-Siersma:
"$\pi : V_D \to D$ has no vanishing cycles along the base locus B"
(in fact, the Milnor fiber of π at a point in B is contractible).

Euler characteristic of arbitrary projective hypersurfaces

Let $A = \mathbb{Q}$ and take Euler characteristics in the specialization sequence.

Euler characteristic of arbitrary projective hypersurfaces

Let $A = \mathbb{Q}$ and take Euler characteristics in the specialization sequence.

Theorem (Parusiński-Pragacz, M.-Saito-Schürmann)

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d, and fix a Whitney stratification $\mathcal{S}$ of V.

Euler characteristic of arbitrary projective hypersurfaces

Let $A = \mathbb{Q}$ and take Euler characteristics in the specialization sequence.

Theorem (Parusiński-Pragacz, M.-Saito-Schürmann)

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d, and fix a Whitney stratification $\mathcal{S}$ of V. Let $W = \{g = 0\} \subset \mathbb{C}P^{n+1}$ be a nonsingular degree d projective hypersurface which is transversal to $\mathcal{S}$.

Euler characteristic of arbitrary projective hypersurfaces

Let $A = \mathbb{Q}$ and take Euler characteristics in the specialization sequence.

Theorem (Parusiński-Pragacz, M.-Saito-Schürmann)

Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface of degree d, and fix a Whitney stratification $\mathcal{S}$ of V. Let $W = \{g = 0\} \subset \mathbb{C}P^{n+1}$ be a nonsingular degree d projective hypersurface which is transversal to $\mathcal{S}$. Then

$$\chi(V) = \chi(W) - \sum_{S \in \mathcal{S}} \chi(S \setminus W) \cdot \mu_S,$$

where

$$\mu_S := \chi\left(\widetilde{H}^*(F_S; \mathbb{Q})\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_S of V at some point in the stratum S of V.

Example (Isolated singularities)

If $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities, then

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\} + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(V)} \mu_x,$$

where μ_x is the Milnor number at $x \in \mathrm{Sing}(V)$.

Example (Isolated singularities)

If $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities, then

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\} + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(V)} \mu_x,$$

where μ_x is the Milnor number at $x \in \mathrm{Sing}(V)$.
In particular, if V is a projective *curve* ($n = 1$), the Betti numbers of V are: $b_0(V) = 1$; $b_2(V) = r$, with $r=$ the number of irreducible components of V; and formula for $\chi(V)$ yields

$$b_1(V) = r + 1 + d^2 - 3d - \sum_{x \in \mathrm{Sing}(V)} \mu_x.$$

Example (Isolated singularities)

If $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities, then

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\} + (-1)^{n+1} \sum_{x \in \mathrm{Sing}(V)} \mu_x,$$

where μ_x is the Milnor number at $x \in \mathrm{Sing}(V)$.
In particular, if V is a projective *curve* ($n = 1$), the Betti numbers of V are: $b_0(V) = 1$; $b_2(V) = r$, with $r=$ the number of irreducible components of V; and formula for $\chi(V)$ yields

$$b_1(V) = r + 1 + d^2 - 3d - \sum_{x \in \mathrm{Sing}(V)} \mu_x.$$

Example (Rational homology manifolds)

If $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then the Lefschetz isomorphism and Poincaré duality over $\mathbb{Q}$ yield $b_i(V) = b_i(\mathbb{C}P^n)$ for all $i \neq n$, while $b_n(V)$ is computed from $\chi(V)$.

Vanishing cohomology

Recall the specialization sequence for $\pi : V_D \to D$, with $V = V_0 = \pi^{-1}(0)$ and smoothing $V_t = \pi^{-1}(t)$ ($t \in D^*$):

$$\to H^k(V;\mathbb{Z}) \overset{sp^k}{\to} H^k(V_t;\mathbb{Z}) \overset{can^k}{\to} H^k(V;\varphi_\pi \mathbb{Z}_{V_D}) \to H^{k+1}(V;\mathbb{Z}) \overset{sp^{k+1}}{\to}$$

The maps sp^k are the "specialization" morphisms in cohomology, while the maps can^k are called "canonical" morphisms.

Vanishing cohomology

Recall the specialization sequence for $\pi : V_D \to D$, with $V = V_0 = \pi^{-1}(0)$ and smoothing $V_t = \pi^{-1}(t)$ ($t \in D^*$):

$$\to H^k(V;\mathbb{Z}) \overset{sp^k}{\to} H^k(V_t;\mathbb{Z}) \overset{can^k}{\to} H^k(V;\varphi_\pi\mathbb{Z}_{V_D}) \to H^{k+1}(V;\mathbb{Z}) \overset{sp^{k+1}}{\to}$$

The maps sp^k are the "specialization" morphisms in cohomology, while the maps can^k are called "canonical" morphisms.

Since the incidence variety $V_D = \pi^{-1}(D)$ deformation retracts to $V = \pi^{-1}(0)$, get:

$$H^k(V;\varphi_\pi\mathbb{Z}_{V_D}) \cong H^{k+1}(V_D, V_t;\mathbb{Z}).$$

Vanishing cohomology

Recall the specialization sequence for $\pi : V_D \to D$, with $V = V_0 = \pi^{-1}(0)$ and smoothing $V_t = \pi^{-1}(t)$ $(t \in D^*)$:

$$\to H^k(V;\mathbb{Z}) \stackrel{sp^k}{\to} H^k(V_t;\mathbb{Z}) \stackrel{can^k}{\to} H^k(V;\varphi_\pi \mathbb{Z}_{V_D}) \to H^{k+1}(V;\mathbb{Z}) \stackrel{sp^{k+1}}{\to}$$

The maps sp^k are the "specialization" morphisms in cohomology, while the maps can^k are called "canonical" morphisms.

Since the incidence variety $V_D = \pi^{-1}(D)$ deformation retracts to $V = \pi^{-1}(0)$, get:

$$H^k(V;\varphi_\pi \mathbb{Z}_{V_D}) \cong H^{k+1}(V_D, V_t;\mathbb{Z}).$$

These groups are called the vanishing cohomology groups of V, denoted by $H^k_\varphi(V;\mathbb{Z})$, and they are the cohomological counterpart of the vanishing homology groups

$$H^\curlyvee_k(V;\mathbb{Z}) := H_k(V_D, V_t;\mathbb{Z})$$

introduced and studied by Siersma-Tibăr for hypersurfaces with 1-dimensional singular loci.

Concentration of vanishing cohomology

Properties of vanishing cycles yield:

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ *be a degree* d *reduced projective hypersurface with* $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. *Then*

$$H^k_{\varphi}(V;\mathbb{Z}) \cong 0 \quad \textit{for} \quad k \notin [n, n+s].$$

Moreover, $H^n_{\varphi}(V;\mathbb{Z})$ *is a free abelian group.*

Concentration of vanishing cohomology

Properties of vanishing cycles yield:

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. Then

$$H^k_{\varphi}(V;\mathbb{Z}) \cong 0 \quad \textit{for} \quad k \notin [n, n+s].$$

Moreover, $H^n_{\varphi}(V;\mathbb{Z})$ is a free abelian group.

Corollary

With the above notations and assumptions, we have that

$$H^{\curlyvee}_k(V;\mathbb{Z}) \cong 0 \quad \textit{for} \quad k \notin [n+1, n+s+1].$$

Moreover, $H^{\curlyvee}_{n+s+1}(V)$ is free.

Consequences for integral cohomology

Corollary

Let $V \subset \mathbb{C}P^{n+1}$ *be a degree* d *reduced projective hypersurface with a singular locus of complex dimension* s*. Then:*

(i) $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for* $k \notin [n, n+s+1]$.

(ii) $H^n(V;\mathbb{Z}) \cong \mathrm{Ker}\,(can^n)$ *is free.*

(iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \mathrm{Coker}\,(can^{n+s})$.

(iv) $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\,(can^k) \oplus \mathrm{Coker}\,(can^{k-1})$ *for* $k \in [n+1, n+s]$, $s \geq 1$.

Consequences for integral cohomology

Corollary

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with a singular locus of complex dimension s. Then:

(i) $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for* $k \notin [n, n+s+1]$.

(ii) $H^n(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^n)$ *is free.*

(iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \mathrm{Coker}\ (can^{n+s})$.

(iv) $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^k) \oplus \mathrm{Coker}\ (can^{k-1})$ *for* $k \in [n+1, n+s]$, $s \geq 1$.

In particular,

$$b_n(V) \leq b_n(V_t) = \frac{(d-1)^{n+2}+(-1)^{n+1}}{d} + \frac{3(-1)^n+1}{2},$$

$$b_k(V) \leq \mathrm{rk}\ H^{k-1}_{\varphi}(V;\mathbb{Z}) + b_k(\mathbb{C}P^n) \quad \text{for } k \in [n+1, n+s+1],\ s \geq 0.$$

Consequences for integral cohomology

Corollary

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with a singular locus of complex dimension s. Then:

- (i) $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for* $k \notin [n, n+s+1]$.
- (ii) $H^n(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^n)$ *is free.*
- (iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \mathrm{Coker}\ (can^{n+s})$.
- (iv) $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^k) \oplus \mathrm{Coker}\ (can^{k-1})$ *for* $k \in [n+1, n+s]$, $s \geq 1$.

In particular,

$$b_n(V) \leq b_n(V_t) = \frac{(d-1)^{n+2}+(-1)^{n+1}}{d} + \frac{3(-1)^n+1}{2},$$

$b_k(V) \leq \mathrm{rk}\ H^{k-1}_{\varphi}(V;\mathbb{Z}) + b_k(\mathbb{C}P^n)$ *for* $k \in [n+1, n+s+1]$, $s \geq 0$.

♣ $s = 0$: Dimca

Consequences for integral cohomology

Corollary

Let $V \subset \mathbb{C}P^{n+1}$ *be a degree* d *reduced projective hypersurface with a singular locus of complex dimension* s*. Then:*

(i) $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for* $k \notin [n, n+s+1]$.

(ii) $H^n(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^n)$ *is free.*

(iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \mathrm{Coker}\ (can^{n+s})$.

(iv) $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^k) \oplus \mathrm{Coker}\ (can^{k-1})$ *for* $k \in [n+1, n+s]$, $s \geq 1$.

In particular,

$$b_n(V) \leq b_n(V_t) = \frac{(d-1)^{n+2}+(-1)^{n+1}}{d} + \frac{3(-1)^n+1}{2},$$

$$b_k(V) \leq \mathrm{rk}\ H^{k-1}_{\varphi}(V;\mathbb{Z}) + b_k(\mathbb{C}P^n) \quad \textit{for } k \in [n+1, n+s+1],\ s \geq 0.$$

- $s = 0$: Dimca
- $s = 1$: Tibăr-Siersma

Torsion issues

The homological counterpart of the above corollary yields that $H_{n+s+1}(V;\mathbb{Z})$ is free.

Torsion issues

The homological counterpart of the above corollary yields that $H_{n+s+1}(V;\mathbb{Z})$ is free.

Recall: $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^k) \oplus \mathrm{Coker}\ (can^{k-1})$ for $k \in [n+1, n+s]$, $s \geq 1$.

Torsion issues

The homological counterpart of the above corollary yields that $H_{n+s+1}(V;\mathbb{Z})$ is free.

Recall: $H^k(V;\mathbb{Z}) \cong \mathrm{Ker}\ (can^k) \oplus \mathrm{Coker}\ (can^{k-1})$ for $k \in [n+1, n+s]$, $s \geq 1$.

Since $H^k(V_t;\mathbb{Z})$ is free for all k, $\mathrm{Ker}\ (can^k) \subseteq H^k(V_t;\mathbb{Z})$ is also free. So the torsion in $H^k(V;\mathbb{Z})$ for $k \in [n+1, n+s+1]$ may come only from the summand $\mathrm{Coker}\ (can^{k-1})$.

More Betti estimates

The ranks of the vanishing cohomology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by using homological algebra techniques.

More Betti estimates

The ranks of the vanishing cohomology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by using homological algebra techniques.

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$.

More Betti estimates

The ranks of the vanishing cohomology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by using homological algebra techniques.

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. For each connected stratum $S_i \subseteq \mathrm{Sing}(V)$ of top dimension s in a Whitney stratification of V, let $F_i^{\pitchfork}$ be its transversal Milnor fiber with Milnor number $\mu_i^{\pitchfork}$.

More Betti estimates

The ranks of the vanishing cohomology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by using homological algebra techniques.

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. For each connected stratum $S_i \subseteq \mathrm{Sing}(V)$ of top dimension s in a Whitney stratification of V, let $F_i^{\pitchfork}$ be its transversal Milnor fiber with Milnor number $\mu_i^{\pitchfork}$. Then:

$$b_{n+s+1}(V) \leq 1 + \sum_i \mu_i^{\pitchfork}$$

and the inequality is strict for $n+s$ even.

More Betti estimates

The ranks of the vanishing cohomology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by using homological algebra techniques.

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. For each connected stratum $S_i \subseteq \mathrm{Sing}(V)$ of top dimension s in a Whitney stratification of V, let $F_i^{\pitchfork}$ be its transversal Milnor fiber with Milnor number $\mu_i^{\pitchfork}$. Then:

$$b_{n+s+1}(V) \leq 1 + \sum_i \mu_i^{\pitchfork}$$

and the inequality is strict for $n+s$ even.

♣ If $s = 0$, i.e., V has only isolated singularities, then $\mu_i^{\pitchfork}$ is just the usual Milnor number of such a singular point of V.

Remark

The upper bound on $b_{n+s+1}(V)$ is sharp: it is achieved for certain quadric hypersurfaces (which have a transversal A_1-singularity).

Remark

The upper bound on $b_{n+s+1}(V)$ is sharp: it is achieved for certain quadric hypersurfaces (which have a transversal A_1-singularity).

If $s = n - 1$, then $b_{n+s+1}(V) = b_{2n}(V) = r$, where r is the number of irreducible components of V.

Remark

The upper bound on $b_{n+s+1}(V)$ is sharp: it is achieved for certain quadric hypersurfaces (which have a transversal A_1-singularity).

If $s = n - 1$, then $b_{n+s+1}(V) = b_{2n}(V) = r$, where r is the number of irreducible components of V. Hence:

Corollary

If the reduced projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, then the number r of irreducible components of V satisfies the inequality:

$$r \leq 1 + \sum_i \mu_i^{\pitchfork}$$

Supplement to the Lefschetz hyperplane section theorem for hypersurfaces

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ *be a reduced projective hypersurface with* $s = \dim \mathrm{Sing}(V)$. *(Set* $s = -1$ *if* V *is nonsingular.)*

Supplement to the Lefschetz hyperplane section theorem for hypersurfaces

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface with $s = \dim \mathrm{Sing}(V)$. (Set $s = -1$ if V is nonsingular.)
Let $H \subset \mathbb{C}P^{n+1}$ be a generic hyperplane (i.e., transversal to a Whitney stratification of V). Then

Supplement to the Lefschetz hyperplane section theorem for hypersurfaces

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ *be a reduced projective hypersurface with* $s = \dim \mathrm{Sing}(V)$. *(Set* $s = -1$ *if* V *is nonsingular.)*
Let $H \subset \mathbb{C}P^{n+1}$ *be a generic hyperplane (i.e., transversal to a Whitney stratification of* V*). Then*

$$H^k(V, V \cap H; \mathbb{Z}) = 0 \ \text{ for } \ k < n \ \text{ and } \ n + s + 1 < k < 2n.$$

Supplement to the Lefschetz hyperplane section theorem for hypersurfaces

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ *be a reduced projective hypersurface with* $s = \dim \mathrm{Sing}(V)$. *(Set* $s = -1$ *if* V *is nonsingular.)*
Let $H \subset \mathbb{C}P^{n+1}$ *be a generic hyperplane (i.e., transversal to a Whitney stratification of* V*). Then*

$$H^k(V, V \cap H; \mathbb{Z}) = 0 \quad \text{for} \quad k < n \quad \text{and} \quad n + s + 1 < k < 2n.$$

Moreover, $H^{2n}(V, V \cap H; \mathbb{Z}) \cong \mathbb{Z}^r$*, where* r *is the number of irreducible components of* V*, and* $H^n(V, V \cap H; \mathbb{Z})$ *is free.*

Supplement to the Lefschetz hyperplane section theorem for hypersurfaces

Theorem (M.-Tibăr-Păunescu)

Let $V \subset \mathbb{C}P^{n+1}$ be a reduced projective hypersurface with $s = \dim \mathrm{Sing}(V)$. (Set $s = -1$ if V is nonsingular.) Let $H \subset \mathbb{C}P^{n+1}$ be a generic hyperplane (i.e., transversal to a Whitney stratification of V). Then

$$H^k(V, V \cap H; \mathbb{Z}) = 0 \quad \text{for} \quad k < n \quad \text{and} \quad n + s + 1 < k < 2n.$$

Moreover, $H^{2n}(V, V \cap H; \mathbb{Z}) \cong \mathbb{Z}^r$, where r is the number of irreducible components of V, and $H^n(V, V \cap H; \mathbb{Z})$ is free.

Remark

The above result can be used to give a new (inductive) proof of Kato's Theorem.

Thank you!