REPORT OF THE MISSION. BUCHAREST, 04/24/2012-05/12/2012.

The first days have been dedicated to the study of the former literature about the problem. In particular the result of Keel and Mckernan [2, Lemma 3.5] that states that for any lc-trivial fibration $f\colon (X,B)\to Z$ with general fibre a rational curve and with $B_Z=0$ the divisor rM_Z is base-point-free.

We concentrate on the case where the base Z of the fibration is a curve. We can assume X smooth. By the Noether-Enriquez theorem, a smooth surface X such that there exists $f\colon X\to Z$ whose general fibre is a rational curve is birationally equivalent over Z to the product $\mathbb{P}^1\times Z$. Then there exists a commutative diagram

$$\begin{array}{ccc} (X,B) & \dashrightarrow & (\mathbb{P}^1\times Z, B') \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \pi_2} & \\ Z & & \end{array}$$

By [1, Theorem 2.6] the moduli part of f and of π_2 are the same, then we can assume that X is a product and consider $f\colon(\mathbb{P}^1\times Z,B)\to Z$.

In this case we have

$$\mathrm{Pic}(\mathbb{P}^1\times Z)\cong f^*\mathrm{Pic}(Z)\oplus\mathbb{Z}[H]$$

where H is a fibre of the projection on $\pi\colon\mathbb{P}^1\times Z\to\mathbb{P}^1$. We can also assume that B is a horizontal divisor. By the structure of the Picard group of $\mathbb{P}^1\times Z$ there exists a divisor δ on Z such that $B\sim 2rH+f^*\delta$. Since B is horizontal, δ is base-point-free on Z. We have

$$\begin{aligned} r(K_{\mathbb{P}^1\times Z}+B) &= r\pi^*K_{\mathbb{P}^1}+rf^*K_Z+2rH+f^*\delta \\ &= rf^*(K_Z+\delta). \end{aligned}$$

Thus $rB_Z+rM_Z\sim\delta$. In particular we obtain the following generalization of Keel-McKernan's result.

Proposition 0.1. *Let $f\colon (X,B)\to Z$ be a lc-trivial fibration whose general fibre is a rational curve. Assume that $B_Z=0$. Then rM_Z is base-point-free.*

We notice that we can see rB as a 1-dimensional family of divisors of degree $2r$ on $\mathbb{P}^1$. Thus we have a natural map

$$Z\to|\mathcal{O}_{\mathbb{P}^1}(1)|\cong\mathbb{P}^{2r}$$

Date: May 31, 2012.

and a commutative diagram

$$\begin{array}{ccc} (\mathbb{P}^1 \times Z, B) & \longrightarrow & (\mathbb{P}^1 \times \mathbb{P}^{2r}, \frac{1}{r}\mathcal{D}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ Z & \longrightarrow & \mathbb{P}^{2r}. \end{array}$$

The morphism F is again an lc-trivial fibration and

$$\mathcal{D} = \left\{ ([z:w],[x_0:\ldots:x_{2r}]) \middle| \sum_{i=0}^{2r} z^i w^{2r-i} x_{2r-i} \right\}.$$

Unfortunately the moduli b-divisor does not descend to $\mathbb{P}^{2r}$. Let $\mu\colon \hat{\mathbb{P}}^{2r} \to \mathbb{P}^{2r}$ be a birational model such that the moduli b-divisor descends to it, that exists by [1, Theorem 2.7], and let $\mathcal{B}$ be the discriminant. Let $\mathcal{I}_m$ be the ideal sheaf $\mu_*\mathcal{O}_{\hat{\mathbb{P}}^{2r}}(-m\mathcal{B})$ defined for all m that clears the denominators of $\mathcal{B}$.
The problem becomes now the following: find an integer $m(r)$ and a linear subsystem $\mathcal{V} \subseteq |m(r)H|$, where H is a hyperplane section of $\mathbb{P}^{2r}$, such that $\mathrm{Bs}\mathcal{V} = \mathcal{I}_{m(r)}$. If $\mathcal{W} \in \mathcal{V}$, then

$$\mu^*\mathcal{W} = \tilde{\mathcal{W}} + m(r)\mathcal{B}.$$

To conclude we should prove that $m(r)M_Z \sim \nu^*\tilde{\mathcal{W}}$ where $\nu\colon Z \to \hat{\mathbb{P}}^{2r}$ is the induced morphism.

A problem that should be related to the sharp bound for the base-point-freeness of M_Z is the study of the automorphisms of pairs $(\mathbb{P}^1, \sum b_i p_i)$ with $\sum b_i p_i = 2$ and $b_i \in \frac{1}{r}\mathbb{Z}$. These are automorphisms of $\mathbb{P}^1$ that permute points with the same coefficient. From the general theory about subgroups of PSL(2) (see [3, Theorem 6.17]) follows this proposition.

Proposition 0.2. *Let $(\mathbb{P}^1, \sum b_i p_i)$ be a pair with $\sum b_i p_i = 2$ and $b_i \in \frac{1}{r}\mathbb{Z}$. Then there are two cases*

- $r = 1 \Rightarrow \mathrm{Aut}(\mathbb{P}^1, p_1 + p_2) \cong \mathbb{C}^* \times \mathbb{Z}/2\mathbb{Z}$*;*
- $r \geq 2 \Rightarrow \mathrm{Aut}(\mathbb{P}^1, \sum b_i p_i) \subseteq \mathrm{PGL}(2)$ *is a finite subgroup belonging to the following set*

  $$\{\mathbb{Z}/k\mathbb{Z}\}_{k \leq 2r-1} \cup \{D_{2k}\}_{k \leq 2r} \cup \{\mathcal{A}_4, \mathcal{S}_4, \mathcal{A}_5\}$$

  where D_{2k} stands for the dihedral group, the group of automorphisms of a regular polyhedron with k vertices.

References

[1] F. Ambro, Shokurov's boundary property, *J. Differential Geometry* **67**, pp 229-255 (2004).
[2] S. Keel, K. Matsuki, J. McKernan, Log abundance theorem for threefolds *Duke Math. J.* **75** (1994), no. 1, 99-119.
[3] M. Suzuki, Group theory *Springer*