We investigate Cohen-Macaulay and unmixed bipartite graphs without isolated vertices, and show that graphs constructed from them by putting appropriate pairs of loops, the so-called quasi-bipartite graphs, retain such properties.

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1. INTRODUCTION

In this article we consider bipartite graphs with no isolated vertices, and expose situations for which algebraic properties of them remain when we add loops in any of their vertex.

Let $\mathcal{G}$ be a graph on $n$ vertices $v_1, \ldots, v_n$. An algebraic object attached to $\mathcal{G}$ is the edge ideal $I(\mathcal{G})$, a monomial ideal of the polynomial ring in $n$ variables $R = K[X_1, \ldots, X_n]$, $K$ a field.

When $\mathcal{G}$ is a loopless graph, $I(\mathcal{G})$ is generated by squarefree monomials of degree two in $R$, $I(\mathcal{G}) = (X_iX_j \mid \{v_i, v_j\} \text{ is an edge of } \mathcal{G})$, but if $\mathcal{G}$ is a graph having loops $\{v_i, v_i\}$, among the generators of $I(\mathcal{G})$ there are also non-squarefree monomials $X_i^2, i = 1, \ldots, n$.

A subset $C$ of the vertex set $\{v_1, \ldots, v_n\}$ of a simple graph $\mathcal{G}$ is said a minimal vertex cover of $\mathcal{G}$ if: (i) every edge of $\mathcal{G}$ is incident with one vertex in $C$, and (ii) no proper subset of $C$ has the first property. If $C$ satisfies condition (i) only, it is called a vertex cover of $\mathcal{G}$.

Remember that a graph $\mathcal{G}$ is bipartite if its vertex set can be partitioned into disjoint subsets $V_1$ and $V_2$ such that every edge of $\mathcal{G}$ joins $V_1$ with $V_2$, and the pair $(V_1, V_2)$ is called a bipartition of $\mathcal{G}$.

A graph $\mathcal{G}$ is said to be Cohen-Macaulay over the field $K$ if the depth and the dimension over $R$ of $R/I(\mathcal{G})$ coincide, unmixed if all its minimal vertex covers have the same cardinality, and that a Cohen-Macaulay graph is unmixed.
Structural aspects of Cohen-Macaulay and unmixed bipartite graphs were studied in [1, 7]. For Cohen-Macaulay or unmixed bipartite graphs without isolated vertices we show under what conditions the graphs having loops associated to them, the so-called quasi-bipartite graphs, preserve such properties.

In particular, we introduce the definitions of quasi-bipartite graph, namely a bipartite graph with loops, and of strong quasi-bipartite graph, that is a complete bipartite graph having loops in every of its vertex. After recalling the combinatorial descriptions given by Herzog and Hibi in [2] and Villarreal in [7] for Cohen-Macaulay and unmixed bipartite graphs respectively, we state that a strong quasi-bipartite graph is Cohen-Macaulay, then unmixed, and we characterize the quasi-bipartite Cohen-Macaulay and unmixed graphs having the corresponding bipartite graphs such properties.

2. COHEN-MACAULAY AND UNMIXED QUASI-BIPARTITE GRAPHS

In this section we present two main results about the behaviour of Cohen-Macaulay and unmixed quasi-bipartite graphs without isolated vertices, when, for the corresponding bipartite graphs, such properties hold.

Let’s introduce some preliminary notions.

Let $G$ be a graph and $V(G) = \{v_1, \ldots, v_n\}$ be the set of its vertices. We put $E(G) = \{\{v_i, v_j\} \mid v_i \neq v_j, v_i, v_j \in V(G)\}$ the set of edges of $G$ and $L(G) = \{\{v_i, v_i\} \mid v_i \in V(G)\}$ the set of loops of $G$. Hence, $\{v_i, v_j\}$ is an edge joining $v_i$ to $v_j$ and $\{v_i, v_i\}$ is a loop of the vertex $v_i$. Set $W(G) = E(G) \cup L(G)$.

If $L(G) = \emptyset$, the graph $G$ is said simple or loopless, otherwise, if $L(G) \neq \emptyset$, $G$ is a graph with loops.

A graph $G$ on $n$ vertices $v_1, \ldots, v_n$ is complete if there exists an edge for all pairs $\{v_i, v_j\}$ of vertices of $G$. It is denoted by $K_n$.

If $V(G) = \{v_1, \ldots, v_n\}$ and $R = K[X_1, \ldots, X_n]$ is the polynomial ring over a field $K$ such that each variable $X_i$ corresponds to the vertex $v_i$, the edge ideal $I(G)$ associated to $G$ is the ideal $\langle X_iX_j \mid \{v_i, v_j\} \in W(G) \rangle \subset R$.

Note that the non-zero edge ideals are those generated by squarefree monomials of degree 2. This implies that $I(G)$ is a graded ideal of $S$ of initial degree 2, that is $I(G) = \oplus_{i \geq 2} (I(G)_i)$. If $W(G) = \emptyset$, then $I(G) = (0)$.

Let $G$ be a graph and $I(G) \subset R$ be its edge ideal. Let $m$ be the irrelevant maximal ideal of $R$. The depth of $R/I(G)$, denoted by depth$(R/I(G))$, is the largest integer $r$ for which there is an homogeneous sequence $f_1, \ldots, f_r$ in $m$ such that $f_i$ is not a zero-divisor of $R/(I(G), f_1, \ldots, f_{i-1})$, for all $1 \leq i \leq r$. Such $r$ is bounded by $\dim(R/I(G))$, where $\dim(R/I(G))$ denotes the Krull dimension of $R/I(G)$.
Definition 2.1. \( \mathcal{G} \) is said to be a Cohen-Macaulay graph over \( K \) (C-M graph for short) if \( \text{depth}(R/I(\mathcal{G})) = \dim(R/I(\mathcal{G})) \).

Remark 2.1. The edge ideal \( I(\mathcal{G}) \) is Cohen-Macaulay if and only if \( R/I(\mathcal{G}) \) is Cohen-Macaulay, i.e., \( \text{depth}(R/I(\mathcal{G})) = \dim(R/I(\mathcal{G})) \).

Let \( G(I(\mathcal{G})) \) be the unique minimal set of monomial generators of the edge ideal \( I(\mathcal{G}) \). A vertex cover of \( I(\mathcal{G}) \) is a subset \( C \) of \( \{X_1, \ldots, X_n\} \) such that each \( u \in G(I(\mathcal{G})) \) is divided by some \( X_i \in C \). Such a vertex cover \( C \) is called minimal if no proper subset of \( C \) is a vertex cover of \( I(\mathcal{G}) \).

Let \( h(I(\mathcal{G})) \) denote the minimal cardinality of the vertex covers of \( I(\mathcal{G}) \).

Proposition 2.1. Let \( \mathcal{G} \) be a graph with loops on the vertex set \( V(\mathcal{G}) = \{v_1, \ldots, v_n\} \). Let \( I(\mathcal{G}) \subseteq S = K[X_1, \ldots, X_n] \) be the edge ideal of \( \mathcal{G} \) such that \( h(I(\mathcal{G})) = n \). Then \( \mathcal{G} \) is a Cohen-Macaulay graph.

Proof. Let \( h(I(\mathcal{G})) = n \), then it follows that \( \dim(S/I(\mathcal{G})) = n - h(I(\mathcal{G})) = 0 \). Being \( \text{depth}(S/I(\mathcal{G})) \leq \dim(S/I(\mathcal{G})) = 0 \). Hence, \( \mathcal{G} \) is Cohen-Macaulay.

We are interested to study the Cohen-Macaulay property for bipartite graphs with loops.

Definition 2.2. A graph \( \mathcal{G} \) with loops is said to be quasi-bipartite if its vertex set \( V \) can be partitioned into disjoint subsets \( V_1 = \{x_1, \ldots, x_n\} \) and \( V_2 = \{y_1, \ldots, y_m\} \), every edge joins a vertex of \( V_1 \) with a vertex of \( V_2 \), and there exists some vertex of \( V \) with a loop ([4]).

Definition 2.3. A graph \( \mathcal{G} \) with loops is called strong quasi-bipartite if all the vertices of \( V_1 \) are joined to any vertex of \( V_2 \) and for each vertex of \( V \) there is a loop ([4]).

The quasi-bipartite graphs determine edge ideals in two sets of variables \( X_1, \ldots, X_m; Y_1, \ldots, Y_n \), where \( m \) is the number of the vertices \( x_1, \ldots, x_m \) and \( n \) the number of the vertices \( y_1, \ldots, y_n \). More precisely, if \( \mathcal{G} \) is a quasi-bipartite graph, then its edge ideal in \( R = K[X_1, \ldots, X_m; Y_1, \ldots, Y_n] \) is

\[
I(\mathcal{G}) = (X_i^2, X_iY_j, Y_k^2 \mid \{x_i, y_j\}, \{x_l, x_t\}, \{y_k, y_k\} \in W(\mathcal{G})).
\]

The following criterion by Herzog and Hibi describes all Cohen-Macaulay bipartite graphs in combinatorial terms.

Theorem 2.1 ([2], Theorem 3.4). Let \( \mathcal{G} \) be a bipartite graph without isolated vertices. Then \( \mathcal{G} \) is Cohen-Macaulay if and only if there is a bipartition \( V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\} \) of \( \mathcal{G} \) such that:

(a) \( \{x_i, y_i\} \in E(\mathcal{G}), \) for all \( i = 1, \ldots, g; \)
(b) if \( \{x_i, y_i\} \in E(\mathcal{G}) \), then \( i \leq j \);

(c) if \( \{x_i, y_j\} \) and \( \{x_j, y_k\} \) are in \( E(\mathcal{G}) \) and \( i, j, k \) are distinct, then \( \{x_i, y_k\} \in E(\mathcal{G}) \).

For quasi-bipartite graphs the following result holds.

**Corollary 2.1.** A quasi-bipartite graph with a loop in each vertex is Cohen-Macaulay.

**Proof.** Let \( \mathcal{G} \) be a quasi-bipartite graph with a loop in each vertex and \( I(\mathcal{G}) \subset R = K[X_1, \ldots, X_m; Y_1, \ldots, Y_n] \) be its edge ideal. Being \( h(I(\mathcal{G})) = m + n \), the thesis follows by Proposition 2.1. \( \square \)

Starting from a Cohen-Macaulay bipartite graph \( \mathcal{G} \) we will show that the property to be Cohen-Macaulay for \( \mathcal{G} \) is preserved when we add pairs of loops to it.

**Theorem 2.2.** Let \( \mathcal{G} \) be a Cohen-Macaulay bipartite graph without isolated vertices on \( V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\} \). If \( \mathcal{G}' \) is the quasi-bipartite graph with 

\[
V(\mathcal{G}') = V(\mathcal{G}), \quad E(\mathcal{G}') = E(\mathcal{G}) \quad \text{and} \quad L(\mathcal{G}') = \{\{x_{i_1}, x_{i_2}\}, \{x_{i_2}, y_{i_1}\}, \ldots, \{x_{i_s}, x_{i_1}\}, \{y_{i_1}, y_{i_2}\}, \ldots, \{y_{i_1}, y_{i_2}\} | i_1, \ldots, i_s \subseteq \{1, \ldots, g\}\},
\]

then \( \mathcal{G}' \) is Cohen-Macaulay.

**Proof.** If \( |L(\mathcal{G}')| = 2g \), the thesis follows by Corollary 2.1.

Now let \( |L(\mathcal{G}')| = 2g - 2t \), for \( g > t > 1 \).

Let \( t = 1 \). \( |L(\mathcal{G}')| = 2g - 2 \) and suppose that \( \{x_g, x_g\}, \{y_g, y_g\} \notin L(\mathcal{G}') \). The proof is by induction on \( g \). For \( g = 2 \), it is either \( I(\mathcal{G}) = (X_1Y_1, X_2Y_2, X_1^2, Y_1^2) \) or \( I(\mathcal{G}) = (X_1Y_1, X_2Y_2, X_1Y_2, X_1^2, Y_1^2) \). Hence, \( \mathcal{G} \) is Cohen-Macaulay. For \( g > 2 \), let \( S = \{x_{r_1}, \ldots, x_{r_p}\} \) be the set of all vertices of \( \mathcal{G} \) adjacent to \( y_g \). Consider the subgraph \( \mathcal{G}' = \mathcal{G} \setminus S \) obtained from \( \mathcal{G} \) by removing the vertices in \( S \). The vertices \( y_{r_1}, \ldots, y_{r_p} \) are isolated vertices of \( \mathcal{G}' \). In fact, if there is an edge \( \{x_i, y_{r_j}\} \in \mathcal{G}' \), with \( i < r_j \), then by Theorem 2.1 one gets that \( \{x_i, y_g\} \in E(\mathcal{G}) \) and \( x_i \) must be a vertex in \( S \). This is a contradiction. Using the induction hypothesis on \( g \), the graphs \( \mathcal{G}' \setminus \{y_{r_1}, \ldots, y_{r_p}\} \) and \( \mathcal{G} \setminus \{x_g, y_g\} \) are Cohen-Macaulay. It follows that \( \mathcal{G} \) is Cohen-Macaulay ([6], Proposition 6.2.2).

Let \( t > 1 \), then \( g > 2 \). Suppose that \( \{x_{g-t+1}, x_{g-t+1}\}, \{y_{g-t+1}, y_{g-t+1}\}, \ldots, \{x_g, x_g\}, \{y_g, y_g\} \notin L(\mathcal{G}) \) and use induction on \( g \). For \( g = 3 \), \( I(\mathcal{G}) \) can be generated by \( \{X_1Y_1, X_2Y_2, X_3Y_3, X_1^2, Y_1^2\} \) or \( \{X_1Y_1, X_2Y_2, X_3Y_3, X_1Y_2, X_1^2, Y_1^2\} \) or \( \{X_1Y_1, X_2Y_2, X_3Y_3, X_1Y_3, X_1^2, Y_1^2\} \) or \( \{X_1Y_1, X_2Y_2, X_3Y_3, X_2Y_3, X_1^2, Y_1^2\} \) or \( \{X_1Y_1, X_2Y_2, X_3Y_3, X_1Y_2, X_2Y_3, X_1^2, Y_1^2\} \) or \( \{X_1Y_1, X_2Y_2, X_3Y_3, X_1Y_2, X_2Y_3, X_1Y_3, X_1^2, Y_1^2\} \).

Hence, \( \mathcal{G} \) is Cohen-Macaulay. Now suppose \( g > 3 \). Let \( S' = \{x_{l_1}, \ldots, x_{l_s}\} \) be the set of all vertices of \( \mathcal{G} \) adjacent to \( y_{g-t+1} \) and consider the subgraph...
\( G'' = \hat{G} \setminus \mathcal{S}' \). As shown in the first part of the proof, the vertices \( y_{l_1}, \ldots, y_{l_q} \) of \( G'' \) are isolated. By induction hypothesis on \( g \), the graphs \( G'' \setminus \{y_{l_1}, \ldots, y_{l_q}\} \) and \( \hat{G} \setminus \{x_{g-t+1}, y_{g-t+1}\} \) are Cohen-Macaulay, and this implies that \( \hat{G} \) is Cohen-Macaulay. \( \square \)

Now, we study the unmixed property for quasi-bipartite graphs.

Recall that a subset \( C \) of \( V(G) \) is said to be a minimal vertex cover of \( G \) if every edge of \( G \) is incident with one vertex in \( C \) and there is no proper subset of \( C \) having such a property. Notice that \( C \) is a minimal vertex cover if and only if \( V(G) \setminus C \) is a maximal independent set. The smallest number of vertices in any minimal vertex cover of \( G \) is called vertex covering number.

**Definition 2.4.** A graph \( G \) is called unmixed if all the minimal vertex covers of \( G \) have the same number of elements.

The notion of unmixed graph is related to some other theoretical and algebraic graph properties. The implication Cohen-Macaulay \( \Rightarrow \) unmixed holds for any graph without isolated vertices [6].

Minimal vertex covers of unmixed bipartite graphs \( G \) with no isolated vertices were first described in [2]. If \( \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\} \) denotes the set of vertices of \( G \), since \( \{x_1, \ldots, x_m\} \) and \( \{y_1, \ldots, y_n\} \) represent minimal vertex covers of \( G \), then \( m = n \). Moreover, by the marriage theorem, there is a perfect matching for \( G \), so it may be assumed that \( \{x_i, y_i\} \) is an edge of \( G \), for all \( i \), and following this fact, each minimal vertex cover of \( G \) is of the form \( \{x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_n}\} \), where \( \{i_1, \ldots, i_n\} = \{1, \ldots, n\} \).

Later Villarreal gave the following combinatorial characterization of all unmixed bipartite graphs with no isolated vertices.

**Theorem 2.3 ([7], Theorem 1.1).** Let \( G \) be a bipartite graph without isolated vertices. Then \( G \) is unmixed if and only if there is a bipartition \( V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\} \) of \( G \) such that:

(a) \( \{x_i, y_i\} \in E(G) \), for all \( i = 1, \ldots, g \);

(b) if \( \{x_i, y_j\} \) and \( \{x_j, y_k\} \) are in \( E(G) \) and \( i, j, k \) are distinct, then \( \{x_i, y_k\} \in E(G) \).

Starting from an unmixed bipartite graph \( G \) we will show that the property to be unmixed for \( G \) is preserved when we add pairs of loops to it.

**Theorem 2.4.** Let \( G \) be an unmixed bipartite graph without isolated vertices on \( V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\} \). If \( \hat{G} \) is the quasi-bipartite graph
with \( V(\hat{G}) = V(G), E(\hat{G}) = E(G) \) and \( L(\hat{G}) = \{ \{x_{i_1}, x_{i_1}\}, \{x_{i_2}, x_{i_2}\}, \ldots, \{x_{i_s}, x_{i_s}\}, \{y_{i_1}, y_{i_1}\}, \{y_{i_2}, y_{i_2}\}, \ldots, \{y_{i_s}, y_{i_s}\} \mid \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, g\} \} \), then \( \hat{G} \) is unmixed.

Proof. For \( |L(\hat{G})| = 2g \), the assertion holds by Corollary 2.1 remembering that Cohen-Macaulay implies unmixed.

If \( |L(\hat{G})| = 2(g - t) \), for \( t = 1, \ldots, g - 1 \), let \( G^* \) be the subgraph of \( \hat{G} \) obtained by removing all vertices of \( \hat{G} \) having loops. Thus, \( G^* \) is a bipartite subgraph of \( G \) with \( 2t \) non isolated vertices, \( x_{i_1}, \ldots, x_{i_t} \in V_1 \) and \( y_{i_1}, \ldots, y_{i_t} \in V_2 \), such that \( \{x_{i_j}, y_{i_j}\} \in E(G^*) \), for \( i_j = 1, \ldots, t \), and when \( \{x_{i_h}, y_{i_k}\}, \{x_{i_k}, y_{i_l}\} \in E(G^*) \) with \( i_h \neq i_k \neq i_l \), then \( \{x_{i_h}, y_{i_k}\} \in E(G^*) \). By Theorem 2.3, \( G^* \) is unmixed and, according to the proof of [7], Theorem 2.1, any minimal vertex cover for it intersects every edge \( \{x_{i_j}, y_{i_j}\} \) in exactly one vertex.

On the other hand, let \( \bar{G} \) denote the subgraph of \( \hat{G} \) whose edge set is given by \( E(\hat{G}) \setminus E(G^*) \). We state that \( \bar{G} \) is also unmixed. In fact, any vertex cover for it must contain at least all the \( 2(g - t) \) vertices with loops of \( \hat{G} \). By construction, every other vertex of \( \bar{G} \) is adjacent to some of such vertices, so a minimal vertex cover for \( \bar{G} \) has cardinality \( 2(g - t) \).

Note that the graph \( \hat{G} \) is the union of the graphs \( G^* \) and \( \bar{G} \).

For showing that \( \hat{G} \) is unmixed, it is enough to prove that all minimal vertex covers for it have \( 2g - t \) vertices.

If \( G^* \) and \( \bar{G} \) are disjoint, a minimal vertex cover for \( \hat{G} \) is just the union set of a minimal vertex cover for \( G^* \) and one for \( \bar{G} \). When \( G^* \) and \( \bar{G} \) are not disjoint, there exist vertices with no loop common to such subgraphs. These vertices are ends of that edges belonging to \( \bar{G} \) whose endpoints are not both with loops. The adding of these edges to \( G^* \) doesn’t increase the cardinality of any minimal vertex cover for \( G^* \) itself, considered as a subgraph of \( \hat{G} \), because a vertex cover for \( \bar{G} \) must intersect the vertices with the loop of such edges. So again, any minimal vertex cover for \( \hat{G} \) is the union set of a minimal vertex cover for \( G^* \) and one for \( \bar{G} \), and it is formed by \( 2(g - t) + t \) vertices.  

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University of Messina,
Department of Mathematics,
Viale Ferdinando Stagno d’Alcontres, 31
98166 Messina, Italy
imbesim@unime.it

University of Messina,
Department of Mathematics,
Viale Ferdinando Stagno d’Alcontres, 31
98166 Messina, Italy
monicalb@dipmat.unime.it