A SUFFICIENT CONDITION FOR A FUNCTION TO SATISFY A WEAK LIPSCHITZ CONDITION

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For a function $f : X \to Y$ let (X, d) and (Y, ρ) be metric spaces consider the following condition: for each $x \in X$ there exists $C_x \in (0, 1)$ such that for each sequence $S = (x_n)_{n\in\mathbb{N}^*} \subseteq X$ satisfying $d(x_n, x_0) \leq C_x^n$, $n \in \mathbb{N}^*$, there exists $M_S \in \mathbb{R}$ such that $\rho(f(x_n), f(x_0)) \leq M_S C_x^n$, $n \in \mathbb{N}^*$. We show that the above condition is sufficient for f to be Lipschitz at each point of X.

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1. INTRODUCTION

Let us start with a basic definition.

Definition. Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called Lipschitz if there exists a constant M such that

$$\rho(f(x), f(y)) \leq M \cdot d(x, y)$$

for all $x, y \in X$.

From the point of view of mathematical analysis, the condition of being Lipschitz may be considered as a weakened version of differentiability, on account of a famous result of Rademacher (see [2]), which states that a Lipschitz function $f : U = \overset{\circ}{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable outside of a Lebesgue null subset of U.

From the point of view of mechanics, a Lipschitz map is one that obeys speed limits.

Lipschitz functions are used in the study of differential equations, measure theory, functional analysis, and in the construction of mathematical models in mechanics, physics and economy.

Lipschitz functions have been intensively studied. Two basic references in the area of Lipschitz functions are [1] and [3].

Therefore, it is important to recognize the Lipschitz functions.

A weak version of the Lipschitz condition is given by the following

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Definition. Let (X, d) and (Y, ρ) be metric spaces and $x \in X$. A function $f : X \to Y$ is called Lipschitz at x if there exists a neighborhood U of x and a constant M such that

$$\rho(f(y), f(x)) \leq Md(y, x)$$

for all $y \in U$.

Clearly, if we consider a function $f : X \to Y$, which is Lipschitz at a point $x \in X$, then for each $C \in (0, 1)$ and each sequence $(x_n)_{n \in \mathbb{N}^*} \subseteq U$ satisfying

$$d(x_n, x) \leq C^n, \quad n \in \mathbb{N}^*,$$

we have

$$\rho(f(x_n), f(x)) \leq MC^n, \quad n \in \mathbb{N}^*.$$

Therefore, it is natural to look for sufficient conditions (in terms of the above discussion) for a function to be Lipschitz at the points of its domain.

In this paper we present a sufficient condition for a function to be Lipschitz at each point of its domain.

2. THE RESULT

THEOREM. *Let (X, d) and (Y, ρ) be two metric spaces. Let $f : X \to Y$ be a function with the following property: for each $x \in X$ there exists $C_x \in (0, 1)$ such that for each sequence $S = (x_n)_{n \in \mathbb{N}^*} \subseteq X$ satisfying the condition $d(x_n, x_0) \leq C_x^n$, $n \in \mathbb{N}^*$, there exists $M_S \in \mathbb{R}$ such that $\rho(f(x_n), f(x_0)) \leq M_S C_x^n$, $n \in \mathbb{N}^*$. Then, f is Lipschitz at any $x \in X$.*

Proof. We shall show that for every $x \in X$ there exist $M_x \in \mathbb{R}$ and $\delta_x \in \mathbb{R}$ such that

$$y \in B(x, \delta_x) \Rightarrow \rho(f(y), f(x)) \leq M_x d(y, x).$$

Let us suppose, for a contradiction that this is not the case. Therefore, there exists $x_0 \in X$ such that for all $M \in \mathbb{R}_+^*$ and $\delta \in \mathbb{R}_+^*$, there exists $x_{M,\delta} \in X$ such that

$$d(x_{M,\delta}, x_0) < \delta$$

and

$$\rho(f(x_{M,\delta}), f(x_0)) > Ld(x_{M,\delta}, x_0).$$

In particular, choosing $M = m \in \mathbb{N}^*$ and $\delta = C_{x_0}^n$, $n \in \mathbb{N}^*$, we get $x_{n,m} \in X$ such that

$$d(x_{n,m}, x_0) < C_{x_0}^n \tag{1}$$

and

$$\rho(f(x_{n,m}), f(x_0)) > md(x_{n,m}, x_0), \tag{2}$$

for all $n, m \in \mathbb{N}^*$.

Inequality (2) implies

$$d(x_{n,m}, x_0) > 0, \quad n, m \in \mathbb{N}^*.$$

Since $\lim_{n\to\infty} C_{x_0}^n = 0$, we can define

$$n_1 = \max\{n \in \mathbb{N}^* \mid 0 < d(x_{1,1}, x_0) \le C_{x_0}^n\},$$
$$n_2 = \max\{n \in \mathbb{N}^* \mid 0 < d(x_{n_1+1,2}, x_0) \le C_{x_0}^n\},$$
$$n_3 = \max\{n \in \mathbb{N}^* \mid 0 < d(x_{n_2+1,3}, x_0) \le C_{x_0}^n\}$$

and, if n_k has been defined, then

$$n_{k+1} = \max\{n \in \mathbb{N}^* \mid 0 < d(x_{n_k+1,k+1}, x_0) \le C_{x_0}^n\}.$$

Let us note that from (1) we have

$$d(x_{n_k+1,k+1}, x_0) < C_{x_0}^{n_k+1},$$

so that

$$n_k + 1 \in \{n \in \mathbb{N}^* \mid 0 < d(x_{n_k+1,k+1}, x_0) \le C_{x_0}^n\},$$

hence $n_k + 1 \le n_{k+1}$, i.e. $n_k < n_{k+1}$, for all $k \in \mathbb{N}^*$.

Therefore, the sequence $(n_k)_{k\in\mathbb{N}^*}$ is strictly increasing.

Let us consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined as

$$\underbrace{x_{1,1}, \dots, x_{1,1}}_{n_1 \text{ times}}, \underbrace{x_{n_1+1,2}, \dots, x_{n_1+1,2}}_{n_2-n_1 \text{ times}}, \dots .$$

By the definition of n_k, for all $k \in \mathbb{N}^*$, we have

$$C_{x_0}^{n_k+1} < d(x_{n_{k-1}+1,k}, x_0) \le C_{x_0}^{n_k}. \tag{3}$$

On the one hand, the sequence $(d(u_n, x_0))_{n\in\mathbb{N}^*}$ is in fact

$$\underbrace{d(x_{1,1}, x_0), \dots, d(x_{1,1}, x_0)}_{n_1 \text{ times}}, \underbrace{d(x_{n_1+1,2}, x_0), \dots, d(x_{n_1+1,2}, x_0)}_{n_2-n_1 \text{ times}}, \dots .$$

Since by (3) we have

$$d(x_{1,1}, x_0) \le C_{x_0}^{n_1} \le C_{x_0}^{n_1-1} \le \dots \le C_{x_0}^1,$$
$$d(x_{n_1+1,2}, x_0) \le C_{x_0}^{n_2} \le C_{x_0}^{n_2-1} \le \dots \le C_{x_0}^{n_1+1},$$
$$\dots$$
$$d(x_{n_{k-1}+1,k}, x_0) \le C_{x_0}^{n_k} \le C_{x_0}^{n_k-1} \le \dots \le C_{x_0}^{n_{k-1}+1},$$
$$\dots$$

we get $d(u_n, x_0) \le C_{x_0}^n$, for all $n \in \mathbb{N}^*$.

On the other hand, the sequence $(\rho(f(u_{n_k}), f(x_0))_{n\in\mathbb{N}^*}$ is

$$\rho(f(x_{1,1}), f(x_0)), \rho(f(x_{n_1+1,2}), f(x_0)), \dots, \rho(f(x_{n_{k-1}+1,k}), f(x_0)), \dots .$$

From (2) we have

$$\rho(f(x_{n_{k-1}+1,k}), f(x_0)) \ge k d(x_{n_{k-1}+1,k}, x_0),$$

so, by (3), we obtain

$$\rho(f(x_{n_{k-1}+1,k}), f(x_0)) \geq kC_{x_0}^{n_k+1} = kC_{x_0}C_{x_0}^{n_k},$$

i.e., $\rho(f(u_{n_k}), f(x_0) \geq kC_{x_0}C_{x_0}^{n_k}$, $k \in \mathbb{N}^*$.

According to the hypothesis, for the sequence $S_0 = (u_n)_{n\in\mathbb{N}^*}$ there exists M_{S_0} such that $\rho(f(u_n), f(x_0)) \leq M_{S_0}C_{x_0}^n$ for all $n \in \mathbb{N}^*$. Then for $k \in \mathbb{N}^*$ such that $kC_{x_0} > M_{S_0}$ we get the contradiction

$$\rho(f(u_{n_k}), f(x_0)) \leq M_{S_0}C_{x_0}^{n_k} < kC_{x_0}C_{x_0}^{n_k} \leq \rho(f(u_{n_k}), f(x_0).$$

Therefore, for each $x \in X$ there exists a neighborhood U of x and a constant M such that

$$\rho(f(y), f(x)) \leq Md(y, x)$$

for all $y \in U$, i.e. f is Lipschitz at x. □

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