Bipotentials for non monotone multivalued operators: fundamental results and applications

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Abstract. This is a survey of recent results about bipotentials representing multivalued operators. The notion of bipotential is based on an extension of Fenchel's inequality, with several interesting applications related to non associated constitutive laws in non smooth mechanics, such as Coulomb frictional contact or non-associated Drücker-Prager model in plasticity.

Relations betweeen bipotentials and Fitzpatrick functions are described. Selfdual lagrangians, introduced and studied by Ghoussoub, can be seen as bipotentials representing maximal monotone operators. We show that bipotentials can represent some monotone but not maximal operators, as well as non monotone operators.

Further we describe results concerning the construction of a bipotential which represents a given non monotone operator, by using convex lagrangian covers or bipotential convex covers. At the end we prove a new reconstruction theorem for a bipotential from a convex lagrangian cover, this time using a convexity notion related to a minimax theorem of Fan.

Mathematics Subject Classifications: 49J53; 49J52; 26B25.

1. Introduction

In the generalized standard material theory of Halphen and Son [21] any constitutive law of a standard material relates a strain rate variable $x \in X$ with a stress-like variable $y \in Y$ by using a dissipation potential ϕ.

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. The dissipation potential ϕ is a convex and lower semicontinuous function defined on X and the associated constitutive law is given by one the following equivalent conditions:

(a) $y \in \partial\phi(x)$, where $\partial\phi$ is the subdifferential of ϕ in the sense of Convex Analysis,

(b) $x \in \partial\phi^*(y)$, where ϕ^* is the Fenchel dual of ϕ with respect to the duality product,

(c) $\phi(x) + \phi^*(y) = \langle x, y \rangle$.

The constitutive laws of standard materials are also called associated laws. From the mathematical viewpoint such laws are cyclically monotone operators. However, there are many non-associated constitutive laws, described by a multivalued operator $T : X \to 2^Y$ **which is not cyclically monotone, in some cases not even monotone**.

A possible way to study non-associated constitutive laws by using convex analysis, proposed first in [33], consists in constructing a "bipotential" function b of two variables, which physically represents the dissipation. See definition 3.1 and the section 3 for the introduction into the subject of bipotentials.

A bipotential function b is bi-convex, satisfies an inequality generalizing Fenchel's one, namely $\forall x \in X, y \in Y, \quad b(x, y) \geq \langle x, y \rangle$, and a relation involving partial subdifferentials of b with respect to variables x, y. In the case of associated constitutive laws the bipotential has the expression $b(x, y) = \phi(x) + \phi^*(y)$ (section 3). The graph of a bipotential b is simply the set $M(b) \subset X \times Y$ of those pairs (x, y) such that $b(x, y) = \langle x, y \rangle$.

A maximal cyclically monotone operator $T : X \to 2^Y$ is represented by a lower semicontinuous and convex "potential" function ϕ, by a well known theorem of Rockafellar. More general, we say that the bipotential b represents the multivalued operator T if the graph of T equals $M(b)$.

There are already many applications of bipotentials to mechanics. Among them we cite: Coulomb's friction law [34], surveyed here in section 9.2, non-associated Drücker-Prager [35] (section 9.1) and Cam-Clay models [36] in soil mechanics, cyclic plasticity ([34],[3]) and viscoplasticity [24] of metals with non linear kinematical hardening rule, Lemaitre's damage law [2], the coaxial laws ([39],[44]), details in sections 8.2 and 9.3.

The notion of a bipotential associated to a multivalued operator is interesting also from a mathematical point of view. We show that bipotentials are related to Fitzpatrick functions associated to a maximally monotone operator. Selfdual lagrangians introduced and studied by Ghoussoub [18] can be seen as bipotentials representing maximal monotone operators (section 5). In section 7 we describe a result from [11] which implies that some monotone non maximal operators can be represented by bipotentials.

Other examples of bipotentials come from inequalities. For example, the Cauchy-Bunyakovsky-Schwarz inequality can be recast as: if $X = Y$ is a Hilbert space then the function $b : X \times X \to \mathbb{R}$ defined by $b(x, y) = \|x\|\|y\|$ is a bipotential. More general, some inequalities involving eigenvalues of real symmetric matrices can be put in a similar form, thus providing more non trivial examples of bipotentials.

In order to better understand the bipotential approach, in [10], [11] we solved two key problems: (a) when the graph of a given multivalued operator can be expressed as the set of critical points of a bipotentials, and (b) a method of construction of a bipotential associated (in the sense of point (a)) to a multivalued, typically non monotone, operator.

At the end of this paper we prove a a new reconstruction theorem for a bipotential from a convex lagrangian cover, this time using a convexity notion related to a minimax theorem of Fan.

2. Notations and first definitions

X and Y are topological, locally convex, real vector spaces of dual variables $x \in X$ and $y \in Y$, with the duality product $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$. The topologies of the spaces X, Y are compatible with the duality product, that is: any continuous linear functional on X (resp. Y) has the form $x \mapsto \langle x, y \rangle$, for some $y \in Y$ (resp. $y \mapsto \langle x, y \rangle$, for some $x \in X$).

We use the notation: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Given a function $\phi : X \to \bar{\mathbb{R}}$, the domain $dom\, \phi$ is the set of points $x \in X$ with $\phi(x) \in \mathbb{R}$. The polar of ϕ, or Fenchel conjugate, $\phi^* : Y \to \bar{\mathbb{R}}$ is defined by: $\phi^*(y) = \sup \{ \langle y, x \rangle - \phi(x) \mid x \in X \}$.

We denote by $\Gamma(X)$ the class of convex and lower semicontinuous functions $\phi : X \to \bar{\mathbb{R}}$. The class of functions $\phi \in \Gamma(X)$ with $dom\, \phi \neq \emptyset$ is denoted by $\Gamma_0(X)$. The class of convex and lower semicontinuous $\phi : X \to \mathbb{R}$ is denoted by $\Gamma(X, \mathbb{R})$.

For any convex and closed set $A \subset X$, its indicator function, χ_A, is defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise} \end{cases}$$

DEFINITION 2.1. *The* **graph of an operator** $T : X \to 2^Y$ *is the set* $G(T)$*:*

$$G(T) = \{ (x, y) \in X \times Y \,:\, y \in T(x) \}$$

To a graph $M \subset X \times Y$ *we associate the multivalued operators:*

$$X \ni x \mapsto m(x) \;=\; \{ y \in Y \mid (x, y) \in M \} \;,$$
$$Y \ni y \mapsto m^*(y) \;=\; \{ x \in X \mid (x, y) \in M \} \;.$$

The **domain of the graph** M *is* $dom(M) = \{x \in X \mid m(x) \neq \emptyset\}$. *The* **image of the graph** M *is the set* $im(M) = \{y \in Y \mid m^*(y) \neq \emptyset\}$.

DEFINITION 2.2. *The* **subdifferential** *of a function* $\phi : X \to \bar{\mathbb{R}}$ *in a point* $x \in dom\,\phi$ *is the (possibly empty) set:*

$$\partial\phi(x) = \{u \in Y \mid \forall z \in X \ \langle z - x, u\rangle \leq \phi(z) - \phi(x)\} \ .$$

In a similar way is defined the subdifferential of a function $\psi : Y \to \bar{\mathbb{R}}$ *in a point* $y \in dom\,\psi$, *as the set:*

$$\partial\psi(y) = \{v \in X \mid \forall w \in Y \ \langle v, w - y\rangle \leq \psi(w) - \psi(y)\} \ .$$

With these notations and definitions we have the Fenchel inequality.

THEOREM 2.3. *Let* $\phi : X \to \bar{\mathbb{R}}$ *be a convex lower semicontinuous function. Then:*

(i) for any $x \in X, y \in Y$ *we have* $\phi(x) + \phi^*(y) \geq \langle x, y\rangle$;

(ii) for any $(x, y) \in X \times Y$ *we have the equivalences:*

$$y \in \partial\phi(x) \iff x \in \partial\phi^*(y) \iff \phi(x) + \phi^*(y) = \langle x, y\rangle \ .$$

DEFINITION 2.4. *An operator* $T : X \to 2^Y$ *is* **monotone** *if for any* $((x, y), (x', y') \in G(T)$ *we have*

$$\langle x - x', y - y'\rangle \geq 0$$

A graph $M \subset X \times Y$ *is monotone if it is the graph of a monotone operator. The graph is* **maximal monotone** *(or the associated operator is maximal monotone) if for any monotone graph* $M' \subset X \times Y$ *such that* $M \subset M'$ *we have* $M = M'$.

An operator $T : X \to 2^Y$ *is cyclically monotone if its graph* $G(T)$ *is cyclically monotone. A graph* M *is* **cyclically monotone** *if for all integer* $m > 0$ *and any finite family of couples* $(x_j, y_j) \in M, j = 0, 1, \ldots, m$,

$$\langle x_0 - x_m, y_m\rangle + \sum_{k=1}^{m}\langle x_k - x_{k-1}, y_{k-1}\rangle \leq 0. \qquad (1)$$

A cyclically monotone graph M *is maximal if it does not admit a strict prolongation which is cyclically monotone.*

3. Bipotentials

DEFINITION 3.1. *A* **bipotential** *is a function* $b : X \times Y \to \bar{\mathbb{R}}$ *with the properties:*

(a) b is convex and lower semicontinuous in each argument;

(b) for any $x \in X, y \in Y$ we have $b(x,y) \geq \langle x, y \rangle$;

(c) for any $(x,y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x,y) = \langle x, y \rangle \ . \quad (2)$$

The **graph of** b *is*

$$M(b) \ = \ \{(x,y) \in X \times Y \ \mid \ b(x,y) = \langle x, y \rangle\} \ . \quad (3)$$

An equivalent definition of a bipotential comes out from the following proposition.

PROPOSITION 3.2. *A function $b : X \times Y \rightarrow \bar{\mathbb{R}}$ is a bipotential if and only if the following conditions are satisfied:*

(A) b is convex and lower semicontinuous in each argument and for any $x \in X, y \in Y$ we have $b(x,y) \geq \langle x, y \rangle$;

(B1) for any $y \in Y$, if the function $z \in X \mapsto (b(z,y) - \langle z, y \rangle)$ has a minimum, then the minimum equals 0;

(B2) for any $x \in X$, if the function $p \in Y \mapsto (b(x,p) - \langle x, p \rangle)$ has a minimum, then the minimum equals 0.

Proof. Condition (A) is the same as conditions (a),(b) from definition 3.1. All we have to prove is: if the function b satisfies condition (A) then condition (c) from definition 3.1 is equivalent with (B1) and (B2).

Assume (A) and take $x \in X$, $y \in Y$ such that $x \in \partial b(x, \cdot)(y)$. This is equivalent with: x is a minimizer of the function

$$z \in X \mapsto (b(z,y) - \langle z, y \rangle)$$

Therefore the statement $x \in \partial b(x, \cdot)(y) \iff b(x,y) = \langle x, y \rangle$ is equivalent with (B1). In the same way we prove that $y \in \partial b(\cdot, y)(x) \iff b(x,y) = \langle x, y \rangle$ is equivalent with (B2).

This simple proposition justifies the introduction of strong bipotentials, which are particular cases of bipotentials. Conditions (B1S) and (B2S) appeared first time as relations (51), (52) [30],

DEFINITION 3.3. *A function $b : X \times Y \rightarrow \bar{\mathbb{R}}$ is a* **strong bipotential** *if it satisfies the conditions:*

(a) b is convex and lower semicontinuous in each argument;

(B1S) for any $y \in Y$ $\inf \{ b(z, y) - \langle z, y \rangle \, : z \in X \} \in \{0, +\infty\}$*;*

(B2S) for any $x \in X$ $\inf \{ b(x, p) - \langle x, p \rangle \, : p \in Y \} \in \{0, +\infty\}$.

4. Operators representable by a bipotential

DEFINITION 4.1. *The non empty set $M \subset X \times Y$ is a* **BB-graph** *(bi-convex, bi-closed) if for all $x \in \ dom(M)$ and for all $y \in \ im(M)$ the sets $m(x)$ and $m^*(y)$ are convex and closed.*

The following theorem (theorem 3.2 [10]) gives a necessary and sufficient condition for the existence of a bipotential associated to a constitutive law M.

THEOREM 4.2. *Given a non empty set $M \subset X \times Y$, there is a bipotential b such that $M = M(b)$ if and only if M is a BB-graph.*

The bipotential mentioned in the previous theorem is denoted by b_M and it has the expression:

$$b_M(x, y) \ = \ \langle x, y \rangle + \chi_M(x, y) \tag{4}$$

In the case of a maximal cyclically monotone graph M, by Rockafellar theorem ([32] Theorem 24.8.) there is an unique separable bipotential associated to M (see section 6 for the definition of separable bipotentials). With the bipotential given by (4) we have two different bipotentials representing the same graph. Therefore, in the larger class made of all bipotentials, in general there is no unicity of the bipotential representing a given BB-graph.

For any BB-graph M, a bipotential b is admissible if $M \subset M(b)$. Then we obviously have $b(x, y) \leq b_M(x, y)$ for any $(x, y) \in X \times Y$. In this sense b_M is the greatest admissible bipotential for the BB-graph M.

5. Fitzpatrick functions; selfdual lagrangians

Let X be a reflexive Banach space and X^* its topological dual. The duality product between X and X^* is the function $\pi : X \times X^* \to \mathbb{R}$, defined by $\pi(x, x^*) = \langle x, x^* \rangle = x^*(x)$.

The space $X \times X^*$ is in duality with itself by the duality product

$$\langle (x, x^*), (y, y^*) \rangle = \langle x, y^* \rangle + \langle y, x^* \rangle$$

Fitzpatrick functions have been introduced in [14]. More on Fitzpatrick functions can be found in [4] and in the book [5].

DEFINITION 5.1. *The* **Fitzpatrick function** *associated to a graph $M \subset X \times X^*$ is the function $f_M : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ given by the Fenchel dual of b_M. Equivalently, f_M is given by:*

$$f_M(x, x^*) = \sup \{ \langle a, x^* \rangle + \langle x, a^* \rangle - \langle a, a^* \rangle \,:\, (a, a^*) \in M \}$$

PROPOSITION 5.2. *(Properties of the Fitzpatrick function) Let $M \subset X \times X^*$ be a graph. Then the associated Fitzpatrick function f_M has the properties:*

(a) f_M is convex and lower semicontinuous,

(b) the graph M is maximal monotone if and only if:

(b1) for any $(x, x^*) \in X \times X^*$ we have $f_M(x, x^*) \geq \langle x, x^* \rangle$ and

(b2) we have equality $f_M(x, x^*) = \langle x, x^* \rangle$ if and only if $(x, x^*) \in M$.

(c) if M is a BB-graph then the function

$$g_M(x, x^*) \;=\; \begin{cases} f_M(x, x^*) \;, & (x, x^*) \in dom\, M \;\times\; im\, M \\ +\infty & otherwise \end{cases}$$

is a strong bipotential.

Proof. By construction any Fitzpatrick function is convex and lower semicontinuous. For proving (b) it is enough to use the following characterization of maximal monotone graphs: M is maximal monotone if and only if

(i1) for any $(x, x^*) \in X \times X^*$ we have

$$\inf \{ \langle x - a, x^* - a^* \rangle : (a, a^*) \in M \} \leq 0$$

(i2) $(x, x^*) \in M$ if and only if

$$\inf \{ \langle x - a, x^* - a^* \rangle : (a, a^*) \in M \} = 0$$

The Fitzpatrick function associated to the graph M can be written as:

$$f_M(x, x^*) \;=\; \langle x, x^* \rangle \;-\; \inf \{ \langle x - a, x^* - a^* \rangle : (a, a^*) \in M \}$$

Therefore (b1), (b2) follow from (i1), (i2) respectively.

In order to prove (c) we have to check (B1S), (B2S) definition 3.3. Let $x \in dom\, M$. Then

$$\inf \{ g_M(x,x^*) - \langle x, x^* \rangle : x^* \in im\, M \} =$$

$$= - \inf \{ \langle x - a, x^* - a^* \rangle : (a,a^*) \in M \, , \, x^* \in im\, M \} \ = \ 0$$

The proof of (B2S) is similar.

Further we describe selfdual lagrangians, with the notations from Ghoussoub [18]. See the mentioned paper and the references therein, especially [19] [20] for more informations on the variational theory associated to selfdual lagrangians. Here we point out that selfdual lagrangians are particular cases of bipotentials.

DEFINITION 5.3. *Let X be a reflexive Banach space. To any function $L \in \Gamma_0(X \times X^*)$ we associate the following operators:*

(i) $\delta L : X \to 2^{X^*}$ *defined by:*

$$\delta L(x) \ = \ \{ x^* \in X^* \ : (x,x^*) \in \partial L(x,x^*) \}$$

(here $\partial L(x,x^*)$ is the subdifferential of L),

(ii) $\bar{\partial} L : X \to 2^{X^*}$ *defined by:*

$$\bar{\partial} L(x) \ = \ \{ x^* \in X^* \ : L(x, x^* \ = \ \langle x, x^* \rangle \}$$

DEFINITION 5.4. *A function $L \in \Gamma_0(X \times X^*)$ is a* **selfdual lagrangian** *if $L = L^*$.*

The following is a slight reformulation of lemma 2.1 and proposition 2.1 [18]

PROPOSITION 5.5. *If L is a selfdual lagrangian such that for some $x_0 \in X$ the function $L(x_0, \cdot)$ is bounded on the balls of X^*, then L is a strong bipotential and we have $M(L) = G(\bar{\partial} L) = G(\delta L)$. Moreover, in this case for any $x^* \in X^*$ there exists $\bar{x} \in X$ such that $x^* \in \delta L(\bar{x})$ and*

$$L(\bar{x}, x^*) \ - \ \langle \bar{x}, x^* \rangle \ = \ \inf \{ L(x,x^*) - \langle x, x^* \rangle : x \in X \} \ = \ 0$$

Proof. As mentioned before, from lemma 2.1 [18] we get that for any selfdual lagrangian L we have $\bar{\partial} L = \delta L$. By definition any selfdual lagrangian is convex, lower semicontinuous and for any $(x,x^*) \in X \times X^*$ we have $L(x,x^*) \geq \langle x, x^* \rangle$, as a consequence of the Fenchel inequality in $X \times X^*$. The fact that L is a strong bipotential (conditions (BS1), (BS2) definition 3.3), as well as the final part of the conclusion are straightforward reformulations of the conclusion of Proposition 2.1 [18].

The following proposition is an application of lemma 3.1 and proposition 3.1 [18].

PROPOSITION 5.6. *Let $M \subset X \times X^*$ be a maximal monotone graph. Then there exists a selfdual lagrangian L_M such that $G(\bar{\delta} L_M) = M$.*

Proof. Let f_M be the Fitzpatrick function of M. Then, according to lemma 3.1 [18] the Fitzpatrick function f_M satisfies the hypothesis of proposition 3.1 [18]. Therefore the selfdual lagrangian defined by

$$L_M(x, x^*) \;=\; \inf \left\{ \frac{1}{2} f_M(x_1, x_1^*) + \frac{1}{2} f_M^*(x_2, x_2^*) + \frac{1}{8}\|x_1 - x_2\|^2 + \right.$$

$$\left. + \frac{1}{8}\|x_1^* - x_2^*\|^2 : (x, x^*) = \frac{1}{2}(x_1, x_1^*) + \frac{1}{2}(x_2, x_2^*) \right\}$$

mentioned in the proof of the proposition 3.1 [18] as “the proximal average” between f_M and f_M^* is the one needed.

6. Separable bipotentials

If $\phi : X \to \mathbb{R}$ is a convex, lower semicontinuous potential, consider the multivalued operator $\partial\phi$ (the subdifferential of ϕ). The graph of this operator is the set

$$M(\phi) \;=\; \{(x, y) \in X \times Y \;\mid\; \phi(x) + \phi^*(y) = \langle x, y \rangle\} \;. \tag{5}$$

$M(\phi)$ is maximal cyclically monotone [32] Theorem 24.8. Conversely, if M is closed and maximally cyclically monotone then there is a convex, lower semicontinuous ϕ such that $M = M(\phi)$.

To the function ϕ we associate the **separable bipotential**

$$b(x, y) = \phi(x) + \phi^*(y).$$

Indeed, the Fenchel inequality can be reformulated by saying that the function b, previously defined, is a bipotential. More precisely, the point (b) (resp. (c)) in the definition of a bipotential corresponds to (i) (resp. (ii)) from Fenchel inequality.

The bipotential b and the function ϕ have the same graph, that is $M(b) = M(\phi)$.

7. Bipotentials for monotone, non maximal graphs

The following two results are from the paper [11] (theorem 3.1 and proposition 3.2).

In the theorem below it is shown that intersections of two maximal monotone graphs are sometimes representable by a bipotential. Therefore there exist bipotentials b with $M(b)$ monotone, but not maximal.

THEOREM 7.1. *Let b_1 and b_2 be separable bipotentials associated respectively to the convex and lower semicontinuous functions $\phi_1, \phi_2 : X \to \mathbb{R} \cup \{+\infty\}$, that is*

$$b_i(x,y) \;=\; \phi_i(x) + \phi_i^*(y)$$

for any $i = 1, 2$ and $(x, y) \in X \times Y$. Consider the following assertions:

(i) $b = max(b_1, b_2)$ is a strong bipotential and $M(b) = M(b_1) \cap M(b_2)$.

(ii') For any $y \in dom\,\phi_1^* \cap dom\,\phi_2^*$ and for any $\lambda \in [0,1]$ we have

$$(\lambda\,\phi_1 \,+\, (1-\lambda)\,\phi_2)^*\,(y) \;=\; \lambda\,\phi_1^*(y) \,+\, (1-\lambda)\,\phi_2^*(y) \tag{6}$$

(ii") For any $x \in dom\,\phi_1 \cap dom\,\phi_2$ and for any $\lambda \in [0,1]$ we have

$$(\lambda\,\phi_1^* \,+\, (1-\lambda)\,\phi_2^*)^*\,(x) \;=\; \lambda\,\phi_1(x) \,+\, (1-\lambda)\,\phi_2(x) \tag{7}$$

Then the point (i) is equivalent with the conjunction of (ii'), (ii"), (for short: (i) $\Longleftrightarrow$ ((ii') AND (ii"))).

In the proof of this theorem we make use of a minimax result by Sion [43]. Notice that in section 12, we use another minimax result of Fan [29] in the proof of theorem 12.4.

The following Proposition shows that in the particular case of X reflexive Banach space and $Y = X^*$ the necessary and sufficient conditions (ii'), (ii") from Theorem 7.1 can be expresses by inf-convolutions.

PROPOSITION 7.2. *Let X be a reflexive Banach space and $Y = X^*$. Consider $\phi_1, \phi_2 \in \Gamma_0(X)$, and for any $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, define $f_1, f_2 \in \Gamma_0(X)$ by $f_1(x) = \alpha\phi_1(x/\alpha)$, $f_2(x) = \beta\phi_2(x/\beta)$, for any $x \in X$. Suppose that for any $x \in dom\,\phi_1 \cap dom\,\phi_2$ and for any $\alpha, \beta \in (0,1)$, $\alpha+\beta = 1$ the subdifferential of the inf-convolution $f_1 \,\Box\, f_2$ is not empty. Then the function b introduced in Theorem 7.1 is a strong bipotential if and only if for any $x \in dom\,\phi_1 \cap dom\,\phi_2$ we have $\partial\phi_1(x) \cap \partial\phi_2(x) \neq \emptyset$.*

8. Bipotentials and inequalities

A source of interesting bipotentials is provided by inequalities. Here we discuss about the Cauchy-Bunyakovsky-Schwarz inequality and about an inequality of Fan concerning eigenvalues of symmetric matrices.

8.1. Cauchy bipotential

Let $X = Y$ be a Hilbert space and let the duality product be equal to the scalar product. Then we define the **Cauchy bipotential** by the formula

$$b(x, y) = \|x\| \; \|y\|.$$

Let us check the Definition (3.1) The point (a) is obviously satisfied. The point (b) is true by the Cauchy-Bunyakovsky-Schwarz inequality. We have equality in the Cauchy-Bunyakovsky-Schwarz inequality $b(x, y) = \langle x, y\rangle$ if and only if there is $\lambda > 0$ such that $y = \lambda x$ or one of x and y vanishes. This is exactly the statement from the point (c), for the function b under study.

The graph of the Cauchy bipotential is not monotone.

8.2. Hill bipotential

Let $S(n)$ be the space of $n \times n$ real symmetric matrices. There is a bipotential expressing that two matrices X and Y have a simultaneous ordered spectral decomposition, [44]. In Mechanics, a constitutive law between two tensors implying that they admit the same eigenvectors is said to be coaxial [39], [44].

The space $S(n)$ is endowed with the scalar product

$$\langle X, Y\rangle = tr\,(XY)$$

Consider the function $b : S(n) \times S(n) \rightarrow \mathbb{R}$ with the expression

$$b(X, Y) \;=\; \lambda_1(X)\lambda_1(Y) + ... + \lambda_n(X)\lambda_n(Y)$$

where for any $X \in S(n)$ $\lambda_1(X) \geq \lambda_2(X) \geq ... \geq \lambda_n(X)$ are the eigenvalues of X ordered from the largest to the smallest. With the help of this function we can write one of Fan's inequalities [28] as: for any $X, Y \in S(n)$ we have

$$b(X, Y) \geq \langle X, Y\rangle$$

with equality if and only if X and Y have the same eigenvectors with preservation of the order of the eigenvalues.

In [44] is proved that for $n = 3$ the function b is a bipotential, called **Hill bipotential** due to applications dealt with by Hill in mechanics. A similar proof, involving majorization, can be done for the case of a general n. The graph of the Hill bipotential is not monotone.

9. Bipotentials in non smooth mechanics

A simple example of a monotone operator in non smooth mechanics is provided by the following model of plasticity of metals. $X = Y$ is the space of $n \times n$ real symmetric traceless matrices with the pairing $\langle x, y\rangle = tr\, xy$ and the associated norm $\|x\| =\mid \langle x, y\rangle \mid^{\frac{1}{2}}$. Let c be a non negative constant. The plasticity operator is defined by

$$T_p\,(0) = K = \{y \in Y \mid \quad \|y\| \ \leq c\ \}$$

otherwise

$$T_p\,(x) = \frac{x}{\|x\|}$$

In plasticity, x is the plastic strain rate tensor, y is the deviatoric stress tensor and c is the yield stress. The closed convex set K is the **plastic domain** and the irreversible or plastic deformations varies when $x \neq 0$. The plastic model is not limited to the metals but can be used also for soil materials.

A more involved example is the **associated Drücker-Prager model** where the variable x is the plastic strain rate tensor and y is the stress tensor (both seen as elements of $S(n)$). With usual notations, the tensors x and y are split into their deviatoric and spheric parts:

$$x = x_d + \frac{1}{3} x_h I, \quad y = y_d + y_h I$$

where I is the identity operator, $x_h = tr\,(x)$ and $y_h = \frac{1}{3} tr\,(y)$. As the decomposition is unique, we can write in short $x = (x_d, x_h)$, $y = (y_d, y_h)$ and the duality pairing is $\langle x, y\rangle = tr\,(x_d y_d) + x_h y_h$.

The convex cone parameterized by the friction angle $\varphi \in \left(0, \frac{\pi}{2}\right)$, the cohesion stress $c > 0$, and $r = 3\sqrt{2}/\sqrt{9 + 12\tan^2\varphi}$ ([7]), with the vertex at $v = \dfrac{c}{\tan\varphi} I$, given by

$$K = \{y \in Y \mid ||y_d|| \leq r(c - y_h \tan\varphi)\}$$

is called the plastic domain.

The multivalued operator corresponding to the associated Drücker-Prager model is defined by: $T_{DP}\,(0) = K$, if $x_h = r||x_d||\tan\varphi$ then

$$T_{DP}\,(x) = \left\{ y \in Y \mid \exists \eta \geq 0,\ y = v + \eta \left(\frac{x_d}{\|x_d\|} - \frac{1}{rc} v \right) \right\} \quad ,$$

if $x_h > r||x_d|| \tan\varphi$ then $T_{DP}(x) = \{v\}$, otherwise $T_{DP}(x) = \emptyset$.

9.1. Drücker-Prager non associated plasticity

The **non associated Drücker-Prager model** is characterized, as previously, by the friction angle $\varphi \in \left(0, \frac{\pi}{2}\right)$, the cohesion stress $c > 0$, and $r = 3\sqrt{2}/\sqrt{9 + 12\tan^2\varphi}$ but also by a new parameter, the dilatancy angle $\theta \in [0, \varphi)$. Once again, $X = Y = S(n)$ and we use the splitting into deviatoric and spheric parts. The associated multivalued operator is T_{na} defined by: $T_{na}(0) = K$, if $x_h = r||x_d|| \tan\theta$ then

$$T_{na}(x) = \left\{ y \in Y \mid \exists \eta \geq 0,\ y = v + \eta \left(\frac{x_d}{\|x_d\|} - \frac{1}{rc} v \right) \right\} \quad ,$$

if $x_h > r||x_d|| \tan\theta$ then $T_{na}(x) = \{v\}$ otherwise $T_{na}(x) = \emptyset$.

If we put $\theta = \varphi$ then we recover the operator T_{DP} of the associated Drücker-Prager model defined above.

The non-associated Drücker-Prager law $y \in T_{na}(x)$ is equivalent with the following differential inclusion:

$$x \ + \ \frac{1}{3}\left(x_h \ + \ r \,\|x\| \,(\tan\phi \ - \ \tan\theta)\right) \ \in \ \partial\chi_K(y)$$

According to [9] [39] [25], this inclusion can be written as $b_p(x, y) = \langle x, y \rangle$, for the bipotential

$$b_p(x, y) \ = \ \chi_K(y) \ + \ \chi_{K_p}(x) \ + \ \frac{c}{\tan\phi} x_h \ +$$

$$+ \ r \,\|x\| \,(\tan\phi \ - \ \tan\theta) \left(y_h \ - \ \frac{c}{\tan\phi} \right)$$

The last term in this expression is a coupling term which gives the implicit character to the constitutive law.

9.2. Coulomb's dry friction law

Another interesting operator comes in relation with the unilateral contact with dry friction or **Coulomb's friction model**. T Consider two bodies Ω_1 and Ω_2 which are in contact at a point M, with $\mathbf{n}$ the unit vector normal to the common tangent plane and directed towards Ω_1. The space $X = \mathbb{R}^3$ is the one of relative velocities between points of contact of two bodies, and the space Y, identified also to $\mathbb{R}^3$, is the one of the contact reaction stresses. The duality product is the usual scalar product. We put

$$(x_n, x_t) \in X = \mathbb{R} \times \mathbb{R}^2, \quad (y_n, y_t) \in Y = \mathbb{R} \times \mathbb{R}^2 \ ,$$

where x_n is the gap velocity, x_t is the sliding velocity, y_n is the contact pressure and y_t is the friction stress. The friction coefficient is $\mu > 0$.

The graph of the law of unilateral contact with Coulomb's dry friction is the union of three sets, respectively corresponding to the 'body separation', the 'sticking' and the 'sliding'.

$$M = \{(x,0) \in X \times Y \mid x_n < 0\} \cup \tag{8}$$
$$\cup \; \{(0,y) \in X \times Y \mid \| y_t \| \leq \mu y_n\} \cup$$
$$\cup \; \left\{(x,y) \in X \times Y \mid x_n = 0, \; x_t \neq 0, \; y_t = \mu y_n \frac{x_t}{\| x_t \|}\right\}$$

It is well known that this graph is not monotone, then not cyclically monotone.

This law can be written as the following differential inclusion ([33] [34] [40] [39]):

$$(x_n \; - \; \mu \, \|x_t\|) \, \mathbf{n} \; + \; x_t \; \in \; \partial \chi_{K_\mu}(y)$$

Let us consider the conjugate cone of the Coulomb's cone:

$$K_\mu^* = \{(x_n, x_t) \in X \mid \mu \| x_t \| + x_n \leq 0\} \, .$$

We shall use also a second pair of conjugate cones:

$$K_0 = \{(y_n, 0) \in Y \mid y_n \geq 0\} \, , \quad K_0^* = \{(x_n, x_t) \in X \mid x_n \leq 0\} \, .$$

The graph of the law of unilateral contact with Coulomb's dry friction is the graph of the following bipotential, previously given in [33]:

$$b(x,y) \; = \mu y_n \| x_t \| + \chi_{K_\mu}(y) + \chi_{K_0^*}(x) \; .$$

9.3. Coaxial laws

Consider the non monotone operator defined by $T_{iso}\,(0) = \mathbb{R}^n$, otherwise

$$T_{iso}\,(x) = \{y \in Y \mid \exists \lambda > 0, \quad y = \lambda x\}$$

An operator $S : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is strongly isotropic if its graph is contained in the graph of T_{iso}. The graph of the non monotone operator T_{iso} is the graph of the Cauchy bipotential.

The eigenvalues of any matrix $x \in S(n)$ can be conventionally ordered from the largest to the smallest: $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x)$. Two elements $x, y \in S(n)$ are said coaxial if they have the same eigenvectors with preservation of the order of the eigenvalues: the largest eigenvalue of x is associated to the eigenvector of y corresponding to the largest eigenvalue of y, and so on. An operator $T : S(n) \to 2^{S(n)}$ is coaxial ([12],[44]) if for any $y \in T(x)$, y and x are coaxial. The graph of any coaxial operator is contained in the non monotone operator defined by $T_H(0) = S(n)$, otherwise

$$T_H(x) = \{y \in Y \mid x \text{ and } y \text{ are coaxial}\}$$

We have seen in section 8.2 that the graph of a coaxial operator is included in the graph of the Hill bipotential.

10. Numerical methods based on the bipotential framework

In applications bipotentials are interesting because of the associated implicit normality rules. The properties of bipotentials allow to discretize an evolution problem into a series of minimization problems concerning the associated **bifunctional**.

For example let us consider a body with reference configuration Ω, with the boundary decomposition $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. At a fixed moment t we have imposed velocities $\dot{u}_t$ on the part Γ_1 of its boundary, imposed forces f_t on Γ_2 and on Γ_3 the body is in unilateral contact with friction with a rigid foundation. The unit normal, pointing outwards, of the boundary $\partial\Omega$ is denoted by $\mathbf{n}$ and for any stress field σ in Ω we denote $\sigma_n = \sigma\, \mathbf{n}$.

Suppose that the body is made by a plastic material described by a bipotential b_p dependent on the strain rate $\varepsilon(\dot{u})$ and the stress σ. The contact with friction is described by the bipotential b_c. associated with the Coulomb law.

For any pair (v, τ) of kinematically admissible velocity field v and statically admissible stress field τ we introduce the bifunctional

$$B(v,\tau) \;=\; \int_\Omega b_p(\varepsilon(v),\tau)\ \mathrm{d}x \;+\; \int_{\Gamma_3} b_c(-v,\tau_n)\ \mathrm{d}s \;-$$

$$-\; \int_{\Gamma_1} \tau_n \cdot \dot{u}_t\ \mathrm{d}s \;-\; \int_{\Gamma_2} f_t \cdot v\ \mathrm{d}s$$

Then, by using integration by parts and the properties of the bipotentials which are involved, one can show that

$$B(v,\tau) \geq 0$$

and that $B(u,\sigma) = 0$ if (u,σ) is the pair formed by the velocity field and associated stress field of the body at moment t.

Therefore we may try to numerically minimize the bifunctional in order to find the solution of the (quasistatic) evolution problem. Alternatively, the bifunctional can be adapted in order to use a predictor/corrector scheme.

As applications we may cite the bound theorems of the limit analysis ([38], [8]) and the plastic shakedown theory ([41], [12], [9], [6]) which can be reformulated by means of weak normality rules. The bipotential method suggests new algorithms, fast and robust, as well as variational error estimators assessing the accurateness of the finite element mesh ([22], [23], [37], [40], [7], [25], [26]). Such algorithms have been proposed and used in applications to the contact mechanics [13], the dynamics of granular materials ([15], [16], [17][42]), the cyclic plasticity of metals [37] and the plasticity of soils ([1], [25]).

A very challenging subject seems to be the extension of the mathematical results of Ghoussoub [19] [20] for the particular case of self-dual lagrangians, to general bipotentials, or at least to bipotentials constructed from bipotential convex covers.

11. Construction of bipotentials

Let $Bp(X,Y)$ be the set of all bipotentials $b : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$. We shall need the following notion of implicitly convex functions.

DEFINITION 11.1. *Let Λ be an arbitrary non empty set and V a real vector space. The function $f : \Lambda \times V \rightarrow \bar{\mathbb{R}}$ is* **implicitly convex** *if for any two elements $(\lambda_1, z_1), (\lambda_2, z_2) \in \Lambda \times V$ and for any two numbers $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ there exists $\lambda \in \Lambda$ such that*

$$f(\lambda, \alpha z_1 + \beta z_2) \ \leq \ \alpha f(\lambda_1, z_1) + \beta f(\lambda_2, z_2) \quad .$$

In the following **bipotential convex covers** are defined, as in definition 4.2 [11].

DEFINITION 11.2. *A* **bipotential convex cover** *of the non empty set M is a function $\lambda \in \Lambda \mapsto b_\lambda$ from Λ with values in the set $Bp(X,Y)$, with the properties:*

(a) The set Λ is a non empty compact topological space,

(b) Let $f : \Lambda \times X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$f(\lambda, x, y) \ = \ b_\lambda(x,y).$$

Then for any $x \in X$ *and for any* $y \in Y$ *the functions* $f(\cdot, x, \cdot) : \Lambda \times Y \to \bar{\mathbb{R}}$ *and* $f(\cdot, \cdot, y) : \Lambda \times X \to \bar{\mathbb{R}}$ *are lower semi continuous on the product spaces* $\Lambda \times Y$ *and respectively* $\Lambda \times X$ *endowed with the standard topology,*

(c) *We have* $M = \bigcup_{\lambda \in \Lambda} M(b_\lambda)$.

(d) *the functions* $f(\cdot, x, \cdot)$ *and* $f(\cdot, \cdot, y)$ *are implicitly convex in the sense of Definition 11.1.*

This notion generalizes the one of a **bi-implicitly convex lagrangian cover**. see Definitions 4.1 and 6.6 [10]. Here we shall give only the definition of a convex lagrangian cover, without the bi-implicit convexity hypothesis.

DEFINITION 11.3. *Let* $M \subset X \times Y$ *be a non empty set. A* **convex lagrangian cover** *of* M *is a function* $\lambda \in \Lambda \mapsto \phi_\lambda$ *from* Λ *with values in the set of lower semicontinuous and convex functions on* X*, with the properties:*

(a) *The set* Λ *is a non empty compact topological space,*

(b) *Let* $f : \Lambda \times X \times Y \to \mathbb{R}$ *be the function defined by*

$$f(\lambda, x, y) = \phi_\lambda(x) + \phi^*_\lambda(y).$$

Then for any $x \in X$ *and for any* $y \in Y$ *the functions* $f(\cdot, x, \cdot)$ *and* $f(\cdot, \cdot, y)$ *are lower semicontinuous from* Λ *with values in the set of lower semicontinuous and convex functions on* X*, endowed with pointwise convergence topology,*

(c) *we have*

$$M = \bigcup_{\lambda \in \Lambda} M(\phi_\lambda) .$$

A bipotential convex cover $\lambda \in \Lambda \mapsto b_\lambda$ such that for any $\lambda \in \Lambda$ the bipotential b_λ is separable is a bi-implicitly convex lagrangian cover, see Definitions 4.1 and 6.6 [10]. . For such covers the sets $M(b_\lambda)$are **maximal cyclically monotone** for any $\lambda \in \Lambda$.

General bipotential convex covers are **not lagrangian**. (see remark 6.1 [10] for a justification of the "lagrangian" term). In the language of convex analysis this means that in general the sets $M(b_\lambda)$ are not cyclically monotone. An example is given in section 5 [11], of a bipotential convex cover of the graph of the Coulomb's dry friction law, which is made by monotone, but not maximally monotone graphs.

In sections 5 and 8 [10] it is explained that not any BB-graph admits a convex lagrangian cover. Moreover, there are BB-graphs admitting (up to reparametrization) only one convex lagrangian cover, as well as BB-graphs which have infinitely many lagrangian covers. The problem of describing the set of all convex lagrangian covers of a BB-graph seems to be difficult.

DEFINITION 11.4. *Let $\lambda \mapsto b_\lambda$ be a bipotential convex cover of the BB-graph M. To the cover we associate the function $b : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ by the formula*

$$b(x,y) \; = \; \inf \{ b_\lambda(x,y) \, : \, \lambda \in \Lambda \}$$

We obtained in theorem 4.6 [11] the following result.

THEOREM 11.5. *Let $\lambda \mapsto \phi_\lambda$ be a bipotential convex cover of the BB-graph M and $b : X \times Y \rightarrow R$ defined as in definition 11.4. Then b is a bipotential and $M = M(b)$.*

In the case of $M = M(\phi)$, with ϕ convex and lower semi continuous (this corresponds to separable bipotentials), the set Λ has only one element $\Lambda = \{\lambda\}$ and we have only one potential ϕ. The associated bipotential from Definition 11.4 is obviously

$$b(x,y) \; = \; \phi(x) + \phi^*(y) \; .$$

This is a bipotential convex cover in a trivial way; the implicit convexity conditions are equivalent with the convexity of ϕ, ϕ^* respectively.

12. One more construction result

For simplicity, in this section we shall work only with lower semicontinuous convex functions ϕ with the property that $\phi \in \Gamma(X, \mathbb{R})$ and its Fenchel dual $\phi^* \in \Gamma(Y, \mathbb{R})$.

We reproduce here the following definition of convexity (in a generalized sense), given by K. Fan [29] p. 42.

DEFINITION 12.1. *Let X, Y be two arbitrary non empty sets. The function $f : X \times Y \rightarrow \mathbb{R}$ is* **convex** *on X* **in the sense of Fan** *if for* *any two elements* $x_1, x_2 \in X$ *and for any two numbers* $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta = 1$ *there exists a* $x \in X$ *such that for all* $y \in Y$*:*

$$f(x,y) \ \leq \ \alpha f(x_1,y) + \beta f(x_2,y).$$

With the help of the previous definition we introduce a new convexity condition for a convex lagrangian cover.

DEFINITION 12.2. *Let* $\lambda \mapsto \phi_\lambda$ *be a convex lagrangian cover of the BB-graph* M*, in the sense of definition 11.3. Consider the functions:*

$$g : X \times \Lambda \times X \to \mathbb{R} \quad , \quad h : Y \times \Lambda \times Y \to \mathbb{R} \quad ,$$

given by $g(x,\lambda,z) = \phi_\lambda(x) - \phi_\lambda(z)$*, respectively* $h(y,\lambda,u) = \phi^*_\lambda(y) - \phi^*_\lambda(u)$*.*

The cover is **Fan bi-implicitly convex** *if for any* $x \in X$*,* $y \in Y$*, the functions* $g(x,\cdot,\cdot)$*,* $h(y,\cdot,\cdot)$ *are convex in the sense of Fan on* $\Lambda \times X$*,* $\Lambda \times Y$ *respectively.*

Recall the following minimax theorem of Fan [29], Theorem 2. In the formulation of the theorem words "convex" and "concave" have the meaning given in definition 12.1 (more precisely f is concave if $-f$ is convex in the sense of the before mentioned definition).

THEOREM 12.3. *(Fan) Let* X *be a compact Hausdorff space and* Y *an arbitrary set. Let* f *be a real valued function on* $X \times Y$ *such that, for every* $y \in Y$*,* $f(\cdot, y)$ *is lower semicontinuous on* X*. If* f *is convex on* X *and concave on* Y*, then we have*

$$\min_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \min_{x \in X} f(x,y) \quad .$$

The difficulty of theorem 11.5 boils down to the fact the class of convex functions is not closed with respect to the inf operator. Nevertheless, by using Fan theorem 12.3 we get the following general result.

THEOREM 12.4. *Let* Λ *be a compact Hausdorff space and* $\lambda \mapsto \phi_\lambda \in \Gamma(X,\mathbb{R})$ *be a convex lagrangian cover of the BB-graph* M*, with* $\phi^*_\lambda \in \Gamma(Y,\mathbb{R})$ *for any* $\lambda \in \Lambda$*, such that:*

(a) for any $x \in X$ *and for any* $y \in Y$ *the functions* $\Lambda \ni \lambda \mapsto \phi_\lambda(x) \in \mathbb{R}$ *and* $\Lambda \ni \lambda \mapsto \phi^*_\lambda(y) \in \mathbb{R}$ *are continuous,*

(b) the cover is Fan bi-implicitly convex in the sense of definition 12.2.

Then the function $b : X \times Y \to \mathbb{R}$ defined by

$$b(x, y) \;=\; \inf \{ \phi_\lambda(x) + \phi^*_\lambda(y) \;\mid\; \lambda \in \Lambda \}$$

is a bipotential and $M = M(b)$.

Proof. For some of the details of the proof we refer to the proof of theorem 4.12 11.5 in [10]. There are five steps in that proof. In order to prove our theorem we have only to modify the first two steps: we want to show that for any $x \in dom(M)$ and any $y \in im(M)$ the functions $b(\cdot, y)$ and $b(x, \cdot)$ are convex and lower semi continuous.

For $(x, y) \in X \times Y$ let us define the function $\overline{xy} : \Lambda \times X \to \mathbb{R}$ by

$$\overline{xy}(\lambda, z) = \langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z) \quad .$$

We check now that $\overline{xy}$ verifies the hypothesis of theorem 12.3. Indeed, the hypothesis (a) implies that for any $z \in X$ the function $\overline{xy}(\cdot, z)$ is continuous. Notice that

$$\overline{xy}(\lambda, z) = \langle z, y \rangle + g(x, \lambda, z) \quad .$$

It follows from hypothesis (b) that the function $\overline{xy}$ is convex on Λ in the sense of Fan.

In order to prove the concavity of $\overline{xy}$ on X, it suffices to show that for any $z_1, z_2 \in X$, for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$, we have the inequality

$$\overline{xy}(\lambda, \alpha z_1 + \beta z_2) \leq \alpha \overline{xy}(\lambda, z_1) + \beta \overline{xy}(\lambda, z_2)$$

for any $\lambda \in \Lambda$. This inequality is equivalent with

$$\langle \alpha z_1 + \beta z_2, y \rangle - \phi_\lambda(\alpha z_1 + \beta z_2) \leq \alpha \left(\langle z_1, y \rangle - \phi_\lambda(z_1) + \beta \left(\langle z_2, y \rangle - \phi_\lambda(z_2) \right.\right.$$

for any $\lambda \in \Lambda$. But this is implied by the convexity of ϕ_λ for any $\lambda \in \Lambda$.

In conclusion the function $\overline{xy}$ satisfies the hypothesis of theorem 12.3. We deduce that

$$\min_{\lambda \in \Lambda} \sup_{z \in X} \overline{xy}(\lambda, z) = \sup_{z \in X} \min_{\lambda \in \Lambda} \overline{xy}(\lambda, z) \quad . \tag{9}$$

Let us compute the two sides of this equality.

For the left hand side term of (9) we have:

$$\min_{\lambda \in \Lambda} \sup_{z \in X} \overline{xy}(\lambda, z) = \min_{\lambda \in \Lambda} \sup_{z \in X} \{ \langle z, y \rangle + \phi_\lambda(x) - \phi_\lambda(z) \} =$$

$$= \min_{\lambda \in \Lambda} \left\{ \phi_\lambda(x) + \sup_{z \in X} \{ \langle z, y \rangle - \phi_\lambda(z) \} \right\} =$$

$$= \min_{\lambda \in \Lambda} \{\phi_\lambda(x) + \phi^*_\lambda(y)\} = b(x,y) \quad .$$

For the right hand side term of (9) we have:

$$\sup_{z \in X} \min_{\lambda \in \Lambda} \overline{xy}(\lambda, z) = \sup_{z \in X} \min_{\lambda \in \Lambda} \{\langle z, y\rangle + \phi_\lambda(x) - \phi_\lambda(z)\} =$$

$$= \sup_{z \in X} \left\{ \langle z, y\rangle - \max_{\lambda \in \Lambda} \{\phi_\lambda(z) - \phi_\lambda(x)\} \right\} \quad .$$

Let $\overline{x} : X \to \mathbb{R}$ be the function

$$\overline{x}(z) = \max_{\lambda \in \Lambda} \{\phi_\lambda(z) - \phi_\lambda(x)\} \quad .$$

Then the right hand side term of (9) is in fact:

$$\sup_{z \in X} \min_{\lambda \in \Lambda} \overline{xy}(\lambda, z) = \overline{x}^*(y) \quad .$$

Therefore we proved the equality:

$$b(x,y) = \overline{x}^*(y) \quad .$$

This shows that the function b is convex and lower semicontinuous in the second argument.

In order to prove that b is convex and lower semicontinuous in the first argument, replace ϕ_λ by ϕ^*_λ in the previous reasoning.

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