Control of nonlinear PDE's

Abstract

This research project summarises two types of control problems for some nonlinear PDE's. The first of them discussed in Sections 1–3 deals with the Navier–Stokes system controlled by a finite-dimensional external force. The second group of problems deals with the magnetohydrodynamics system and other related equations with a localised control. In both cases, we give a precise formulation of the problems we wish to study and describe some expected results.

1 Feedback control by a finite-dimensional body force

Let us consider the Navier–Stokes system

$$\begin{cases} \partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla u = f(t,x), \\ \qquad\qquad\qquad\qquad \operatorname{div} u = 0, \end{cases} \tag{1}$$

where u and p are unknown velocity field and pressure, ν is the kinematic viscosity, and f is an external force. Suppose that the space variable x belongs to the torus $\mathbb{T}^d$ with $d = 2$ or 3, and denote by V the space of H^1-smooth divergence-free vector fields on $\mathbb{T}^d$. Let us assume that the right-hand side of (1) is represented in the form

$$f(t,x) = h(t,x) + \eta(t,x), \tag{2}$$

where h is a given function and η is a control taking on values in a finite-dimensional subspace $E \subset V$. Equations (1), (2) are supplemented with the initial condition

$$u(0,x) = u_0(x), \tag{3}$$

where $u_0 \in V$. It was proved in the recent papers [AS05, AS06, Shi06, Shi07] that if E sufficiently large, then problem (1)–(3) is controllable in the following sense.

Controllability : For any $T > 0$ and $\varepsilon > 0$, any vector-fields $\hat{u}, u_0 \in V$, and any finite-dimensional space $F \subset V$ there is a control $\eta \in C^\infty(0,T;E)$ such that the problem (1)–(3) has a unique solution $u(t,x)$ (in an appropriate functional class), which satisfies the relations

$$\|u(T,\cdot)-\hat{u}\|_1<\varepsilon,$$
$$\mathsf{P}_F u(T,\cdot)=\mathsf{P}_F\hat{u},$$

where $\|\cdot\|_1$ is the H^1 norm and $\mathsf{P}_F : V \to V$ stands for the orthogonal projection onto F.

Our first goal is to study

PROBLEM I : Feedback stabilisation to a given bounded solution.

In other words, given an initial function $u_0 \in V$ and a time-bounded solution $\hat{u}(t,x)$ of problem (1), (2) with $\eta \equiv 0$, we wish to construct an operator $K : V \to E$ such that problem (1)–(3) with $\eta = K(u)$ has a unique solution $u(t,x)$, which satisfies the condition

$$\|u(t,\cdot)-\hat{u}(t,\cdot)\|_1 \to 0 \quad \text{as } t \to +\infty. \tag{4}$$

Feedback stabilisation to a given stationary solution was studied in the paper [BT04], which establishes the existence of a linear stabiliser K with range in a finite-dimensional space depending on the stationary solution. Furthermore, it follows from the existence of finitely many determining modes (see [FP67]) that for any $\nu > 0$ one can construct a finite-dimensional feedback control K with range in a space E_ν such that (4) holds for any bounded solution $\hat{u}$. Our goal is to establish a similar result with a control space not depending on ν. This result would be important in ergodic theory for stochastic Navier–Stokes equations.

2 Finite-dimensional boundary control

Let us consider the Navier–Stokes system (1) supplemented with the boundary condition

$$u\big|_{\partial D}=\eta(t,x), \tag{5}$$

where η is a control function. This problem is rather well understood in the case when there are no further restrictions on η. In particular, it was proved in [Cor96, CF96, FÈ99] (see also the references in [Cor07]) that problem (1), (5) is globally exactly controllable. Furthermore, feedback stabilisation was studied in [Fur01, Fur04, BLT06$_1$, Ray06, Ray07]. Our aim here is to investigate the problem of controllability by a finite-dimensional control force.

PROBLEM II : Approximate controllability.

We wish to find a finite-dimensional space $E \subset L^2(\partial D)$ (not depending on ν) such that for any $\varepsilon > 0$, $T \gg 1$ and any divergence-free vector fields $u_0, \hat{u} \in H^1$ vanishing on the boundary there is a solution $u(t,x)$ of (1) defined on the interval $[0,T]$ such that

$$u(t,\cdot)\big|_{\partial D}\in E \quad \text{for any } t\in[0,T].$$

3 Exact controllability

Let us return to problem (1), (2). It is well known that in the case of torus (both 2D or 3D), this problem is exactly controllable by an external force η supported in a given open subset $\omega \subset \mathbb{T}^d$; see [Cor96, CF96, FÈ99]. We wish to investigate

Problem III : Exact controllability by a finite-dimensional body force.

Let us describe this problem in more details. We fix a function h and a constant $T > 0$ and denote by $\mathcal{B}_T$ the set of points $\hat{u}$ that are representable in the form $\hat{u} = v(T, \cdot)$, where $v(t, x)$ is a solution of (1) with $f = h$. For any divergence-free smooth vector field u_0 and any $\hat{u} \in \mathcal{B}_T$, we seek a control $\eta(t)$ with range in a finite-dimensional space $E \subset L^2(\mathbb{T}^d, \mathbb{R}^d)$ such that the solution of (1)–(3) is defined on the time interval $[0, T]$ and satisfies the relation $u(T, \cdot) = \hat{u}$. Note that in the case of the Euler system ($\nu = 0$), the exact controllability does not hold. Namely, it is proved in [Shi08] that the complement of the set of accessibility $\mathcal{A}_T(u_0)$ from a given initial point u_0 is everywhere dense in the phase space. Since the Euler equations are time-reversible, the set $\mathcal{B}_T$ coincides with the phase space and therefore is much bigger than $\mathcal{A}_T(u_0)$. Due to regularising property of the Navier–Stokes dynamics, the set $\mathcal{B}_T$ is a proper subset of the space of smooth functions, and the property of exact controllability becomes much more delicate.

The important point of the three problems described above is that their solution will certainly use the structure of nonlinear term, and the corresponding properties are not likely to hold for the linearised Navier–Stokes system.

4 Exact boundary controllability

In [FÈ99], the *global* exact boundary controllability is established for both the Navier-Stokes and Boussinesq equations. This is a striking result because, when the control is distributed inside, only *local* exact controllability results are known for the Navier-Stokes and Boussinesq equations (see [Im98, Im01, FCGIP04, HPS05, HPS06$_1$]). Similar results of local exact internal controllability are established for the magnetohydrodynamic (MHD) equations as well (see [BHPS03, BHPS05, HPS06$_2$, HPS07]). The key ingredient of the proofs consists in appropriate Carleman estimates for the adjoint linearized equations. The MHD equations describe the motion of a viscous incompressible conducting fluid in a magnetic field and consist in a subtle and elegant coupling of the Navier-Stokes equations of viscous incompressible fluid flow and the Maxwell equations of electromagnetic field:

$$\begin{cases} \partial_t y + \langle y, \nabla \rangle\, y - \langle B, \nabla \rangle\, B - \nu \Delta y + \nabla p + \nabla(\frac{1}{2}\, B)^2 = f, \\ \partial_t B + (y, \nabla) B - \langle B, \nabla \rangle\, y + \eta\, \operatorname{curl}(\operatorname{curl} B) = 0, \\ \operatorname{div} y = 0, \ \operatorname{div} B = 0. \end{cases} \tag{6}$$

Here y and B are the velocity and magnetic fields, p is the pressure, ν is the kinematic viscosity, η is the magnetic resistivity and f is an external force.

Inspired by [FÈ99], we expect to obtain a global exact boundary controllability result for the MHD system by following a strategy made up by three steps: 1. One establishes a local exact controllability result for the MHD equations on the torus $\mathbb{T}^d$ (with $d = 2$ or 3). 2. One proves the approximate controllability for the MHD equations on $\mathbb{T}^d$. 3. Combining the first two results, we obtain the global exact controllability for the MHD system on $\mathbb{T}^d$. By a periodic extension procedure, one reduces the global exact boundary controllability for the MHD equations in bounded domains to the global exact controllability for the MHD equations on the torus. So we have to investigate the following problems.

Problem IV : Local exact controllability for the MHD system on the torus.

More specifically, let $(\hat{y}, \hat{B}, \hat{p})$ be a solution of equations (6), considered on $\mathbb{T}^d$ with $d = 2$ or 3. Let ω be an open subset of $\mathbb{T}^d$ and $T > 0$. We expect that the following assertion is true. If the target solution $(\hat{y}, \hat{B}, \hat{p})$ is sufficiently regular, then there is $\varepsilon > 0$ such that for any $y_0, B_0 \in (H^1(\mathbb{T}^d))^d$ with $\operatorname{div} y_0 = \operatorname{div} B_0 = 0$ which satisfy

$$\|y_0 - \hat{y}(\cdot, 0)\|_{(H^1(\mathbb{T}^d))^d} + \|B_0 - \hat{B}(\cdot, 0)\|_{(H^1(\mathbb{T}^d))^d} \le \varepsilon$$

there exist $(u, v) \in (L^2(\mathbb{T}^d \times (0,T)))^{2d}$ and $(y, B, p, q) \in (H^{2,1}(\mathbb{T}^d \times (0,T)))^{2d} \times (L^2(0,T; H^1(\mathbb{T}^d)))^2$ such that

$$\begin{cases} \partial_t y + \langle y, \nabla \rangle\, y - \langle B, \nabla \rangle\, B - \nu \Delta y + \nabla p + \nabla(\frac{1}{2}\, B^2) = f + \chi_\omega u, \\ \partial_t B + \langle y, \nabla \rangle\, B - \langle B, \nabla \rangle\, y + \eta\, \operatorname{curl}(\operatorname{curl} B) + \nabla q = \chi_\omega v, \\ \operatorname{div} y = 0, \ \operatorname{div} B = 0, \\ y(\cdot, 0) = y_0, \ B(\cdot, 0) = B_0, \end{cases} \tag{7}$$

and

$$y(\cdot, T) = \hat{y}(\cdot, T), \quad B(\cdot, T) = \hat{B}(\cdot, T).$$

Here χ_ω is the characteristic function of ω.

Problem V : Approximate controllability for the MHD system on the torus.

Let us describe the problem in more details. Let $(\hat{y}, \hat{B}, \hat{p})$ be a solution of (6), considered on $\mathbb{T}^d$, and let ω be an open subset of $\mathbb{T}^d$. We shall try to show that, if $(\hat{y}, \hat{B}, \hat{p})$ is sufficiently regular, then for any $y_0, B_0 \in (H^1(\mathbb{T}^d))^d$ with $\operatorname{div} y_0 = \operatorname{div} B_0 = 0$ and any $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and the vector functions $(u, v) \in (L^2(\mathbb{T}^d \times (0,T)))^{2d}$ and $(y, B, p, q) \in (H^{2,1}(\mathbb{T}^d \times (0,T)))^{2d} \times (L^2(0,T; H^1(\mathbb{T}^d)))^2$ which satisfy equations (7) and

$$\|y(\cdot,T) - \hat{y}(\cdot,T)\|_{(H^1(\mathbb{T}^d))^d} + \|B(\cdot,T) - \hat{B}(\cdot,T)\|_{H^1(\mathbb{T}^d))^d} \le \varepsilon.$$

Moreover, we have

$$\lim_{\varepsilon \to 0} T(\varepsilon) = 0.$$

Problem VI : Global exact controllability for the MHD equations in bounded domains.

More specifically, let Ω be a bounded open set of $\mathbb{R}^d$ ($d = 2$ or 3) with a sufficiently smooth boundary $\partial\Omega$, and let $T > 0$ be a fixed time. Consider a solution $(\hat{y}, \hat{B}, \hat{p})$ of equations (6), viewed as equations in $\Omega \times (0,T)$. We expect to show that, if $(\hat{y}, \hat{B}, \hat{p})$ is sufficiently regular, then for any $y_0, B_0 \in (H^1(\Omega))^d$ with $\operatorname{div} y_0 = \operatorname{div} B_0 = 0$ there exist $(\alpha, \beta) \in (L^2(0,T; (H^{3/2}(\partial\Omega))^d))^2$ and $(y, B, p) \in (H^{2,1}(\Omega \times (0,T)))^{2d} \times L^2(0,T; H^1(\Omega))$ which satisfy equation (6) in $\Omega \times (0,T)$, the boundary conditions

$$y = \alpha, \; B = \beta \;\text{ on } \partial\Omega \times (0,T),$$

the initial conditions

$$y(\cdot,0) = y_0, \; B(\cdot,0) = B_0 \text{ in } \Omega,$$

and the final conditions

$$y(\cdot,T) = \hat{y}(\cdot,T), B(\cdot,T) = \hat{B}(\cdot,T) \text{ in } \Omega.$$

5 Feedback stabilization

We study stabilization for fluid dynamics equations (Navier-Stokes, magneto-hydrodynamics, Boussinesq). As an example we consider controlled MHD system (6).

Given a stationary solution $(\hat{y}, \hat{B})$ we search for a feedback controller (linear or nonlinear)

$$(u(t), v(t)) = K(y(t), B(t)),$$

such that the system becomes stable around the stationary state. One of the reasons for taking controllers localized in subdomains is the fact that one may extend the results to boundary controllers by the usual technique of considering the problem in a larger domain.

Feedback stabilization results for Navier-Stokes equations were obtained in [Bar03, $\mathrm{BLT06}_1$, $\mathrm{BLT06}_2$, BT04, Fur01, Fur04, $\mathrm{Lef08}_2$]. Results concerning local feedback stabilization for the MHD system were obtained in [$\mathrm{Lef08}_1$].

There are several issues concerning the stabilization of Navier-Stokes type systems:

PROBLEM VII : The study of the domain of attraction for Navier-Stokes equations controlled by feedback stabilization laws. The study of the existence of global linear feedback stabilizers, either internal or boundary.

PROBLEM VIII : Stabilization of the system with different boundary conditions.

Various boundary conditions for the velocity field of the fluid may be encountered in practice, the most usual being the nonslip boundary condition

$$y = 0$$

and Navier-slip boundary conditions

$$\sum_{i,j=1}^{3} N_i \left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) T_j = 0, \quad y \cdot N = 0,$$

where N is the outer normal vector to the boundary and T is a generic tangent vector. The problems in different boundary settings need separate treatment, especially when deriving Carleman type observability inequalities.

PROBLEM IX : Stabilization in unbounded domains, particularly in channels.

PROBLEM X : Investigation of the stabilization properties and the domain of stabilization by using very particular laws of feedback, more realistic from the applications point of view.

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