The Control Variational Method for Beams in Contact with Deformable Obstacles

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Abstract

We consider a mathematical model which describes the equilibrium of an elastic beam in contact with two obstacles. The contact is modeled with a normal compliance type condition in such a way that the penetration is allowed but is limited. We state the variational formulation of the problem and prove an existence and uniqueness result for the weak solution. Then, we provide an alternative approach to the model, based on the control variational method. Necessary and sufficient optimality conditions are derived, together with an approximation property. Finally, we extend our results to some versions of the model which describe the contact with a single obstacle, including a time-dependent case.

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Key words: Euler-Bernoulli beam, normal compliance, Signorini's condition, adhesion, weak solution, control variational method, optimal state.

1 Introduction

The control variational method for differential equations was introduced in [1, 15]. A comprehensive presentation for this new variational method, together with various examples and applications, may be found in the recent monograph [9]. The main new idea in this method is to perform the minimization of the energy of the system via the optimal control theory, which represents an extension of the arguments via the calculus of variations, used in the classical variational method. This new general framework is very flexible and may offer several different solutions for the same problem, as shown in [16]. It is relevant both from the theoretical and the numerical point of view. In particular, it replaces the solution of nonlinear differential equations of order four by the solution of linear equations of lower order and, moveover, it provides regularity results.

The interest in contact problems involving beams lies in the fact that their mathematical analysis may provide insight into the possible types of behaviour of the

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solutions and on occasions leads to decoupling of some of the equations, thus simplifying the approach. Moreover, one may use such models as tests and benchmarks for computer schemes meant for simulation of complicated multidimensional contact problems. Models, analysis and simulations of contact problems for beams can be found in [3, 7, 8, 12] and the references therein.

The aim of this paper is to illustrate the use of the control variational method in the study of various models that describe the equilibrium of an Euler-Bernoulli beam in possible contact with one or two obstacles. First, we consider a mathematical model describing the process of contact of a beam in the presence of two obstacles. We model the contact with normal compliance in such a way that the penetration is limited, which gives one of the traits of novelty of this paper. The Signorini unilateral condition represents a particular case of our contact condition and can be recovered from it. In the variational formulation, the problem leads to an elliptic variational inequality whose unique solvability is obtained by arguments of monotone operators. A second trait of novelty of the paper consists in the fact that, besides the use of the standard arguments above, we analyse the model by using the control variational method. During the analysis we develop arguments that can be useful in the study of various models of contact for beams, both in the elliptic and in the evolutionary case. We provide examples of such kind of models which describe the contact of a beam with a single obstacle, including a time-dependent model with adhesion.

The rest of the paper is structured as follows. In Section 2 we present the model of the contact problem with two obstacles. Then, in Section 3 we list the assumptions on the data, derive the variational formulation and prove an existence and uniqueness result. In Section 4 we analyze the model via the control variational method; this analysis leads us to provide existence, characterization and approximation results. In Section 5 we continue the study with three models describing the contact with a single obstacle. The first two models are time-independent; for them we present results related to our approach, including a regularity poperty. The third model is time-dependent and describes the adhesive contact problem with a deformable obstacle.

2 The model

The physical setting and the process are as follows. An elastic beam occupies in the reference configuration the interval $[0, L]$ of the Ox axis, is clamped at its left end and the right end is free. The beam is acted upon by an applied force of (linear) density $f = f(x)$ where x is the spatial variable. For $x \in [0, L]$, denote by $u = u(x)$ the vertical displacement of the beam and, when the meaning is clear, we do not indicate explicitly the dependence of various variables on x. The beam may arrive in contact with two obstacles S_1 and S_2, situated at a distance $g_1 \leq 0$ and $g_2 \geq 0$ on the Ox axis, respectively. The obstacles are deformable and, therefore, the penetrations are allowed, but are limited. The physical setting is depicted in Fig. 2.1.

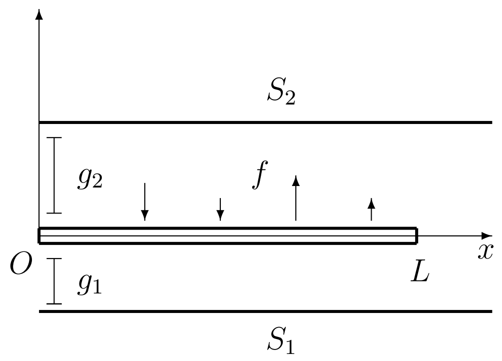

Fig. 2.1. A beam in potential contact with two obstacles.

We use the Euler-Bernoulli model for the beam and we denote $A = EI$, where I is the beam's moment of inertia and E its Young modulus. We have

$$\frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f + \xi \tag{2.1}$$

which is the classical equilibrium equation of the beam, in which ξ represents the contact force.

Next, since the penetration is limited, the vertical displacement satisfies the unilateral constraint

$$k_1 \le u \le k_2, \tag{2.2}$$

where k_1 and k_2 are functions of x which satisfy $k_1 \le g_1$ and $k_2 \ge g_2$. When $g_1 < u < g_2$ then there is no contact between the beam and the obstacles and therefore the contact force vanishes. Thus

$$g_1 < u < g_2 \quad \Longrightarrow \quad \xi = 0. \tag{2.3}$$

When $u \le g_1$ the beam is in contact with the obstacle S_1. In this case the obstacle reacts with a normal force ξ directed upward, $\xi \ge 0$. We assume that the reaction ξ satisfies

$$k_1 < u \le g_1 \implies \xi = -p_1(g_1 - u), \tag{2.4}$$

$$u = k_1 \implies \xi \ge -p_1(g_1 - u), \tag{2.5}$$

where $p_1 : \mathbb{R}_+ \to \mathbb{R}$ is a given nonpositive function. Condition (2.4) shows that when $k_1 < u \le g_1$, then the reaction ξ is uniquely determined by the penetration $g_1 - u$. Also, condition (2.5) implies that when $u = k_1$, then the reaction is not uniquely determined, but is submitted to the restriction $\xi \ge -p_1(u - g_1)$. Such conditions show that the contact follows a normal compliance condition up to the limit k_1 and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap k_1. For this reason we refer to the contact condition (2.4), (2.5) as a normal compliance contact condition with finite penetration and unilateral constraint, and we conclude that the obstacle S_1 has an elastic-rigid behavior.

A similar situation arise when $u > g_2$, i.e. when the beam arrives in contact with the obstacle S_2. In this case the obstacle reacts with a normal force ξ directed downward, $\xi \le 0$. The equilibrium equation is still (2.1) and the reactive normal force ξ satisfies

$$g_2 \le u < k_2 \implies \xi = -p_2(u - g_2), \tag{2.6}$$

$$u = k_2 \implies \xi \le -p_2(u - g_2), \tag{2.7}$$

where $p_2 : \mathbb{R}_+ \to \mathbb{R}$ is a given nonnegative function. Conditions (2.6) and (2.7) show that the contact with S_2 follows a normal compliance condition up to the limit k_2 and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap k_2. We conclude that the obstacle S_2 has an elastic-rigid behavior, too.

Details on the normal compliance contact condition as well as on the Signorini contact condition can be found in [5, 11] and the reference therein. Note that a contact condition similar to that used above, which combine the normal compliance condition and the Signorini condition, was used in [6], in the study of a dynamic frictionless contact problem with elastic-viscoplastic materials.

We restate now conditions (2.3)–(2.7) in a different way, which will be more convenient for the variational analysis of the problem. To this end, we consider the function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(r) = \begin{cases} p_1(g_1 - r) & \text{if } r \le g_1, \\ 0 & \text{if } g_1 < r < g_2, \\ p_2(r - g_2) & \text{if } g_2 \le r. \end{cases} \tag{2.8}$$

It is easy to see that conditions (2.3), (2.4) and (2.6) may be written, in an equivalent form, as follows:

$$k_1 < u < k_2 \implies \xi = -p(u). \tag{2.9}$$

Also, conditions (2.5) and (2.7) may be written as

$$u = k_1 \implies \xi + p(u) \ge 0, \tag{2.10}$$

$$u = k_2 \implies \xi + p(u) \le 0, \tag{2.11}$$

respectively. Finally, since the beam is rigidly attached at its left we impose the condition

$$u(0) = \frac{du}{dx}(0) = 0 \tag{2.12}$$

and, since no moments act on the free end of the beam, we have

$$\frac{d^2u}{dx^2}(L) = \frac{d^3u}{dx^3}(L) = 0. \tag{2.13}$$

We collect the equations and conditions above to obtain the classical formulation of the contact problem.

Problem P**.** *Find a displacement field* $u : [0, L] \to \mathbb{R}$ *which satisfies conditions* (2.1), (2.2), (2.9)–(2.11) *in* $(0, L)$, *together with the boundary conditions* (2.12) *and* (2.13).

The variational analysis of the contact problem P will be provided in Sections 3 and 4 by using arguments of variational inequalities and the control variational method, respectively.

3 Existence and uniqueness

We turn now to derive a weak or variational formulation of Problem P. To this end we assume in what follows that

$$A \in L^\infty(0, L), \quad \exists\, m > 0 \text{ such that } A(x) \geq m \ \text{ a.e. } x \in (0, L), \tag{3.1}$$
$$f \in L^2(0, L), \tag{3.2}$$
$$g_1 \leq 0, \quad g_2 \geq 0, \tag{3.3}$$
$$k_1 \in L^2(0, L), \quad k_1(x) \leq g_1 \ \text{ a.e. } x \in (0, L), \tag{3.4}$$
$$k_2 \in L^2(0, L), \quad k_2(x) \geq g_2 \ \text{ a.e. } x \in (0, L). \tag{3.5}$$

Also, the normal compliance functions $p_1 : \mathbb{R}_+ \to \mathbb{R}$ and $p_2 : \mathbb{R}_+ \to \mathbb{R}$ satisfy

$$\left\{ \begin{array}{l} \text{(a) There exists } L_1 > 0 \text{ such that} \\ \qquad |p_1(r) - p_1(s)| \leq L_1\, |r - s| \quad \forall\, r,\, r \in \mathbb{R}_+. \\ \text{(b) } (p_1(r) - p_1(s)) \cdot (r - s) \leq 0 \quad \forall\, r,\, s \in \mathbb{R}_+. \\ \text{(c) } p_1(r) \leq 0 \quad \forall\, r \geq 0 \quad \text{and} \quad p_1(0) = 0. \end{array} \right. \tag{3.6}$$

$$\left\{ \begin{array}{l} \text{(a) There exists } L_2 > 0 \text{ such that} \\ \qquad |p_2(r) - p_2(s)| \leq L_2\, |r - s| \quad \forall\, r,\, r \in \mathbb{R}_+. \\ \text{(b) } (p_2(r) - p_2(s)) \cdot (r - s) \geq 0 \quad \forall\, r,\, s \in \mathbb{R}_+. \\ \text{(c) } p_2(r) \geq 0 \quad \forall\, r \geq 0 \quad \text{and} \quad p_2(0) = 0. \end{array} \right. \tag{3.7}$$

We remark that the assumptions (3.6) and (3.7) on $p_1(\cdot)$ and $p_2(\cdot)$ are fairly general. The main severe restriction comes from condition (a) which requires that the functions are Lipschitz continuous. From mechanical point of view, condition (b) express the fact that the magnitude of the reaction force increases with the penetration. One standard example is provided by the functions

$$p_1(r) = -\mu_1\, r_+, \qquad p_2(r) = \mu_2\, r_+$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are stiffness coefficients and r_+ denotes the positive part of r, i.e. $r_+ = max\,\{0, r\}$. Clearly these functions satisfy assumptions (3.6) and (3.7), respectively.

It is easy to see that, under assumptions (3.6) and (3.7), the function p defined by (2.8) is a Lipschitz continuous monotone function which vanishes at the origin, i.e. it satisfies

$$\begin{cases} \text{(a) There exists } \widetilde{L} > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \leq \widetilde{L}\,|r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(b) } (p(r_1) - p(r_2)) \cdot (r_1 - r_2) \geq 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(b) } p(0) = 0. \end{cases} \tag{3.8}$$

Here, the Lipschitz constant $\widetilde{L}$ is given by $\widetilde{L} = \max\{L_1, L_2\}$, where L_1 and L_2 are the Lipschitz constants of the functions p_1 and p_2, respectively.

In what follows we use standard notation for L^p and Sobolev spaces and the subscripts x and xx will represent the first and the second derivatives with respect to x, respectively. We introduce the closed subspace of $H^2(0, L)$ given by

$$V = \{\ v \in H^2(0, L)\ :\ v(0) = v_x(0) = 0\ \}, \tag{3.9}$$

and, below, we denote by 0_V the zero element of V. We note that there exists $c > 0$ such that $\|v\|_{L^2(0,L)} \leq c\,\|v_x\|_{L^2(0,L)}$ for all $v \in H^1(0, L)$ satisfying $v(0) = 0$, thus,

$$\|v\|_{H^2(0,L)} \leq c\,\|v_{xx}\|_{L^2(0,L)} \qquad \forall\, v \in V. \tag{3.10}$$

We consider now the inner product on V given by

$$(u, v)_V = (u_{xx}, v_{xx})_{L^2(0,L)}, \tag{3.11}$$

and let $\|\cdot\|_V$ be the associated norm. Using (3.10) we find that $\|\cdot\|_{H^2(0,L)}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, (\cdot,\cdot)_V)$ is a real Hilbert space.

In addition, we consider the bilinear form $a : V \times V \to \mathbb{R}$, the functional $j : V \times V \to \mathbb{R}$, and the set of admissible displacement fields K, defined by

$$a(u, v) = \int_0^L A\, u_{xx} v_{xx}\, dx \qquad \forall\, u,\, v \in V, \tag{3.12}$$

$$j(u, v) = \int_0^L p(u)\, v\, dx \qquad \forall\, u,\, v \in V, \tag{3.13}$$

$$K = \{\ v \in V\ :\ k_1 \leq v \leq k_2 \quad \text{in} \quad (0, L)\ \}. \tag{3.14}$$

We note that by (3.1) and (3.8) it follows that the integrals in (3.12) and (3.13) are well-defined; moreover, by conditions (3.4) and (3.5) it follows that K is nonempty since, for instance,

$$0_V \in K. \tag{3.15}$$

Assume now that u is a regular solution of Problem P. Then (2.2) implies that $u \in K$. Let v be an arbitrary element in K. It follows from (2.1) that

$$\int_0^L \frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big)(v - u)\, dx = \int_0^L f\,(v - u)\, dx + \int_0^L \xi\,(v - u)\, dx$$

and, performing two integrations by parts and using the boundary conditions (2.12), (2.13) yields

$$\int_0^L Au_{xx}(v_{xx} - u_{xx})\,dx = \int_0^L f\,(v-u)\,dx + \int_0^L \xi\,(v-u)\,dx. \tag{3.16}$$

On the other hand, using (2.9)–(2.11) and the definition of K we deduce that

$$\xi(v-u) \geq -p(u)(v-u) \quad \text{in} \quad (0, L),$$

which implies that

$$\int_0^L \xi\,(v-u)\,dx \geq -\int_0^L p(u)(v-u)\,dx. \tag{3.17}$$

We combine now (3.16) and (3.17), then we use notation (3.12) and (3.13) to obtain the following variational formulation of Problem P.

Problem P_V**.** *Find a displacement field* u *such that*

$$u \in K, \qquad a(u, v-u) + j(u, v-u) \geq (f, v-u)_{L^2(0,L)} \qquad \forall\, v \in K. \tag{3.18}$$

We have the following existence and uniqueness result, which provides the unique weak solvability of the contact problem P.

Theorem 1. *Assume that* (3.1)–(3.7) *hold. Then there exists a unique solution* $u^* \in V$ *to the variational problem* P_V*.*

Proof. Let $P : V \to V$ be the operator given by

$$(Pu, v)_V = a(u, v) + j(u, v) \qquad \forall\, v \in V. \tag{3.19}$$

We use assumption (3.1), definition (3.11) and the properties (3.8) of the function p to see that P is a strongly monotone Lipschitz continuous operator on V. Moreover, by (3.2) it follows that there exists a unique element $\tilde{f} \in V$ such that

$$(\tilde{f}, v)_V = (f, v)_{L^2(0,L)} \qquad \forall\, v \in V. \tag{3.20}$$

Also, from assumptions (3.4) and (3.5) it follows that K is a nonempty closed convex subset of V. We use now a standard result on elliptic variational inequality to see that there exists a unique element $u^* \in V$ such that

$$u^* \in K, \qquad (Pu^*, v - u^*)_V \geq (\tilde{f}, v - u^*)_V \qquad \forall\, v \in K. \tag{3.21}$$

Theorem 1 is now a consequence of (3.19)–(3.21). ■

4 Analysis via the control variational method

In this section we indicate an alternative approach to the problem P_V, based on optimal control arguments. Everywhere below we assume that (3.1)–(3.7) hold. We denote by l the inverse of A, i.e. $l = A^{-1}$, and note that, by condition (3.1), it follows that $l \in L^\infty(0,L)$. Also, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a primitive of p, $\varphi' = p$, and let $z \in H^2(0,L)$ be the solution of the problem

$$\frac{d^2 z}{dx^2} = f \quad \text{in} \quad (0,L), \qquad z(L) = \frac{dz}{dx}(L) = 0. \tag{4.1}$$

We start by introducing the following optimal control problem:

$$\min_{[u,h]\in K\times L^2(0,L)} \Big\{ \frac{1}{2}\int_0^L l h^2\,dx + \int_0^L \varphi(u)\,dx \Big\}, \tag{4.2}$$

$$u_{xx} = lz + lh \quad \text{in} \quad (0,L). \tag{4.3}$$

The solvability of the optimal problem (4.2)–(4.3) and its link with the variational problem P_V is given by the following result.

Theorem 2. *Assume that* (3.1)–(3.7) *hold. Then, problem* (4.2)–(4.3) *has a unique optimal pair* $[u^*, h^*] \in K \times L^2(0,L)$. *Moreover,* u^* *satisfies the variational inequality* (3.18).

Proof. Note that the cost functional in (4.2) is strictly convex (as $\varphi' = p$ and p is monotone, see (3.8)(b)) and coercive in h. Therefore, the existence and the uniqueness of the optimal pair $[u^*, h^*] \in K \times L^2(0,L)$ follows from standard arguments.

We consider now admissible variations of the form $u^* + \lambda(v - u^*)$ and $h^* + \lambda(k - h^*)$ in which λ is an arbitrary element of $[0,1]$, v is an arbitrary element of K, and $k \in L^2(0,L)$ is such that

$$v_{xx} = lz + lk \quad \text{in} \quad (0,L). \tag{4.4}$$

Since $[u^*, h^*]$ is the optimal pair of the problem (4.2)–(4.3), it follows that

$$\frac{1}{2}\int_0^L l(h^*)^2\,dx + \int_0^L \varphi(u^*)\,dx$$
$$\le \frac{1}{2}\int_0^L l(h^* + \lambda(k-h^*))^2\,dx + \int_0^L \varphi(u^* + \lambda(v-u^*))\,dx.$$

We divide this inequality by $\lambda > 0$ and then we pass to the limit as $\lambda \to 0$ to obtain

$$0 \le \int_0^L l h^*(k-h^*)\,dx + \int_0^L p(u^*)(v-u^*)\,dx.$$

Next, we replace h^* and k by using (4.3) and (4.4), respectively, and operate integrations by part in the resulting inequality, by using (4.1). As a result we obtain

$$\begin{aligned}0 &\le \int_0^L (Au^*_{xx} - z)(v_{xx} - u^*_{xx})\,dx + \int_0^L p(u^*)(v-u^*)\,dx\\ &= \int_0^L Au^*_{xx}(v_{xx} - u^*_{xx})\,dx + \int_0^L p(u^*)(v-u^*)\,dx - \int_0^L f(v-u^*)\,dx\end{aligned}$$

and, using the notation (3.12) and (3.13) it follows that

$$a(u^*, v-u^*) + j(u^*, v-u^*) \ge (f, v-u^*)_{L^2(0,L)},$$

which concludes the proof. ■

A simple calculation implies that

$$\begin{aligned}\frac{1}{2}\int_0^L lh^2\,dx + \int_0^L \varphi(u)\,dx &= \frac{1}{2}\int_0^L A(u_{xx} - lz)^2\,dx + \int_0^L \varphi(u)\,dx\\ &= \frac{1}{2}\int_0^L A(u_{xx})^2\,dx + \int_0^L \varphi(u)\,dx - \int_0^L fu\,dx + \frac{1}{2}\int_0^L lz^2\,dx\end{aligned}$$

which shows that, up to a constant (provided by the last term), the cost functional in (4.2) represents the usual energy associated to Problem P_V. We conclude from here that the classical variational approach is a special case of the control variational method presented above.

Next, we approximate the control problem with the optimization problem

$$\min_{[u,h]\in K\times L^2(0,L)} \Big\{\frac{1}{2}\int_0^L lh^2\,dx + \int_0^L \varphi(u)\,dx + \frac{1}{2\varepsilon}\int_0^L (u_{xx} - lh - lz)^2\,dx\Big\}, \tag{4.5}$$

in which $\varepsilon > 0$. We have the following existence, uniqueness and convergence result.

Theorem 3. *Assume that* (3.1)–(3.7) *hold. Then, for each $\varepsilon > 0$, the problem* (4.5) *has a unique minimizer $[u_\varepsilon, h_\varepsilon] \in K \times L^2(0,L)$. Moreover, as $\varepsilon \to 0$, the following convergences hold:*

$$u_\varepsilon \to u^* \quad \text{strongly in } V, \qquad\qquad h_\varepsilon \to h^* \quad \text{strongly in } L^2(0,L).$$

Proof. Let $\varepsilon > 0$. The existence of a unique minimizer $[u_\varepsilon, h_\varepsilon]$ for (4.5) follows from arguments similar to those used in Theorem 2. Clearly $[u^*, h^*]$ is admissible for (4.5) and, moreover, we have the inequality

$$\begin{aligned}&\frac{1}{2}\int_0^L lh_\varepsilon^2\,dx + \int_0^L \varphi(u_\varepsilon)\,dx\\ &\quad+\frac{1}{2\varepsilon}\int_0^L ((u_\varepsilon)_{xx} - lh_\varepsilon - lz)^2\,dx \le \frac{1}{2}\int_0^L l(h^*)^2\,dx + \int_0^L \varphi(u^*)\,dx.\end{aligned} \tag{4.6}$$

Denote by r_ε the function defined by

$$r_\varepsilon = \frac{1}{\varepsilon}\big((u_\varepsilon)_{xx} - lh_\varepsilon - lz\big) \tag{4.7}$$

and note that this equality implies that

$$(u_\varepsilon)_{xx} = lh_\varepsilon + lz + \varepsilon r_\varepsilon. \tag{4.8}$$

Recall also that φ is bounded from below by an affine functional. Then, using (4.6)–(4.8) it follows that

$$\{h_\varepsilon\} \qquad \text{is a bounded sequence in } L^2(0,L), \tag{4.9}$$

$$\{\varepsilon^{\frac{1}{2}} r_\varepsilon\} \qquad \text{is a bounded sequence in } L^2(0,L), \tag{4.10}$$

$$\{u_\varepsilon\} \qquad \text{is a bounded sequence in } V, \tag{4.11}$$

$$\varepsilon r_\varepsilon \to 0 \qquad \text{in } L^2(0,L), \tag{4.12}$$

as $\varepsilon \to 0$. It follows from (4.9) and (4.11) that there exists a pair $[\hat{h}, \hat{u}] \in L^2(0,L)\times V$ such that, passing to subsequences, again denoted $\{h_\varepsilon\}$ and $\{u_\varepsilon\}$, we have

$$h_\varepsilon \to \hat{h} \quad \text{weakly in } L^2(0,L), \tag{4.13}$$

$$u_\varepsilon \to \hat{u} \quad \text{weakly in } V. \tag{4.14}$$

Inequality (4.6), the convergences (4.13), (4.14) and the weak lower semicontinuity of convex functions yield

$$\frac{1}{2}\int_0^L l(\hat{h})^2\,dx + \int_0^L \varphi(\hat{u})\,dx \le \frac{1}{2}\int_0^L l(h^*)^2\,dx + \int_0^L \varphi(u^*)\,dx. \tag{4.15}$$

Moreover, by (4.12) it follows that the pair $[\hat{u}, \hat{h}]$ satisfies (4.3). Therefore, $[\hat{u}, \hat{h}]$ is an admissible pair for the control problem (4.2), (4.3) and, in addition, (4.15) shows that it is optimal. The uniqueness of the optimal control, guaranteed by Theorem 2, implies that $\hat{h} = h^*$ and $\hat{u} = u^*$. Thus, by (4.13) and (4.14) and lower semicontinuity arguments we have

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\int_0^L l(h_\varepsilon)^2\,dx \ge \frac{1}{2}\int_0^L l(h^*)^2\,dx,$$

$$\liminf_{\varepsilon\to 0} \int_0^L \varphi(u_\varepsilon)\,dx \ge \int_0^L \varphi(u^*)\,dx$$

and, taking into account (4.15), we obtain that

$$\frac{1}{2}\int_0^L l(h_\varepsilon)^2\,dx \to \frac{1}{2}\int_0^L l(h^*)^2\,dx \qquad \text{as } \varepsilon \to 0. \tag{4.16}$$

A well known convergence criterion in Hilbert spaces combined with (4.16), (4.13) and equality $\hat{h} = h^*$ implies that $h_\varepsilon \to h^*$ strongly in $L^2(0,L)$ as $\varepsilon \to 0$. We use now (4.8), (4.12) and equality $u^*_{xx} = lh^* + lz$ to obtain that that $u_\varepsilon \to u^*$ strongly in V as $\varepsilon \to 0$. Since the limit is unique, we deduce that the strong convergences above are valid for the whole sequences $\{h_\varepsilon\}$ and $\{u_\varepsilon\}$, which concludes the proof. ■

A characterization of the optimal pair of the control problem (4.2), (4.3) is provided by the following result.

Theorem 4. *Assume that* (3.1)–(3.7) *hold. Then, the element* $[u^*, h^*] \in K \times L^2(0,L)$ *is the optimal pair of the control problem* (4.2), (4.3) *if and only if there exists* $r^* \in L^2(0,L)$ *such that*

$$0 \le \int_0^L lh^*(k-h^*)\,dx + \int_0^L p(u^*)(v-u^*)\,dx + \int_0^L r^*(v_{xx} - lk - lz)\,dx \qquad (4.17)$$

for all $k \in L^2(0,L)$ *and* $v \in K$.

Proof. We take admissible variations of the form $u_\varepsilon + \lambda(v-u_\varepsilon)$, $h_\varepsilon + \lambda(k-h_\varepsilon)$, where λ is an arbitrary element of $[0,1]$, v belongs to K and $k \in L^2(0,L)$, and obtain

$$\begin{aligned}\frac{1}{2}\int_0^L lh_\varepsilon^2\,dx &+ \int_0^L \varphi(u_\varepsilon)\,dx + \frac{1}{2\varepsilon}\int_0^L ((u_\varepsilon)_{xx} - lh_\varepsilon - lz)^2\,dx \\ &\le \frac{1}{2}\int_0^L l[h_\varepsilon + \lambda(k-h_\varepsilon)]^2\,dx + \int_0^L \varphi(u_\varepsilon + \lambda(v-u_\varepsilon))\,dx \\ &\quad + \frac{1}{2\varepsilon}\int_0^L \big((u_\varepsilon)_{xx} + \lambda(v-u_\varepsilon) - l(h_\varepsilon + \lambda(k-h_\varepsilon) - lz\big)^2\,dx.\end{aligned}$$

Then, we divide the previous inequality by $\lambda > 0$ and pass to the limit as $\lambda \to 0$. As a result we find that

$$0 \le \int_0^L lh_\varepsilon(k-h_\varepsilon)\,dx + \int_0^L p(u_\varepsilon)(v-u_\varepsilon)\,dx + \int_0^L r_\varepsilon(v_{xx} - (u_\varepsilon)_{xx} - lk + lh_\varepsilon)\,dx. \quad (4.18)$$

We use (4.18), (4.7) and inequality $-\varepsilon|r_\varepsilon|^2 \le 0$ to deduce that

$$0 \le \int_0^L lh_\varepsilon(k-h_\varepsilon)\,dx + \int_0^L p(u_\varepsilon)(v-u_\varepsilon)\,dx + \int_0^L r_\varepsilon(v_{xx} - lk - lz)\,dx. \qquad (4.19)$$

Now, we use (3.15) and test in (4.19) with $v = 0_V$ and $k = -z + w$, where w is an arbitrary element in $L^2(0,L)$ which satisfies $|w|_{L^2(0,L)} \le 1$. We infer that

$$0 \le \int_0^L lh_\varepsilon(w - z - h_\varepsilon)\,dx + \int_0^L p(u_\varepsilon)u_\varepsilon\,dx - \int_0^L r_\varepsilon lw\,dx.$$

As the first two terms are bounded for $\varepsilon > 0$, Theorem 3 and the continuity of $p(\cdot)$ yield that the sequence $\{r_\varepsilon\}$ is bounded in $L^2(0,L)$ as w is arbitrary in the unit ball of $L^2(0,L)$. Then, there exists an element $r^* \in L^2(0,L)$ and a subsequence of the sequence $\{r_\varepsilon\}$, again denoted $\{r_\varepsilon\}$, which converge weakly to r^* in $L^2(0,L)$.

Next, we pass to the limit in (4.19) and use Theorem 3 combined with the continuity of $p(\cdot)$ in order to obtain the necessity of (4.17). The sufficiency of this condition follows by choosing $[v,k]$ admissible for the optimal control problem (4.2), (4.3). Then, we have

$$0 \le \int_0^L lh^*(k-h^*)\,dx + \int_0^L p(u^*)(v-u^*)\,dx$$

and, using the definition of the subdifferential $\partial\varphi(u^*)$, we conclude the proof. ■

The inequality (4.19) gives the first order optimality condition for the approximating optimization problem (4.5) and this condition is necessary and sufficient, too. Note that the function $r^* \in L^2(0,L)$ is the Lagrange multiplier associated to the state equation (4.3). Similar arguments have been used in [2] in order to derive first order optimality conditions in the study of abstract control problems for evolution equations. Note also that conditions (4.17) or (4.19) express the fact that the gradient of the minimized functional (4.5) (or (4.2)), is zero (or has a certain orientation with respect to the constraints) at the minimizer. This information may be exploited in gradient algorithms for the corresponding minimization problems. We conclude that the control variational method provides new possibilities for constructing approximation methods for boundary value problems.

5 Versions of the model and related results

In this section we present three versions of the model and state results similar to those presented in the previous section, obtained by using the control variational method. The physical setting is similar to that described in Section 2. The difference arise from the fact that now we have a single obstacle, denoted S, situated at a distance $g \le 0$ from the Ox axis, as shown in Fig. 5.1. Formally, this physical setting can be recovered from the physical setting depicted in Fig. 2.1 by taking $S_1 = S$, $g_1 = g$, $g_2 = \infty$.

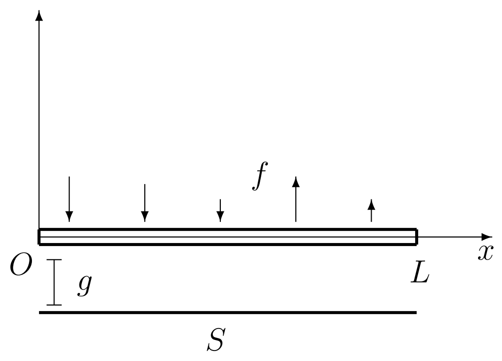

Fig. 5.1. A beam in potential contact with a single obstacle.

The first two models are time-independent. In the first one we assume that the contact is modeled with the classical normal compliance condition, i.e. the penetration is not limited. Therefore, the classical formulation of the problem is the following.

Problem P^{nc}**.** *Find a displacement field* $u : [0, L] \to \mathbb{R}$ *such that*

$$\frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f + \xi \quad \text{in} \quad (0, L), \tag{5.1}$$

$$u > g \Longrightarrow \xi = 0, \quad u \le g \Longrightarrow \xi = -p_1(g - u) \quad \text{in} \quad (0, L), \tag{5.2}$$

$$u(0) = \frac{du}{dx}(0) = 0, \tag{5.3}$$

$$\frac{d^2u}{dx^2}(L) = \frac{d^3u}{dx^3}(L) = 0. \tag{5.4}$$

Here, $p_1 : \mathbb{R}_+ \to \mathbb{R}$ is a given nonpositive function which describes the reaction of the obstacle.

In the second model we assume that the penetration is not allowed and, therefore, the contact is described with the Signorini condition. The classical formulation of the problem is the following.

Problem P^{Sig}**.** *Find a displacement field* $u : [0, L] \to \mathbb{R}$ *such that*

$$\frac{d^2}{dx^2}\Big(A\,\frac{d^2u}{dx^2}\Big) = f + \xi \quad \text{in} \quad (0, L), \tag{5.5}$$

$$u \ge g, \quad \xi \ge 0, \quad \xi(g - u) = 0 \quad \text{in} \quad (0, L), \tag{5.6}$$

$$u(0) = \frac{du}{dx}(0) = 0, \tag{5.7}$$

$$\frac{d^2u}{dx^2}(L) = \frac{d^3u}{dx^3}(L) = 0. \tag{5.8}$$

We turn to the variational formulation to Problems P^{nc} and P^{Sig}. To this end we assume that A and f satisfy (3.1) and (3.2), respectively, and p_1 satisfies (3.6). Also, we consider the function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(r) = \begin{cases} p_1(g - r) & \text{if } r \le g, \\ 0 & \text{if } r > g. \end{cases} \tag{5.9}$$

We use the space V, (3.9), the bilinear form (3.12) and the function (3.13) where p is now given by (5.9). Also, we consider the set

$$K = \{\, v \in V \ :\ v \ge g \quad \text{in} \ \ (0, L) \,\}. \tag{5.10}$$

Then, the variational formulation of problems P^{nc} and P^{Sig} can be obtained by using the arguments presented in Section 3 and are the following.

Problem P_V^{nc}. *Find a displacement field u such that*

$$u \in V, \qquad a(u,v) + j(u,v) = (f,v)_{L^2(0,L)} \qquad \forall\, v \in V. \tag{5.11}$$

Problem P_V^{Sig}. *Find a displacement field u such that*

$$u \in K, \qquad a(u,v-u) \geq (f,v-u)_{L^2(0,L)} \qquad \forall\, v \in K. \tag{5.12}$$

Note that Problem P_V^{nc} can be recovered from Problem P_V by taking $g_1 = g$, $k_1 = -\infty$, $g_2 = k_2 = \infty$ and $p_2 \equiv 0$. Also, Problem P_V^{Sig} can be recovered from Problem P_V by taking $g_1 = k_1 = g$, $g_2 = k_2 = \infty$ and $p_2 \equiv 0$. Therefore, we conclude that the control variational method in Section 4 works and can be used to provide the existence of the solutions of problems P_V^{nc} and Problem P_V^{Sig}, respectively. In particular, in the study of Problem P_V^{nc} we introduce the optimal control problem

$$\min_{[u,h]\in V\times L^2(0,L)} \Big\{ \frac{1}{2}\int_0^L lh^2\,dx + \int_0^L \varphi(u)\,dx \Big\}, \tag{5.13}$$

$$u_{xx} = lz + lh \quad \text{in} \quad (0,L), \tag{5.14}$$

$$u(0) = u_x(0) = 0. \tag{5.15}$$

Here, again, $l = A^{-1}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is such that $\varphi' = p$ and $z \in H^2(0,L)$ is the solution of the problem

$$\frac{d^2z}{dx^2} = f \quad \text{in} \quad (0,L), \qquad z(L) = \frac{dz}{dx}(L) = 0. \tag{5.16}$$

The solvability of the optimal problem (5.13)–(5.15) and its link with the variational problem P_V^{nc} is given by the following result.

Theorem 5. *Assume that* (3.1), (3.2) *and* (3.6) *hold and let $g \leq 0$. Then, problem* (5.13)–(5.15) *has a unique optimal pair $[u^*,h^*] \in H^2(0,L)\times L^2(0,L)$ and, moreover, u^* satisfies* (5.11).

The interest in the optimal control method also arises from the fact that it provides regularity results. To illustrate this, we turn again to the optimal control problem (5.13)$-$(5.15). We introduce the adjoint system and the adjoint state $r \in H^2(0,L)$ given by

$$r_{xx} = -p(u^*) \quad \text{in} \ \ (0,L), \tag{5.17}$$

$$r(L) = r_x(L) = 0. \tag{5.18}$$

Performing variations around the optimal pair (as in the proof of Theorem 2) and some integration by parts, we find that

$$r + h^* = 0 \quad \text{in} \ \ (0,L). \tag{5.19}$$

Relation (5.19) expresses the fact that the gradient of the cost functional (5.13) (as a function of h alone) is zero in the minimum point $h^* \in L^2(0, L)$. The left-hand side of (5.19) represents this gradient and can be used in the iterative procedures (the gradient methods) for the solution of (5.13)–(5.15). By Theorem 5 we see that the optimal control method above provides an alternative solution method for the original problem (5.1)–(5.4), involving just the equations (5.14) and (5.17), that may be integrated directly.

We also use relation (5.19) to note that h^* has the same regularity as r, i.e. $h^* \in H^3(0, L)$. Here, we use the fact that $p(\cdot)$ is a Lipschitz continuous function and the regularity of u^*, r from the state, respectively the adjoint equation. If l is smooth enough, by (5.14) we also obtain that $u^* \in H^5(0, L)$, which represents a regularity property for the solution of Problem P_V^{nc}.

We now turn to the analysis of Problem P_V^{Sig} and, to this end, we introduce the optimal control problem

$$\min_{h\in L^2(0,L)} \Big\{ \int_0^L l h^2 dx \Big\}, \tag{5.20}$$

subjected to (5.14), (5.15) and to the constraint

$$u \geq g \quad \text{in} \;\; (0, L). \tag{5.21}$$

The solvability of this optimal problem and its link with the variational problem P_V^{Sig} is given by the following result.

Theorem 6. *Assume that* (3.1) *and* (3.2) *hold and let* $g \leq 0$. *Then, problem* (5.20), (5.14)*,* (5.15)*,* (5.21) *has a unique optimal pair* $[\hat{u}, \hat{h}] \in K \times L^2(0, L)$ *and, moreover,* $\hat{u}$ *satisfies* (5.12)*.*

More details in the study of the contact problems P^{nc} and P^{Sig}, including complete proofs of Theorems 5 and 6 and can be found in [14].

We end this section with the description of a time-dependent model for contact with a single obstacle. The physical setting is similar to that depicted in Fig. 5.1. The difference arises from the fact that now there is no gap between the beam and the foundation i.e. $g = 0$ and, moreover, the adhesion of the contact surfaces is taken into account and is modelled by a time-dependent variable, the bonding field.

Let $[0, T]$ be the time interval of interest, $T > 0$. For $x \in [0, L]$ and $t \in [0, T]$, we denote by $u = u(x, t)$ the vertical displacement of the beam, by $\beta(x, t)$ the bonding field and, below, we indicate the dependence of the variables on x and t. Following [4], the bonding field describes the fractional density of active bonds on the contact surface and, as a fraction, its values are restricted to $0 \leq \beta(x, t) \leq 1$. When $\beta(x, t) = 1$ the adhesion is complete and all the bonds are active; when $\beta(x, t) = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta(x, t) < 1$ the adhesion is partial and only a fraction $\beta(x, t)$ of the bonds is active. We refer the reader to the extensive bibliography on the modelling and analysis of contact problems with adhesion in [4, 10, 11, 13].

The classical formulation of the problem is as follows.

Problem P^a**.** *Find a displacement field* $u : [0, L] \times [0, T] \to \mathbb{R}$ *such that, for all* $t \in [0, T]$,

$$\frac{\partial^2}{\partial x^2}\Big(A(x)\,\frac{d^2u}{dx^2}(x,t)\Big) = f(x,t) + \xi(x,t) \quad \text{for all } x \in (0, L), \tag{5.22}$$

$$\left.\begin{array}{l} u(x,t) > 0 \Longrightarrow \xi(x,t) = -\gamma(x)\beta^2(x,t)R(u(x,t)) \\ u(x,t) \le 0 \Longrightarrow \xi(x,t) = -p_1(-u(x,t)) \end{array}\right\} \text{ for all } x \in (0, L), \tag{5.23}$$

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0, \tag{5.24}$$

$$\frac{\partial^2 u}{\partial x^2}(L,t) = \frac{\partial^3 u}{\partial x^3}(L,t) = 0. \tag{5.25}$$

Condition (5.23) represents the normal compliance condition with adhesion. Here p_1 is the normal compliance function, γ is a positive function and R is the truncation operator given by

$$R(r) = \begin{cases} 0 & \text{if } \ r < 0, \\ r & \text{if } \ 0 \le r \le l_0, \\ l_0 & \text{if } \ r > l_0. \end{cases} \tag{5.26}$$

where $l_0 > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [10]). The introduction of the truncation operator R is motivated mainly by mathematical reasons, but it is also related to the observation that for some glues when the extension is more than l_0, the glue extends plastically without offering additional tensile traction. However, by choosing l_0 very large, we recover the case where the traction is linear in the extension. We note that when there is contact (i.e. when $u(x,t) \le 0$), then condition (5.23) is similar to condition (5.2). Nevertheless, when there is separation (i.e. when $u(x,t) > 0$), then (5.23) shows that the adhesive normal traction in the point x at the moment t is $\gamma(x)\beta^2(x,t)R(u(x,t))$; it is tensile and proportional, with proportionality coefficient $\gamma(x)$, to the square of the adhesion field, and to the vertical displacement, as long as it does not exceed the bond length l_0.

In [4, 10, 11, 13] the evolution of the bonding field is described by the equation

$$\frac{\partial \beta}{\partial t}(x,t) = -\big(\gamma(x)\beta(x,t)R^2(u(x,t)) - \epsilon(x)\big)_+ \quad \text{for all} \quad (x,t) \in (0,L) \times (0,T)$$

in which ϵ is a positive coefficient and, again, r_+ denotes the positive part of r. Nevertheless, an examination of the results in these references shows that the study of contact problems with adhesion is usually carried out by considering intermediate problems in which the bonding field is known, followed by a fixed point argument. For this reason, in Problem P^a we consider β as given and note that this restrictive assumption leads to a simplified contact model with adhesion. Its analysis via the control variational method has some interest in its own, since it lies the background for more complicate problems in which the bonding field is unknown.

We turn now to derive a weak or variational formulation of Problem P^a. To this end we assume in what follows that A satisfies (3.1), p_1 satisfies (3.6) and :

$$f \in W^{1,\infty}(0,T;L^2(0,L)), \tag{5.27}$$

$$\gamma \in L^\infty(0,L), \quad \gamma(x) \geq 0 \ \text{ a.e. } x \in (0,L), \tag{5.28}$$

$$\beta \in W^{1,\infty}(0,T;L^2(0,L)), \tag{5.29}$$

$$0 \leq \beta(x,t) \leq 1 \ \text{ a.e. } x \in (0,L), \quad \forall\, t \in [0,T]. \tag{5.30}$$

In what follows we use the space (3.9) with the inner product (3.11) and the associated norm $\|\cdot\|_V$. In addition, we consider the bilinear form $a : V \times V \to \mathbb{R}$ defined by (3.12), the function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(r) = \begin{cases} p_1(-r) & \text{if } r \leq 0, \\ 0 & \text{if } r > 0, \end{cases} \tag{5.31}$$

and the functional $j : [0,T] \times V \times V \to \mathbb{R}$ defined by

$$j(t,u,v) = \int_0^L \big(p(u(x)) + \gamma(x)\beta^2(x,t)R(u(x))\big)\, v(x)\, dx \qquad \forall\, u,\, v \in V.$$

Using (5.26) and (5.31) it is easy to see that the contact condition (5.23) can be written into the equivalent form

$$\xi(x,t) = -p(u(x,t)) - \gamma(x)\beta^2(x,t)R(u(x,t)) \quad \text{for all } x \in (0,T) \text{ and } t \in [0,T].$$

Therefore, by a standard procedure based on two integrations by parts, we obtain the variational formulation of Problem P^a.

Problem P_V^a. *Find a displacement field $u : [0,T] \to V$ such that*

$$a(u(t),v) + j(t,u(t),v) = (f(t),v)_{L^2(0,L)} \quad \forall\, v \in V,\ t \in [0,T]. \tag{5.32}$$

For each $t \in [0,T]$, let us introduce the optimal control problem

$$\min_{[u(t),h(t)] \in V \times L^2(0,L)} \Big\{ \frac{1}{2}\int_0^L l(x)h^2(x,t)\, dx + \int_0^L \varphi(x,t,u(x,t))\, dx \Big\}, \tag{5.33}$$

$$u_{xx}(x,t) = l(x)z(x,t) + l(x)h(x,t) \quad \text{for all } x \in (0,L), \tag{5.34}$$

$$u(0,t) = u_x(0,t) = 0. \tag{5.35}$$

Here, $l(x) = A^{-1}(x)$ and $\varphi(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is such that

$$\frac{\partial \varphi}{\partial r}(x,t,r) = p(r) + \gamma(x)\beta^2(x,t)R(r)$$

for all $x \in (0, L)$ and $r \in \mathbb{R}$. Moreover, $z(\cdot, t) \in H^2(0, L)$ is the solution of the problem

$$\frac{\partial^2 z(x,t)}{\partial x^2} = f(x,t) \quad \text{for all } x \in (0,L), \qquad z(L,t) = \frac{\partial z}{\partial x}(L,t) = 0. \tag{5.36}$$

The solvability of the optimal problem (5.33)–(5.35) and its link with the variational problem P_V^a is given by the following result.

Theorem 7. *Assume that* (3.1), (3.6), (5.27)–(5.30) *hold. Then, for each* $t \in [0, T]$, *problem* (5.33)–(5.35) *has a unique optimal pair* $[\widetilde{u}(t), \widetilde{h}(t)] \in H^2(0, L) \times L^2(0, L)$. *The functions* $\widetilde{u}$ *and* $\widetilde{h}$ *belong to* $W^{1,\infty}(0, T; V)$ *and* $W^{1,\infty}(0, T; L^2(0, L))$, *respectively, and, moreover,* $\widetilde{u}$ *satisfies* (5.32).

The proof of Theorem 7 follows from results similar to those used in Section 4 and, for this reason, is omitted.

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