INVARIANTS OF STABLY TRIVIAL VECTOR BUNDLES WITH CONNECTION

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ABSTRACT. We define a Chern–Simons invariant of connections on stably trivial vector bundles over smooth manifolds, taking values in 3-forms modulo closed forms with integral cohomology class. We show an additivity property of this invariant for connections defined on a direct sum of bundles, under a certain block-diagonality condition on the curvature. As a corollary, we deduce an obstruction for conformally immersing a n-dimensional Riemannian manifold in a translation manifold of dimension $n+1$.

1. INTRODUCTION

By a striking result of Chern and Simons [3], the Chern–Simons invariant of a compact oriented Riemannian 3-manifold (X, g) conformally immersed in the flat Euclidean space $\mathbb{R}^4$ must vanish modulo $\mathbb{Z}$, showing, for instance, that the real projective space $\mathbb{R}P^3$ cannot be conformally immersed in $\mathbb{R}^4$. This result was extended in a recent paper by Čap, Flood and Mettler [1] to equiaffine immersions, a notion that we recall below. Motivated by those results, we prove in this note a more general vanishing result for the Chern–Simons invariant of a stably trivial real vector bundle with connection over an arbitrary smooth manifold (Theorem 12). When the bundle is the tangent bundle of a Riemannian manifold, we derive an obstruction for conformal immersions as hypersurfaces in translation manifolds:

Theorem 1. *Let $f : X^n \to Y^{n+1}$ be a conformal immersion of an oriented Riemannian manifold (X, g) of dimension n in a $n+1$-dimensional translation manifold Y. Then, for every smooth singular 3-cycle $C \in C_3(X, \mathbb{Z})$, the evaluation of $\mathfrak{CS}(\nabla^X)$ on C must vanish modulo integers:*

$$\int_C \mathfrak{CS}(\nabla) \in \mathbb{Z}.$$

The definition of a translation manifold is recalled in Section 7. We should stress that X is not assumed to be parallelizable; to define the Chern–Simons invariant of ∇^X it suffices that TY be trivial, which entails that TX is stably trivial. As a consequence, we deduce, for instance, that 3-dimensional lens spaces cannot be conformally embedded in any (possibly incomplete) Riemannian 4-manifold with trivial global holonomy.

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We first review Chern–Simons invariants of trivial bundles using a principal-bundle-free formalism, based on the calculus of matrix-valued forms. We then extend the definition to stably trivial bundles with connection; prove the additivity property; and finally deduce various obstructions to immersions into translation manifolds. In the Appendix we show that a certain 3-form on the general linear group, constructed in terms of the Maurer–Cartan form, is integral, and we compute the Riemannian Chern–Simons invariant of 3-dimensional lens spaces.

For several reasons, we do not treat here higher-dimensional Chern–Simons forms: first, because a large part of the current interest in Chern–Simons invariants seems to concentrate in dimension 3; secondly, since the tangent bundle of oriented manifolds is trivial in dimension 3, but in general not stably trivial in higher dimensions; and thirdly, because there are technical difficulties in extending the additivity property (Lemma 5) beyond dimension 3. With these limitations, the paper is essentially self-contained, and could hopefully also be used as an efficient introduction to the subject, free of the customary reliance on the topology of classifying spaces.

2. Matrix-valued forms

Let X be a smooth manifold. Denote by $\mathcal{M}_{kk'}$ the space of real matrices with k rows and k' columns, and let $X \times \mathcal{M}_{kk'} \to X$ be the trivial vector bundle. A *matrix-valued form* on X is a section in the tensor product bundle $\Lambda^* X \otimes_{\mathbb{R}} \mathcal{M}_{kk'}$. For $k = k' = 1$ this is the usual algebra of differential forms on X.

Besides the evident associative product (denoted below by concatenation)

$$(\Lambda^p X \otimes \mathcal{M}_{kk'}) \times (\Lambda^q X \otimes \mathcal{M}_{k'k''}) \mapsto \Lambda^{p+q} X \otimes \mathcal{M}_{kk''},$$

there are two other natural operations on matrix-valued forms.

First, we have the first-order differential operator d, the exterior differential on X twisted by the trivial connection (also denoted d) on the trivial vector bundle $\mathcal{M}_{kk'}$:

$$\mathrm{d} : \Lambda^p X \otimes \mathcal{M}_{kk'} \to \Lambda^{p+1} X \otimes \mathcal{M}_{kk'}, \qquad \mathrm{d}(\alpha \otimes A) = (\mathrm{d}\alpha) \otimes A + (-1)^p \alpha \mathrm{d}A.$$

Here $\alpha \in \Lambda^p X$ is a form and $A : X \to \mathcal{M}_{kk'}$ is a section in $X \times \mathcal{M}_{kk'}$. The operator d has the signed Leibniz property

$$\mathrm{d}(uv) = \mathrm{d}(u)v + (-1)^{\deg(u)} u\mathrm{d}(v),$$

where, by definition, the degree of a matrix-valued form $u \in \Lambda^p X \otimes \mathcal{M}_{kk'}$ is $\deg(u) = p$.

The second natural operation is the trace, defined when $k = k'$:

$$\mathrm{tr} : \Lambda^p X \otimes \mathrm{gl}_k(\mathbb{R}) \to \Lambda^p X, \qquad \mathrm{tr}(\alpha \otimes A) = \alpha \mathrm{tr}(A).$$

Although matrix-valued forms do not form a super-commutative algebra, their trace has the sign-commutation property

$$\mathrm{tr}(uv) = (-1)^{pq} \mathrm{tr}(vu).$$

for every $u \in \Lambda^p X \otimes \mathcal{M}_{kk'}$ and $v \in \Lambda^q X \otimes \mathcal{M}_{k'k}$.

Combining these two operations, it is useful to note the identity

$$\mathrm{d}(\mathrm{tr}(u)) = \mathrm{tr}(\mathrm{d}(u)).$$

The same properties hold for complex matrix-valued forms. We shall distinguish between the two cases by a subscript in the complex trace $\mathrm{tr}_{\mathbb{C}}$, in order to distinguish it from its real counterpart.

3. Chern–Simons forms and invariants

Let E be a topologically trivial real vector bundle of rank n with connection ∇ over X. Once we fix a frame in E, we write $\nabla = \mathrm{d} + \omega$ for some matrix-valued 1-form ω, and we define the Chern–Simons 3-form by

$$\mathfrak{cs}(\omega) = -\frac{1}{16\pi^2}\mathrm{tr}\left(\omega\mathrm{d}\omega + \tfrac{2}{3}\omega^3\right) \in \Lambda^3(X).$$

Denote by $\Omega = \mathrm{d}\omega + \omega^2 \in \Lambda^2(X, \mathrm{gl}_n\mathbb{R})$ the matrix-valued curvature 2-form of ∇ in the fixed frame. One can alternately describe the Chern–Simons form as

$$\mathfrak{cs}(\omega) = -\frac{1}{16\pi^2}\mathrm{tr}\left(\omega\Omega - \tfrac{1}{3}\omega^3\right).$$

A standard computation gives

$$\mathrm{d}(\mathfrak{cs}(\omega)) = -\frac{1}{16\pi^2}\mathrm{tr}(\Omega^2) \in \Lambda^4(X).$$

Definition 2. A complex k-form $\omega \in \Lambda^k(X) \otimes_{\mathbb{R}} \mathbb{C}$ on a smooth manifold X is said to be *integral* if it is closed, and its cohomology class belongs to the image of the natural map $H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{C})$. The space of integral k-forms on X is denoted $\Lambda^k_{\mathbb{Z}}(X)$.

Equivalently, $\omega \in \Lambda^k_{\mathbb{Z}}(X)$ if and only if $\int_C \omega \in \mathbb{Z}$ for every oriented smooth singular k-cycle C with integer coefficients.

In a different trivialization obtained by (right-)multiplying the given frame in E by a map $a : X \to \mathrm{GL}_n(\mathbb{R})$, the connection 1-form becomes $\omega' = a^{-1}\omega a + a^{-1}\mathrm{d}a$. By an easy computation using the properties of matrix-valued forms, the associated Chern–Simons form changes into

$$\mathfrak{cs}(a^{-1}\omega a + a^{-1}\mathrm{d}a) = \mathfrak{cs}(\omega) + \frac{1}{48\pi^2}\mathrm{tr}\left((a^{-1}\mathrm{d}a)^3\right) - \frac{1}{16\pi^2}\mathrm{d}(\mathrm{d}a a^{-1}\omega) \in \Lambda^3(X). \tag{1}$$

The form $a^{-1}\mathrm{d}a$ is the pull-back by a of the Maurer–Cartan form $\mu \in \Lambda^1(\mathrm{GL}_n(\mathbb{R}), \mathrm{gl}_n\mathbb{R})$. It is a fact, detailed in Appendix A for the sake of completeness, that for every $n \geq 3$, the form $\frac{1}{48\pi^2}\mathrm{tr}(\mu^3)$ is closed, defines an integral cohomology class on $\mathrm{GL}_n(\mathbb{R})$, and its evaluation on the 3-homology cycle $\mathrm{SO}(3) \subset \mathrm{GL}_n(\mathbb{R})$ equals -1, thus motivating the choice of the normalization constant. Equation (1) thus shows that $\mathfrak{cs}(\omega')$ agrees with $\mathfrak{cs}(\omega)$ up to an integral 3-form. It follows that the class of $\mathfrak{cs}(\omega)$ modulo $\Lambda^3_{\mathbb{Z}}(X)$ gives rise to a well-defined element $\mathfrak{CS}(\nabla) \in \Lambda^3(X)/\Lambda^3_{\mathbb{Z}}(X)$, independent of the trivialization, that we call the *Chern–Simons invariant* of ∇. Whenever the first Pontriaghin form $\mathrm{tr}(\Omega^2)$ of ∇ vanishes, $\mathfrak{cs}(\omega)$ is closed, and its cohomology class is independent of the choice of trivialization up to an integral class.

3.1. **Vanishing results.** The vanishing of a Chern–Simons invariant means that it is represented by a closed Chern–Simons form whose associated de Rham cohomology class is integral, i.e., it lives in the image of the natural map $H^3(X,\mathbb{Z}) \to H^3(X,\mathbb{C})$. The above discussion ensures that the vanishing of a Chern–Simons invariant is independent of the choice of trivialization used for defining the connection 1-form, so it is a property of the connection, rather than of the connection *form* in some trivialization.

Here are some evident instances of vanishing results:

Lemma 3. (1) *The Chern–Simons invariant of a trivial connection vanishes.*
(2) *The Chern–Simons form of a flat connection on a trivial vector bundle of rank* 1 *vanishes.*
(3) *If* $\omega \in \Lambda^1(X)$ *defines an involutive distribution* $\ker(\omega) \subset TX$ *of codimension* 1 *in* X*, i.e.,* $\omega \wedge \mathrm{d}\omega = 0$*, then the Chern–Simons form of the connection* $\mathrm{d} + \omega$ *on the trivial bundle* $\mathbb{R} \times X$ *vanishes.*

Proof. By definition, in a parallel frame, the connection 1-form of a trivial connection vanishes, so both terms in the Chern–Simons form vanish as well.

If the rank of the bundle is 1, the term ω^3 vanishes identically by anti-symmetry. The term $\omega\Omega$ also vanishes if the curvature is 0, proving the second claim.

The third claim is similar, in that the term $\Omega = \mathrm{d}\omega$ does not need to vanish anymore, however the Frobenius integrability condition implies $\omega\Omega = 0$. □

3.2. **Complex Chern–Simons forms and invariants.** For trivial complex vector bundles endowed with a $\mathbb{C}$-linear connection, more or less the same definition makes sense using the complex trace, resulting in a complex-valued 3-form on X. In that case, the form $\frac{1}{24\pi^2}\mathrm{tr}_{\mathbb{C}}(\mu^3)$ is integral on $\mathrm{GL}_n(\mathbb{C})$ by Lemma 17, so the natural normalization constant for the Chern–Simons invariant is in this case $1/8\pi^2$:

$$\mathfrak{cs}_{\mathbb{C}}(\omega) = -\tfrac{1}{8\pi^2}\mathrm{tr}_{\mathbb{C}}\left(\omega\mathrm{d}\omega + \tfrac{2}{3}\omega^3\right) \in \Lambda^3(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

This form has the same invariance properties as its real counterpart, eventually defining a class in $\mathfrak{CS}_{\mathbb{C}}(\nabla) \in (\Lambda^3(X) \otimes \mathbb{C})/\Lambda^3_{\mathbb{Z}}(X)$ independent of trivializations, called the complex Chern–Simons invariant of ∇.

Since every complex vector bundle has a subjacent real structure, it is natural to inquire about a possible relationship between the real and the complex Chern–Simons invariants of a trivial complex vector bundle with complex connection. This relation is simply:

Lemma 4. $\mathfrak{CS}(\nabla) = \Re(\mathfrak{CS}_{\mathbb{C}}(\nabla))$.

Proof. For $A \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ define $A_{\mathbb{R}} \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ to be the image of A through the natural inclusion $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \hookrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. It is evident that the inclusion is an algebra map, and that $\mathrm{tr}(A_{\mathbb{R}}) = 2\Re(\mathrm{tr}_{\mathbb{C}}(A))$. Then

$$\left(\omega\mathrm{d}\omega + \tfrac{2}{3}\omega^3\right)_{\mathbb{R}} = \left(\omega_{\mathbb{R}}\mathrm{d}\omega_{\mathbb{R}} + \tfrac{2}{3}\omega_{\mathbb{R}}^3\right).$$

The different normalization constants in the definitions of $\mathfrak{cs}$ and $\mathfrak{cs}_{\mathbb{C}}$ account for the factor 2 relating the real and the complex traces. □

By tensoring with $\mathbb{C}$, every trivial real vector bundle E with connection gives rise to a complex vector bundle $E_{\mathbb{C}}$ endowed with a complex connection. Any trivialization of E will induce a complex trivialization of $E \otimes_{\mathbb{R}} \mathbb{C}$, and the connection forms in E and $E_{\mathbb{C}}$ are the same $\mathrm{gl}_n(\mathbb{R})$-valued 1-form ω. From the difference in normalization constants, it follows that $\mathfrak{cs}(\omega) = \frac{1}{2}\mathfrak{cs}_{\mathbb{C}}(\omega)$.

4. Stable Chern–Simons invariants

Let E_1, E_2 be two trivial vector bundles (real or complex) over X with connections ∇_1, ∇_2. We say that a connection ∇ on $E := E_1 \oplus E_2$ *extends* ∇_1 and ∇_2 if

$$\Pi_1 \nabla \Pi_1 = \nabla_1, \qquad \Pi_2 \nabla \Pi_2 = \nabla_2 \tag{2}$$

where Π_1, Π_2 are the projectors on E_1, respectively E_2, in E. If we fix trivializations of E_1 and E_2, we write $\nabla_1 = \mathrm{d} + \omega_1$, $\nabla_2 = \mathrm{d} + \omega_2$, and therefore $\nabla = \mathrm{d} + \omega$, where

$$\omega = \begin{bmatrix} \omega_1 & A \\ B & \omega_2 \end{bmatrix}.$$

Here A, respectively B are some matrix-valued 1-forms of suitable dimensions. Denote by R the curvature of ∇, and by Ω its matrix-valued 2-form in the fixed trivialization. For $i, j \in \{1, 2\}$ denote by R_{ij} the $E_j \to E_i$ component of the curvature:

$$R_{ij} = \Pi_i R \Pi_j.$$

Lemma 5. *If $R_{12} = 0$ and $R_{21} = 0$, then in the real case*

$$\mathfrak{cs}(\omega) = \mathfrak{cs}(\omega_1) + \mathfrak{cs}(\omega_2),$$

while in the complex case $\mathfrak{cs}_{\mathbb{C}}(\omega) = \mathfrak{cs}_{\mathbb{C}}(\omega_1) + \mathfrak{cs}_{\mathbb{C}}(\omega_2)$.

The proof is formally the same in both cases, so we disregard the subscript $\mathbb{C}$ in the trace whenever the bundles are complex.

Proof. Compute the various ingredients of the curvature and of the Chern–Simons form of ω as follows:

$$\begin{aligned}
\omega^2 &= \begin{bmatrix} \omega_1^2 + AB & \omega_1 A + A\omega_2 \\ B\omega_1 + \omega_2 B & \omega_2^2 + BA \end{bmatrix} \\
\Omega &= \begin{bmatrix} \mathrm{d}\omega_1 + \omega_1^2 + AB & \mathrm{d}A + \omega_1 A + A\omega_2 \\ \mathrm{d}B + B\omega_1 + \omega_2 B & \mathrm{d}\omega_2 + \omega_2^2 + BA \end{bmatrix} \\
\omega\Omega &= \begin{bmatrix} \omega_1 \mathrm{d}\omega_1 + \omega_1^3 + \omega_1 AB + A\mathrm{d}B + AB\omega_1 + A\omega_2 B & * \\ * & B\omega_1 A + B\mathrm{d}A + BA\omega_2 + \omega_2 \mathrm{d}\omega_2 + \omega_2^3 + \omega_2 BA \end{bmatrix} \\
\omega^3 &= \begin{bmatrix} \omega_1^3 + \omega_1 AB + AB\omega_1 + A\omega_2 B & * \\ * & B\omega_1 A + \omega_2 BA + \omega_2^3 + BA\omega_2 \end{bmatrix}.
\end{aligned}$$

We have omitted the off-diagonal terms in $\omega\Omega$ and in ω^3, replacing them with the symbol $*$, as they are eventually irrelevant for the trace. From the above, we compute the traces appearing in the Chern–Simons form:

$$\begin{aligned}
\operatorname{tr}(\omega^3) &= \operatorname{tr}(\omega_1^3) + \operatorname{tr}(\omega_2^3) + 3\operatorname{tr}(\omega_1 AB) + 3\operatorname{tr}(\omega_2 BA)\\
\operatorname{tr}(\omega\Omega) &= \operatorname{tr}(\omega_1\Omega_1) + \operatorname{tr}(\omega_2\Omega_2) + \operatorname{tr}(A\mathrm{d}B) + \operatorname{tr}(B\mathrm{d}A) + 3\operatorname{tr}(\omega_1 AB) + 3\operatorname{tr}(\omega_2 BA)\\
\mathfrak{cs}(\omega) &= \mathfrak{cs}(\omega_1) + \mathfrak{cs}(\omega_2) + \operatorname{tr}[A(\mathrm{d}B + B\omega_1 + \omega_2 B)] + \operatorname{tr}[B(\mathrm{d}A + \omega_1 A + A\omega_2)].
\end{aligned}$$

We now recognise in the last two terms the off-diagonal terms in the curvature: $\Omega_{21} = \mathrm{d}B + B\omega_1 + \omega_2 B$, $\Omega_{12} = \mathrm{d}A + \omega_1 A + A\omega_2$. These terms vanish by hypothesis, proving our claim. $\square$

A similar result appears in [6] for the Simons invariants of complete metrics of constant positive sectional curvature. Note also a related *multiplicative* statement for Cheeger-Simons differential characters in [2, Theorem 4.7].

As a consequence, we get an additivity property of Chern–Simons invariants of trivial bundles with connections. More specifically, if E_1, E_2 are trivial bundles with connections and ∇ is a connection on $E := E_1 \oplus E_2$ extending ∇_1 and ∇_2 as in (2), then, whenever the curvature endomorphism of ∇ is diagonal with respect to the splitting, we have

$$\mathfrak{CS}(\nabla) = \mathfrak{CS}(\nabla_1) + \mathfrak{CS}(\nabla_2) \mod \Lambda^3_{\mathbb{Z}}(X).$$

We shall prove a stable version of the above identity shortly. Let us recall that a *stably trivial vector bundle* is a bundle which admits a trivial complement in a trivial bundle. More precisely, E is stably trivial if there exists E' trivial with $E \oplus E'$ also trivial.

Proposition 6. *Let $E \to X$ be a stably trivial real (respectively complex) vector bundle with connection ∇^E, and E' a trivial complement (i.e., such that $E \oplus E'$ is trivial) endowed with a trivial connection ∇'. Then the invariant $\mathfrak{CS}(\nabla \oplus \nabla')$ (respectively $\mathfrak{CS}_{\mathbb{C}}(\nabla \oplus \nabla')$) is independent of the choice of (E', ∇').*

Proof. We only treat the real case, the proof in the complex case being formally identical. Choose another trivial complement E'' with trivial connection ∇'' such that $E \oplus E''$ is trivial.

Notice that $\nabla \oplus \nabla' \oplus \nabla''$ is the direct sum connection on $E \oplus E' \oplus E''$ with respect to the splitting in the direct sum of the trivial bundles $E \oplus E'$ and E'', so in particular its curvature is diagonal with respect to this splitting. We use Lemma 5 to deduce that, once we fix trivializations of $E \oplus E'$ and of E'',

$$\mathfrak{cs}(\nabla \oplus \nabla' \oplus \nabla'') = \mathfrak{cs}(\nabla \oplus \nabla') + \mathfrak{cs}(\nabla'').$$

But the last term vanishes by Lemma 3 since ∇'' is trivial, hence we have proved the equality $\mathfrak{cs}(\nabla \oplus \nabla' \oplus \nabla'') = \mathfrak{cs}(\nabla \oplus \nabla')$ in the chosen trivializations. Passing to the residue modulo $\Lambda^3_{\mathbb{Z}}(X)$, we get $\mathfrak{CS}(\nabla \oplus \nabla' \oplus \nabla'') = \mathfrak{CS}(\nabla \oplus \nabla')$.

Similarly, fixing trivializations of $E \oplus E''$ and of E', we have $\mathfrak{cs}(\nabla \oplus \nabla' \oplus \nabla'') = \mathfrak{cs}(\nabla \oplus \nabla'')$, and we deduce that independently of trivializations, $\mathfrak{CS}(\nabla \oplus \nabla' \oplus \nabla'') = \mathfrak{CS}(\nabla \oplus \nabla'')$.

We deduce the equality $\mathfrak{CS}(\nabla \oplus \nabla') = \mathfrak{CS}(\nabla \oplus \nabla'')$. $\square$

Definition 7. The *Chern–Simons invariant* $\mathfrak{CS}(\nabla)$ of a stably trivial vector bundle E with connection ∇ is defined as the class $\mathfrak{CS}(\nabla \oplus \nabla') \in \Lambda^3(X)/\Lambda^3_{\mathbb{Z}}(X)$, for any trivial vector bundle E' endowed with a trivial connection ∇' such that $E \oplus E'$ is trivial. (By Proposition 6, that class is independent of the choices involved.)

5. Conformal invariance

If (X^3, g) is a parallelizable compact 3-Riemanian manifold, the Chern–Simons invariant of the Levi-Civita connection is known to be conformally invariant [3]. We extend this result when the tangent bundle TX is only *stably* trivial as follows:

Proposition 8. *Let (X^n, g) be a Riemannian manifold of dimension n with stably trivial tangent bundle, and ∇ its Levi-Civita connection. Fix a trivial complement of rank k, L, to TX, with its trivial connection d, and a global frame s in the trivial bundle $TX \oplus L$ over X. Let $\omega \in C^\infty(X, \mathrm{gl}_{n+k})$ be the connection 1-form of $\nabla \oplus \mathrm{d}$ in the fixed framing. Then the Chern–Simons 3 form $\mathfrak{cs}(\omega)$ is conformally invariant up to exact forms on X.*

Proof. Let $f \in C^\infty(X)$ be a conformal factor, ∇^t the connection associated to the conformal metric $g_t = e^{2tf}g$, and $\omega_t \in C^\infty(X, \mathrm{gl}_{n+k})$ the connection 1 form in the fixed framing. An easy computation valid for for every 1-parameter variation of the Chern–Simons form gives:

$$\begin{aligned}\partial_t \mathfrak{cs}(\omega_t)_{|t=0} &= \mathrm{tr}(\dot\omega_t \mathrm{d}\omega_t + \omega_t \mathrm{d}\dot\omega_t + 2\dot\omega_t\omega_t^2)_{|t=0} \\ &= \mathrm{d}\mathrm{tr}(\dot\omega\omega) + 2\mathrm{tr}(\dot\omega(\mathrm{d}\omega + \omega^2)).\end{aligned}$$

The first term is exact. In the second term, $\mathrm{d}\omega + \omega^2$ is the matrix of the curvature of the connection $\nabla \oplus \mathrm{d}$ in the fixed framing s. A direct application of Koszul's formula shows, in our case of conformal variations, that the variation of the Levi-Civita connection is

$$\partial_t(\nabla^t_U V) = U(f)V + V(f)U - g(U, V)\nabla f,$$

in particular it is independent of t. So $\dot\omega$ is the matrix in the fame s of the block-diagonal endomorphism (with respect to the splitting $TX \oplus L$)

$$\begin{bmatrix} \partial_t(\nabla^t)_{|t=0} & 0 \\ 0 & 0 \end{bmatrix}$$

One can compute $\mathrm{tr}(\dot\omega(\mathrm{d}\omega + \omega^2))$ in any local basis, that we now choose to be compatible with the splitting:

$$\mathrm{tr}(\dot\omega(\mathrm{d}\omega + \omega^2)) = \mathrm{tr}\left(\partial_t(\nabla^t)_{|t=0} R\right)$$

where R is the curvature endomorphism of ∇. So the bundle L does not contribute at all to this term. In any orthonormal frame $\{e_j\}$ on X,

$$\begin{aligned}\partial_t(\nabla^t)_{|t=0} &= \mathrm{d}f \otimes I + \sum_i e^i \otimes \left(\mathrm{d}f \otimes e_i - e^j \otimes \nabla f\right) \\ R &= \tfrac{1}{4} \sum_{j,k,\alpha,\beta} R_{jk\alpha\beta} e^j \wedge e^k \otimes \left(e^\alpha \otimes e_\beta - e^\beta \otimes e_\alpha\right).\end{aligned}$$

Using the anti-symmetry of the Riemann curvature coefficients, we get $\operatorname{tr}((\nabla f \otimes I)R) = 0$. The remaining term gives

$$\sum_{i,j,k,l} R_{jkil}\partial_l(f),$$

which vanishes by the first Bianchi identity. $\square$

6. Vanishing results

Theorem 9. *Let $E \to X$ be a stably trivial vector bundle endowed with a connection ∇^E. Assume that there exists a line bundle $L \to X$ with a flat connection ∇^L, and a trivial connection ∇ on the bundle $E \oplus L$ such that*

$$\nabla^E = \Pi_E \nabla \Pi_E, \qquad \nabla^L = \Pi_L \nabla \Pi_L,$$

where Π_E and $\Pi_L = 1 - \Pi_E$ are the projectors on the E, respectively L factors in $E \oplus L$. Then $\mathfrak{CS}(\nabla^E) = 0$.

Proof. The conditions of Lemma 5 are clearly satisfied since ∇ is trivial. Moreover, $\mathfrak{CS}(\nabla) = 0$ by the same reason. Also, $\mathfrak{CS}(\nabla^L) = 0$ by Lemma 3. It follows from Lemma 5 that $\mathfrak{CS}(\nabla^E) = 0$. $\square$

From this, we recover some classical, and some more recent, vanishing results modulo $\mathbb{Z}$.

Corollary 10 (Chern–Simons [3]). *Let $f : X^3 \to \mathbb{R}^4$ be a conformal immersion of an oriented Riemannian compact 3-fold (X, g) in the flat Euclidean space $\mathbb{R}^4$, and ω the connection 1-form of the Levi-Civita connection of g in some global frame on X. Then*

$$\int_X \mathfrak{cs}(\omega) \in \mathbb{Z}.$$

Proof. First, assume that f is an isometric immersion. The tangent bundle of every orientable 3-manifold is trivial [9], so it makes sense to consider the Chern–Simons form for the Levi-Civita connection in some trivialization. The pull-back of $T\mathbb{R}^4$ through the isometric immersion is the direct sum of TX (since the map f_* is injective on each tangent space) and of the normal line bundle NX. This last bundle is trivial together with its induced connection, since it admits a parallel section (the unit normal vector field compatible with the orientations on TX and $T\mathbb{R}^4$). Moreover, the connection induced on TX by projecting the trivial connection on $f^*T\mathbb{R}^4$ is the Levi-Civita connection. Hence for every trivialization of TX we obtain by Theorem 9 that $\mathfrak{cs}(\omega) \in \Lambda^3_{\mathbb{Z}}(X)$. By pairing with the fundamental homology class $[X] \in H_3(X, \mathbb{Z})$ we get the conclusion in the case where f is isometric.

In general, replace g by the pull-back of the euclidean metric through f. Since f is conformal, this metric is conformal to g, so the Chern–Simons invariant does not change. $\square$

A notion more general than isometric immersions is that of *equiaffine* immersions, see [8]. Recall that a connection on the tangent bundle of a manifold X^n is called equiaffine (or unimodular) if it preserves a volume form. Equivalently, an equiaffine connection induces the trivial connection on the top form bundle $\Lambda^n X$. Yet another way of defining it would be to ask that the global holonomy group be a subgroup of $\mathrm{SL_n}$. An equiaffine immersion of a manifold endowed with a torsion-free equiaffine connection ∇^{TX} is by definition an immersion $f : X \to \mathbb{R}^m$ together with a complement E' of TX inside $f^*T\mathbb{R}^m$, such that for the projector Π_{TX} defined by E', the connection on TX induced from $f^*\nabla^{\mathbb{R}^4}$ is precisely ∇:

$$\nabla = \Pi_{TX} f^* \nabla^{\mathbb{R}^4} \Pi_{TX}.$$

A generalization of the result of Chern and Simons was recently found for equiaffine immersions:

Theorem 11 (Čap-Flood-Mettler [1])**.** *Let X^3 be a compact oriented 3-fold with a torsion-free connection ∇ on TX which preserves a volume form. If (X, ∇) admits an equiaffine immersion in $\mathbb{R}^4$ then*

$$\int_X \mathfrak{cs}(\nabla) \in \mathbb{Z}.$$

The proof in [1] is based on a notion of "flat embeddings" of principal bundles with structure group G inside some larger Lie group $\tilde{G}$. Theorem 11 is in fact a particular case of a more general vanishing result:

Theorem 12. *Let X^n be a smooth manifold, and ∇^X a connection on TX. Assume that:*

(1) *the local holonomy of ∇^X is a subgroup of $\mathrm{SL}_n(\mathbb{R})$;*
(2) *there exists an immersion $f : X \to Y$ in a manifold Y^{n+1} and a trivial connection ∇^Y on TY;*
(3) *there exists a rank-1 complement E of TX inside f^*TY such that*

$$\nabla^X = \Pi_{TX} f^*(\nabla^Y) \Pi_{TX},$$

where $\Pi_{TX} : f^*TY \to TX$ is the projection on the first factor in $f^*TY = TX \oplus E$.

Then the Chern–Simons invariant $\mathfrak{CS}(\nabla^X)$ vanishes.

Proof. The connection induced by ∇^X on the form bundle $\Lambda^n X$ has holonomy in the group $\{1\}$, hence it is flat and the line bundle $\Lambda^n X$ is trivial. Let thus μ^X be a parallel volume form on X for ∇^X, and choose a nonzero section e in E such that $\mu^X \wedge e$ is parallel for the trivial connection $f^*(\nabla^Y)$ on $\Lambda^{n+1} f^*TY$. (The section v_{n+1} is unique up to a locally constant multiplicative factor.) Since $\nabla^X = \Pi_{TX} f^*(\nabla^Y)\Pi_{TX}$, it follows that $\Pi_E f^* \nabla^Y e = 0$, implying that the induced connection on the line bundle E

$$\nabla^E = \Pi_E f^*(\nabla^Y) \Pi_E$$

is flat. The conclusion follows now from Theorem 9. □

Theorem 11 follows from Theorem 12 by taking X to be compact, 3-dimensional and oriented (so in particular parallelizable), $Y = \mathbb{R}^4$ equipped with is standard flat connection, and E the normal line bundle. Since in Theorem 11 the connection ∇^X is assumed to preserve a global volume form, the *global* holonomy group of ∇^X must be a subgroup of $\mathrm{SL}_3(\mathbb{R})$. On an oriented 3-manifold X, a 3-form belongs to $\Lambda^3_{\mathbb{Z}}(X)$ if and only if its integral on X belongs to $\mathbb{Z}$.

Note that the method of proof in [1] relies on mapping the principal frame bundle of X into a principal bundle with total space a larger Lie group; this method does not generalise in an obvious way to the setting of Theorem 12.

7. Applications to conformal immersions in translation manifolds

We conclude this note with an obstruction to conformal immersions in translation manifolds.

7.1. **Translation manifolds.** By definition, a *translation manifold structure* on a topological manifold Y denotes an atlas $\mathcal{T}$ whose transition maps consist of Euclidean translations. Since translations are oriented isometries of $\mathbb{R}^n$, Y is oriented and inherits from the Euclidean space a flat Riemannian metric g^Y. This metric has trivial holonomy on Y, a parallel global frame being defined at any $p \in Y$ by the preimage through any chart in $\mathcal{T}$ near p of the standard frame on $\mathbb{R}^n$. Since translations preserve the standard frame, this is well-defined independently of the chart in $\mathcal{T}$. Conversely, a Riemannian manifold with trivial global holonomy must in particular be parallelizable (because one can fix a global parallel frame) and locally isometric to $\mathbb{R}^n$ (since flat). Out of all local isometries with $\mathbb{R}^n$, select those that map the fixed parallel frame onto the standard frame of $\mathbb{R}^n$, obtaining in this way a translation manifold structure.

In real dimension 2 there exists a related notion of *translation surface*, which means a compact Riemann surface endowed with a holomorphic 1-form α. Outside the zero locus of α, the translation surface has a natural translation atlas whose charts consist of local primitives of α, and a flat metric defined by $\Re(\alpha \otimes \overline{\alpha})$, with conical singularities at the zero locus, with angle an integer multiple of 2π. According to our definition, the complement of the zero locus of α in a translation surface is a translation manifold of dimension 2. Note, however, that a general translation manifold cannot always be compactified by adding some conical points. Note also that translation manifolds are oriented and flat, but the converse does not hold. A free quotient of $\mathbb{R}^n$ by a discrete subgroup of isometries is a translation manifold if and only if all group elements are translations. In general, a translation manifold is not required to be complete, so it need not immerse in some quotient of $\mathbb{R}^n$.

7.2. **Obstructions to conformal immersions in translation manifolds.**

Proof of Theorem 1. By Proposition 8, the Chern–Simons form in a fixed frame is conformally invariant up to exact forms on X, so we can assume that the metric on X is chosen such that the immersion in Y is isometric. Having trivial holonomy means that we can find a global frame on Y parallel for ∇^Y; clearly in this case the Chern–Simons invariant of ∇^Y vanishes (Lemma 3). By Theorem 12, $\mathfrak{CS}(\nabla^X) \equiv 0 \mod \Lambda^3_{\mathbb{Z}}(X)$ so by integration along C we get an integer. $\square$

In dimension $n = 3$ we obtain:

Corollary 13. *If an oriented Riemannian compact 3-fold (X, g) immerses in a 4-dimensional translation manifold, then*

$$\int_X \mathfrak{CS}(\nabla^g) \in \mathbb{Z}.$$

7.3. **Examples.** The Chern–Simons invariant of the real projective space $\mathbb{R}P^3 = \mathrm{SO}(3)$ equals $1/2 + \mathbb{Z}$. More generally, the Riemannian Chern–Simons invariant of the lens space $L_{p;a,b}$ (see Appendix B) with its spherical metric equals $-1/p$ (Proposition 18). Lens spaces are locally isometric to $\mathbb{S}^3 \hookrightarrow \mathbb{R}^4$. They also admit immersions in $\mathbb{R}^4$, which however do not respect the conformal class. Despite these facts, it follows from Corollary 13 and the computation in Appendix B that lens spaces cannot be conformally immersed in any Riemannian 4-manifold with trivial holonomy.

Appendix A. De Rham cohomology of the general linear group

We relegated to this section the proof of the fact, crucial for the definition of Chern–Simons invariants, that the closed 3-form $\frac{1}{48\pi^2}\mathrm{tr}(\mu^3)$, constructed from the Maurer–Cartan form on $\mathrm{GL}_n^+(\mathbb{R})$, is integral for every n. (This is standard for $n = 3$.) We prove at the same time the analogous result for the form $\frac{1}{24\pi^2}\mathrm{tr}_{\mathbb{C}}(\mu^3)$ on $\mathrm{GL}_n(\mathbb{C})$.

A.1. **Retraction onto the maximal compact subgroups.** The general linear group $\mathrm{GL}_n(\mathbb{C})$ retracts onto $\mathrm{U}(n)$ by the deformation

$$\Phi : [0,1] \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}), \qquad \Phi_s(A) = (AA^*)^{-s/2}A.$$

The restriction of Φ to $\mathrm{GL}_n(\mathbb{R})^+$ defines a deformation retraction of $\mathrm{GL}_n(\mathbb{R})^+$ onto $\mathrm{SO}(n)$. Thus the inclusions $\mathrm{U}(n) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SO}(n) \hookrightarrow \mathrm{GL}_n(\mathbb{R})^+$ induce isomorphisms in (co)homology.

A.2. **Cohomology classes constructed from the Maurer–Cartan form.** Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a linear Lie group. The Maurer–Cartan 1-form $\mu = g^{-1}\mathrm{d}g \in \Lambda^1(G) \otimes \mathrm{gl}_n\mathbb{R}$ is defined by $\mu_g(V) = g^{-1}V$ for $g \in G, V \in T_gG$. It satisfies the equation

$$\mathrm{d}\mu = \mu^2$$

which is equivalent to the vanishing of the curvature of the unique connection on the trivial G-bundle over a point. It follows that the form $\mathrm{tr}(\mu^3)$ is closed:

$$\mathrm{dtr}(\mu^3) = \mathrm{tr}\left(\mathrm{d}(\mu^3)\right) = \mathrm{tr}\left(\mathrm{d}(\mu)\mu^2) - \mu\mathrm{d}(\mu)\mu + \mu^2\mathrm{d}(\mu)\right) = 3\mathrm{tr}\left(\mathrm{d}(\mu)\mu^2)\right) = 3\mathrm{tr}(\mu^4) = 0.$$

The last equality holds because on one hand, $\mu^4 = \mu^3\mu = \mu\mu^3$, while on the other hand, from the trace property, $\mathrm{tr}(\mu^3\mu) = -\mathrm{tr}(\mu\mu^3)$.

The case where G is a subgroup in $\mathrm{GL}_n(\mathbb{C})$ is entirely similar, the only difference being that in that case, $\mathrm{tr}_{\mathbb{C}}(\mu^3)$ is a complex-valued closed 3-form on G.

Note that the Maurer–Cartan form μ on $G \subset \mathrm{GL}_n(\mathbb{R})$ coincides with the pullback to G of the Maurer–Cartan form of the group $\mathrm{GL}_n(\mathbb{R})$ itself. A similar statement holds for complex linear groups.

A.3. **Right regular representation of the quaternion algebra.** The field $\mathbb{K}$ of quaternions is isomorphic to $\mathbb{C}^2$ as complex vector spaces, by the identification $z_1 + z_2 j \leftrightarrow (z_1, z_2)$. This isomorphism is an isometry when we endow $\mathbb{K}$ with the Hermitian inner product

$$\langle q_1, q_2 \rangle_{\mathbb{C}} = \mathfrak{C}(q_1 \overline{q_2}),$$

where $\mathfrak{C}(z_1 + z_2 j) := z_1$ denotes the complex part of the quaternion $z_1 + z_2 j$. $\mathbb{K}$ is also isometric to $\mathbb{R}^4$ as real vector spaces by the identification $(a, b, c, d) \leftrightarrow a + bi + cj + dk$, with respect to the scalar product $\langle q_1, q_2 \rangle = \Re(q_1 \overline{q_2})$.

Define a $\mathbb{R}$-algebra morphism

$$R : \mathbb{K} \to \mathrm{End}_{\mathbb{K}}(\mathbb{K}), \qquad R(q)v := v\bar{q}.$$

By composing R with the chain of inclusion maps $\mathrm{End}_{\mathbb{K}}(\mathbb{K}) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{K}) \hookrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{K})$, we get $\mathbb{R}$-algebra maps $R_{\mathbb{C}} : \mathbb{K} \to \mathrm{End}_{\mathbb{C}}(\mathbb{K})$, respectively $R_{\mathbb{R}} : \mathbb{K} \to \mathrm{End}_{\mathbb{R}}(\mathbb{K})$. Since $R(q)$ is an isometry for $|q| = 1$, by restricting to the Lie group of unit-length quaternions we get Lie group morphisms

$$R_{\mathbb{C}} : \mathbb{S}^3 \to \mathrm{U}(2), \qquad R_{\mathbb{R}} : \mathbb{S}^3 \to \mathrm{O}(4), \qquad R_{\mathbb{C}}(q) = R_{\mathbb{R}}(q) = (v \mapsto v\bar{q}).$$

Furthermore, the group morphism $\det \circ R_{\mathbb{C}} : \mathbb{S}^3 \to \mathbb{S}^1$ factors through the abelianization of $\mathbb{S}^3$, which is $\{\pm 1\}$. Since $\mathbb{S}^3$ is connected, it follows that $R_{\mathbb{C}}$, respectively $R_{\mathbb{R}}$, take values in $\mathrm{SU}(2)$, respectively $\mathrm{SO}(4)$. In fact $R_{\mathbb{C}} : \mathbb{S}^3 \to \mathrm{SU}(2)$ turns out to be an isomorphism, while $R_{\mathbb{R}} : \mathbb{S}^3 \hookrightarrow \mathrm{SO}(4)$ is an embedding.

A.4. **3-homology of $\mathrm{SO}(n)$ and $\mathrm{U}(n)$.** Let e_1 denote the first element in the standard basis of $\mathbb{R}^n$.

Lemma 14. *The Lie group* $\mathrm{SO}(4)$ *is the semidirect product of the subgroups* $\mathrm{SO}(3) = \mathrm{Stab}(e_1)$ *acting on* $R_{\mathbb{R}}(\mathbb{S}^3)$.

Proof. For every $n \geq 2$ consider the principal fibre bundle with structure group $\mathrm{SO}(n-1) = \mathrm{Stab}(e_1)$

$$\begin{array}{ccc} \mathrm{SO}(n) & \text{——} & \mathrm{SO}(n-1) \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathbb{S}^{n-1} & & \end{array} \tag{3}$$

where $\pi(A) = Ae_1$. When $n = 4$, the embedding

$$s : \mathbb{S}^3 \to \mathrm{SO}(4), \qquad s(q) = R_{\mathbb{R}}(q^{-1}) \tag{4}$$

is a global section in the fibration π, giving rise to a product decomposition of manifolds

$$\mathrm{SO}(4) = s(\mathbb{S}^3) \cdot \mathrm{SO}(3).$$

For clarity, this means that the every element in the group $\mathrm{SO}(4)$ can be written uniquely as the product of elements of the subgroups $s(\mathbb{S}^3) = R_{\mathbb{R}}(\mathbb{S}^3)$, respectively $\mathrm{SO}(3)$. This decomposition is in fact a semidirect product since $R_{\mathbb{R}}(\mathbb{S}^3)$ is normal in $\mathrm{SO}(4)$. Indeed, $\mathrm{SO}(3) = \mathrm{Stab}(e_1)$ is also the image through Ad of $\mathbb{S}^3$, and

$$\mathrm{Ad}_{a^{-1}} R_{\mathbb{R}}(q) \mathrm{Ad}_a = R_{\mathbb{R}}(\mathrm{Ad}_{a^{-1}}(q)). \qquad \square$$

By the Künneth formula, $H_3(\mathrm{SO}(4), \mathbb{Z})$ is hence generated by the two 3-cycles of the subgroups $s(\mathbb{S}^3)$ and $\mathrm{SO}(3)$, the latter being embedded in $\mathrm{SO}(4)$ in the lower right corner.

Lemma 15. *For $n \geq 5$, the map $H_3(\mathrm{SO}(n-1), \mathbb{Z}) \to H_3(\mathrm{SO}(n), \mathbb{Z})$ induced by inclusion is surjective.*

Proof. Represent any 3-homology class on $\mathrm{SO}(n)$ by a smooth singular 3-cycle C. By dimensional reasons, $\pi \circ C$ must avoid at least a point $p \in \mathbb{S}^{n-1}$. Let $(\Phi_t)_{0 \leq t \leq 1}$ be a deformation retract of $\mathbb{S} \setminus \{p\}$ onto $-p$. Using a connection in the fibration (3), we lift the retraction horizontally, to define a retraction by deformation of $\pi^{-1}(\mathbb{S} \setminus \{p\})$ to the fiber over $-p$, hence to $\mathrm{SO}(n-1)$. It follows that the cycle C is cohomologous to a cycle in $\mathrm{SO}(n-1) \subset \mathrm{SO}(n)$, as claimed. $\square$

The case of $\mathrm{GL}_n(\mathbb{C})$ is similar but simpler, using the principal fibration

$$\begin{tikzcd} \mathrm{U}(n) \arrow[r, dash] \arrow[d, "\pi"] & \mathrm{U}(n-1) \\ \mathbb{S}^{2n-1} & \end{tikzcd} \tag{5}$$

For $n = 2$ this fibration is trivial because it admits the section $s_{\mathbb{C}} = R_{\mathbb{C}} \circ \mathrm{inv}$ as in (4), inducing a product decomposition

$$\mathrm{U}(2) = s_{\mathbb{C}}(\mathbb{S}^3) \cdot \mathrm{U}(1),$$

which turns out to be a semidirect product decomposition of $\mathrm{U}(2) = \mathrm{U}(\mathbb{K})$ as $\mathrm{U}(1) = \mathrm{Stab}(1)$ acting by conjugation on $R_{\mathbb{C}}(\mathbb{S})$. It follows by Künneth that $H_3(\mathrm{U}(2), \mathbb{Z})$ is infinite cyclic, with generator the cycle $s(\mathbb{S}^3)$. For larger n, the lacunarity of the Leray-Serre homology spectral sequence of the fibration (5) shows that the inclusion $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n)$ induces an isomorphism in H_3, so for every n the 3-homology of $\mathrm{U}(n)$ is generated by the cycle $\mathbb{S}^3$.

A.5. **Integrality.**

Lemma 16. *For every N, the form $\frac{1}{48\pi^2}\mathrm{tr}(\mu^3)$ belongs to $\Lambda^3_{\mathbb{Z}}(\mathrm{GL}_n^+(\mathbb{R}))$.*

Proof. Since $\mathrm{SO}(n)$ is a deformation retract of $\mathrm{GL}_n^+(\mathbb{R})$, we have to check that the evaluation of $\frac{1}{48\pi^2}\mathrm{tr}(\mu^3)$ on every 3-cycle with integer coefficients in $\mathrm{SO}(n)$ is integral. This is trivially true (for dimensional reasons) when $n \leq 2$.

For every $n \geq 3$ we have the cycle consisting of (a triangulation of) the subgroup $\mathrm{SO}(3) \hookrightarrow \mathrm{SO}(n)$ embedded diagonally in the right lower corner. One checks immediately that the restriction of μ to $\mathrm{SO}(3)$ is the Maurer–Cartan form of that subgroup, composed with the diagonal inclusion of spaces of matrices $\mathrm{gl}(3) \hookrightarrow \mathrm{gl}(n)$. This inclusion is moreover compatible with the trace maps. We now claim that

$$\mathrm{tr}(\mu^3)_{|\mathrm{SO}(3)} = -48\mathrm{vol}, \tag{6}$$

where vol is the standard volume form on $\mathbb{S}^3$. To see this, consider the adjoint representation

$$\mathrm{Ad} : \mathbb{S}^3 \to \mathrm{GL}(\mathbb{K}), \qquad \mathrm{Ad}_q(v) = qvq^{-1}. \tag{7}$$

It decomposes into the trivial representation on $\mathbb{R}$ (which is the center of $\mathbb{K}$) and a orthogonal representation on the span over $\mathbb{R}$ of the imaginary quaternions $\{i, j, k\}$, identified with $\mathbb{R}^3$, that we denote by $\mathrm{Ad} : \mathbb{S}^3 \to \mathrm{SO}(3)$. The group morphism Ad is surjective, a $2 : 1$ covering map with kernel $\{\pm 1\}$, inducing an isomorphism $\mathrm{ad} : \mathbb{R}^3 \to \mathrm{so}(3)$ at the level of Lie algebras. The group $\mathbb{S}^3 \subset \mathbb{K}$ with its standard bi-invariant metric admits an orthonormal frame (I, J, K) of left-invariant vector fields defined at $q \in \mathbb{S}^3$ by $I_q = qi \in T_q\mathbb{S}^3 \subset \mathbb{K}$, etc. In this basis, the map ad takes the form

$$\mathrm{ad}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \qquad \mathrm{ad}(J) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \qquad \mathrm{ad}(K) = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this, we compute

$$\mathrm{tr}(\mathrm{ad}(I)\mathrm{ad}(J)\mathrm{ad}(K)) = -8, \qquad \mathrm{tr}(\mathrm{ad}(I)\mathrm{ad}(K)\mathrm{ad}(J)) = 8.$$

Note that $\mathrm{tr}(\mu^3)$ is left-invariant, so it is a multiple (to be determined) of the standard volume form induced from $\mathbb{S}^3$ by the $2 : 1$ covering map (7). Moreover, from the above identities, $\mathrm{Ad}^*\mathrm{tr}(\mu^3)(I, J, K) = 48$. Together, these two facts end the proof of (6).

When $n = 3$, the cycle $\mathrm{SO}(3)$ clearly generates $H_3(\mathrm{SO}(3), \mathbb{Z})$.

For $n = 4$, we have seen that $\mathrm{SO}(4)$ is diffeomorphic to the Cartesian product of the submanifolds $\mathrm{SO}(3)$ and $s(\mathbb{S}^3)$, so $H_3(\mathrm{SO}(4), \mathbb{Z})$ is free abelian with *two* generators, the extra one being the subgroup $s(\mathbb{S}^3)$, or equivalently[1] $R_\mathbb{R}(\mathbb{S}^3)$. The form $R_\mathbb{R}^*\mathrm{tr}(\mu^3)$ is left-invariant, hence a constant multiple of the standard volume form. Write $R_\mathbb{R}^*\mu \in \Lambda^1(\mathbb{S}^3, \mathrm{gl}_4(\mathbb{R}))$ in the standard basis of $\mathbb{K}$ as $R_\mathbb{R}^*\mu(V) = R_\mathbb{R}(V)$ for every $V \in T\mathbb{S}^3$. Since $R_\mathbb{R}$ is multiplicative, for every permutation $\sigma \in \Sigma_3$,

$$R_\mathbb{R}(\sigma(I))R_\mathbb{R}(\sigma(J))R_\mathbb{R}(\sigma(K)) = R_\mathbb{R}(\sigma(I)\sigma(J)\sigma(K)) = R_\mathbb{R}(-\mathrm{sign}(\sigma)) = -\mathrm{sign}(\sigma)\mathrm{Id}_4,$$

where $\mathrm{sign}(\sigma)$ is the signature of the permutation σ, therefore $\mathrm{tr}\left((R_\mathbb{R}^*\mu^3)(I, J, K)\right) = -24$, showing that

$$\mathrm{tr}(\mu^3)_{|R_\mathbb{R}(\mathbb{S}^3)} = -24\mathrm{vol}. \tag{8}$$

Finally, for $n \geq 5$, by Lemma 15 the cycles $\mathrm{SO}(3)$ and $s(\mathbb{S}^3)$ still generate[2] $H_3(\mathrm{SO}(n), \mathbb{Z})$. Since $\mathrm{vol}(\mathbb{S}^3) = 2\pi^2$ and $\mathrm{vol}(\mathrm{SO}(3)) = \pi^2$, it follows from (6) and (8) that $\frac{1}{48\pi^2}\mathrm{tr}(\mu^3)$ evaluated on these cycles equals -1, respectively 1. □

[1] The 3-cycles $s(\mathbb{S}^3)$ and $R_\mathbb{R}(\mathbb{S}^3)$ in SO(4) differ by the orientation-changing involution $q \mapsto q^{-1}$ of $\mathbb{S}^3$.

[2] Albeit not freely, but this is irrelevant for the sake of our argument.

Lemma 17. *For every n, the form $\frac{1}{24\pi^2}\mathrm{tr}_{\mathbb{C}}(\mu^3)$ belongs to $\Lambda^3_{\mathbb{Z}}(\mathrm{GL}_n(\mathbb{C}))$.*

Proof. This is similar to the real case using the same arguments as above, only simpler since for every n the homology space $H_3(\mathrm{GL}_n(\mathbb{C}))$ is free of rank 1, generated by the cycle $\mathbb{S}^3 \to \mathrm{SU}(2) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$. The matrix-valued form μ^3 evaluated on the orthonormal frame $\{I, J, K\}$ equals 3! times $R_{\mathbb{C}}(I)R_{\mathbb{C}}(J)R_{\mathbb{C}}(K) = R_{\mathbb{C}}(IJK) = R_{\mathbb{C}}(-1) = -\mathrm{Id}_2$, so $\mathrm{tr}_{\mathbb{C}}(\mu^3)(I, J, K) = 12$. Since $\mathrm{vol}(\mathbb{S}^3) = 2\pi^2$, it follows that

$$\int_{\mathbb{S}^3} \mathrm{tr}_{\mathbb{C}}(\mu^3) = -24\pi^2. \qquad \square$$

Appendix B. Riemannian Chern–Simons invariant of lens spaces

Chern–Simons invariants of flat $\mathrm{SU}(2)$ connections on 3-dimensional lens spaces have been computed by Kirk-Klassen [5]. Here we compute the real Chern–Simons invariant of the Levi-Civita connection.

B.1. **The sphere $\mathbb{S}^3$.** The Chern–Simons form of the Levi-Civita connection ∇ on $\mathbb{S}^3$ in the orthonormal frame (I, J, K) was computed in the original paper of Chern and Simons [3]:

$$\mathfrak{cs}(\nabla; (I, J, K)) = -\frac{1}{2}\mathrm{vol}.$$

We refer e.g. to [7] for the easy proof, which can also be derived from the results of the previous sections. As a corollary, $\mathfrak{CS}(\nabla) = 0 \mod \mathbb{Z}$.

B.2. **Lens spaces.** The *lens space* $L_{p;q_1,q_2}$ is defined as the quotient of $\mathbb{S}^3 \subset \mathbb{C}^2$ by the cyclic group Γ of order p generated by the isometry

$$(z_1, z_2) \mapsto (e^{2\pi i q_1/p} z_1, e^{2\pi i q_2/p} z_2)$$

for some integers p, q_1, q_2. If q_1, q_2 are both relatively prime to p, the action is free, so the lens space $L_{p;q_1,q_2}$ is an oriented compact Riemannian 3-fold with constant sectional curvature 1. Denote by $\pi : \mathbb{S}^3 \to L_{p;q_1,q_2}$ the natural $p : 1$ covering map.

Proposition 18. *The Chern–Simons invariant of the Levi-Civita connection for the standard spherical metric on $L_{p;q_1,q_2}$ equals $-\frac{1}{p} \mod \mathbb{Z}$.*

Proof. The vector field I defined by right multiplication with $i \in \mathbb{K}$ is invariant by Γ, so its orthogonal complement, the span of J and K, forms an invariant oriented real vector bundle of rank 2 with inner product, which can therefore be naturally equipped with a complex line bundle structure. Being Γ-invariant, this bundle descends to a complex line bundle E over $L_{p;q_1,q_2}$. Since $H^2(L_{p;q_1,q_2}, \mathbb{Z}) = 0$, the Chern class of E vanishes so E is trivial. A trivialization of E by a global section U of length 1 defines a trivialization of the underlying real vector bundle by an orthonormal frame (U, V), where V is the rotation of U by angle $\pi/2$. Together with I, these vector fields form an orthonormal frame on $L_{p;q_1,q_2}$. Let $a : \mathbb{S}^3 \to \mathrm{SO}(3)$ be the matrix relating the pull-back of the frame (I, U, V) to $\mathbb{S}^3$, denoted (π^*I, π^*U, π^*V), and (I, J, K). Since $I = \pi^*I$

is the first vector in both frames, the matrix a relating them is block-diagonal, orthogonal, with 1 in the upper left corner, hence a rotation in the bottom right corner. In other words, a factors through $\mathrm{SO}(2)$, embedded in $\mathrm{SO}(3)$ as the stabilizer of e_1.

Use now formula (1) to relate the Chern–Simons forms of the Levi-Civita connection on $\mathbb{S}^3$ in the two trivializations (π^*I, π^*U, π^*V) and (I, J, K).

Since the map a factors through the 1-dimensional subgroup $\mathrm{SO}(2)$, the 3-form $\mathrm{tr}((a^{-1}\mathrm{d}a)^3) = a^*\mathrm{tr}(\mu^3)$ vanishes. Moreover, by the Stokes formula, $\int_{\mathbb{S}^3} \mathrm{d}\mathrm{tr}(\mathrm{d}aa^{-1}\omega) = 0$. In conclusion,

$$\int_{\mathbb{S}^3} \mathfrak{cs}(\nabla; (\pi^*I, \pi^*U, \pi^*V)) = \int_{\mathbb{S}^3} \mathfrak{cs}(\nabla; (I, J, K)) = -1. \tag{9}$$

Since π is an isometric covering map of degree p, we have the equality of 3-forms on $\mathbb{S}^3$

$$\pi^*\mathfrak{cs}(\nabla; (I, U, V)) = \mathfrak{cs}(\nabla; (\pi^*I, \pi^*U, \pi^*V)).$$

Hence the left-hand side of (9) equals p times the integral on $L_{p;q_1,q_2}$ of the Chern–Simons form $\mathfrak{cs}(\nabla; (I, U, V))$, leading to

$$\int_{L_{p;q_1,q_2}} \mathfrak{cs}(\nabla; (I, U, V)) = -\frac{1}{p}. \qquad \square$$

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REFERENCES

[1] A. Čap, K. Flood, T. Mettler, *Flat extensions of principal connections and the Chern–Simons 3-form*, arXiv:2409.12811, to appear in Annales H. Lebesgue.

[2] J. Cheeger, J. Simons, *Differential characters and geometric invariants*, In: Geometry and Topology. Lect. Notes Math. **1167**, Springer (1985). https://doi.org/10.1007/BFb0075216

[3] S.-S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. Math. **99** (1974), 48–69.

[4] C. Guillarmou, S. Moroianu, *Chern–Simons line bundle on Teichmüller space*, Geometry & Topology **18** (2014), 327–377.

[5] P.A. Kirk, E.P. Klassen, *Chern–Simons invariants of 3-manifolds and representation spaces of knot groups*, Math. Ann. **287**, 343–367 (1990). https://doi.org/10.1007/BF01446898

[6] J. Millson, *Examples of nonvanishing Chern–Simons invariants*, J. Diff. Geom. **10** (1974), 589–600.

[7] S. Moroianu, *The Cotton tensor and Chern–Simons invariants in dimension 3: an introduction*, Bul. Acad. Ştiinţe Moldova. Mat. **78** no. 2 (2015), 3–20.

[8] K. Nomizu,T. Sasaki, *Affine differential geometry*, Cambridge Tracts in Mathematics **111**, Cambridge University Press (1994)

[9] E. Stiefel, Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten, Comment. Math. Helv. **8** (1935), no. 1, 305–353.

[10] R. Thom, Espaces fibrés en sphéres et carrés de Steenrod, Ann. Sci. Ecole Norm. Sup. **69** (1952), 109–182.

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