

INDEX AND HOMOLOGY OF PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. We prove a local index formula for cusp-pseudodifferential operators on a manifold with boundary. This is known to be equivalent to an index formula for manifolds with cylindrical ends, and hence we obtain a new proof of the classical Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary, as well as an extension of Melrose's b -index theorem. Our approach is based on an unpublished paper by Melrose and Nistor "*Homology of pseudo-differential operators I. Manifolds with boundary*" [39]. We therefore take the opportunity to review some of the results from that paper from the perspective of subsequent research on the Hochschild and cyclic homologies of algebras of pseudodifferential operators and of their applications to index theory.

INTRODUCTION

About ten years ago, three papers have been written on the homological invariants of algebras of pseudodifferential operators on singular spaces (conical manifolds and manifolds with boundary). These papers by Melrose [37], by Melrose and Nistor [39], and by Schrohe [51], have had a significant impact on two related subjects of research: Hochschild and cyclic homology of algebras of pseudodifferential operators and index theory on singular and non-compact manifolds. One of the goals of the present work is to complete the results from [39], as well as to provide an up-to-date account of the unpublished results of that paper. We try to provide extensive references to subsequent work. We complete the results of [39] by fully relating the results of that paper with the Atiyah-Patodi-Singer index theorem for Dirac operators.

Most of the results from [39] on Hochschild homology are now published in greater generality see [9, 10, 28, 29] (some of these papers were motivated at least in part by [39]). On the other hand, the index formulas of [39] have since then been improved. An index formula simplifying the Index Theorem of [39] appeared in an unpublished manuscript [25], later generalized to manifolds with corners [26]. A similar index formula for b -operators on manifolds with corners has been found by Loya [32, 33], extending earlier work by Piazza [50]. The method of proof used in the present paper is a very simple instance of the method used in [27]. Again, the ideas of [39] were at the origin of [27]. In turn, [39] relied in part on the approach in [45] to index theorems using the boundary map in cohomology.

Many quite significant related results on traces, homology, and index theorems in the framework of pseudodifferential operators on singular spaces were obtained more recently by Grubb [15], Grubb and Schrohe [17, 16], Nazaikinskii, Savin, Schulze, and Sternin [43], Nest and Schrohe [44], Schrohe [52], Nistor [48, 49]. The related problem of multiplicative determinants and multiplicative defects was studied by Scott and Wojciechowski [54], Scott [53], Lescure and Paycha [30], and by others. Progress has also been achieved on the analysis underlying the index problem for cusp-pseudodifferential operators. For example, a simple definition using groupoids has been given in [46] and in [23] it was proved that the cusp algebra is closed under holomorphic functional calculus, which is a result of Melrose.

The index formula in [39] was motivated, in part, by the desire to get a better understanding of the Atiyah-Patodi-Singer index theorem [8] and to eventually extend it to other singular (or

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non-compact) spaces. The Atiyah-Patodi-Singer index theorem can be used to obtain a formula for the index of elliptic boundary value problems on domains with conical points [22], in the same way as the usual Atiyah-Singer index theorem [7] can be used to obtain an index formula for elliptic boundary problems [6] on smooth domains. The relevance of non-smooth domains to practical problems (Structural Analysis in Civil Engineering and the related numerical methods) makes it desirable to extend these results to other types of domains, especially polyhedral domains in three dimensions. The homological approach to index theorems [13, 45], the excision property in periodic cyclic homology [14], and especially the approach presented in this paper show that the index of a Fredholm pseudodifferential operator on a Lie manifold [2] can always be represented by a local term that is given by the Atiyah-Patodi-Singer integrand and some non-local terms that can be expressed completely in terms of the behavior at infinity of the operator.

We work in the following geometric context: X is a compact manifold with boundary (for notational simplicity, we shall assume that X is connected with non-empty boundary); $E \rightarrow X$ is a smooth Hermitian vector bundle with a fixed metric; $x : X \rightarrow [0, \infty)$ is a boundary-defining function; $\partial X \times [0, \epsilon) \hookrightarrow X$ is a product decomposition of X near the boundary; and g is a Riemannian metric on X° of the following form near the boundary. Let $M := \partial X$, then

$$(1) \quad g = a \left(\left(\frac{dx}{x^2} + \alpha(x) \right)^2 + g^M(x) \right),$$

where $a : [0, \epsilon) \times M \rightarrow (0, \infty)$ is a smooth family of strictly positive functions, $\alpha : [0, \epsilon) \rightarrow \Omega^1(M)$ is a smooth function with values 1-forms on M , and $g^M : [0, \epsilon) \rightarrow T^*M \otimes T^*M$ is a smooth function with values Riemannian metrics on $M := \partial X$. The metric g then turns out to be what is sometimes called a “cusp-metric.”

The natural class of differential operators associated to a cusp metric is the class of cusp differential operators. These are differential operators generated as an algebra near the boundary by $x^2\partial_x$ and P_y , where $P_y : [0, \epsilon) \rightarrow \text{Diff}(M)$ is a smooth family of differential operators on $[0, \epsilon)$ with values differential operators on M . No condition is imposed on cusp differential operators away from the boundary. All geometric operators associated to a cusp metric are cusp differential operators. This includes the k -form Laplacian Δ_c and the Dirac operator associated to a spin^c structure on X . Let D be a cusp differential operator acting between sections of some bundles on X . If D is Fredholm, then its generalized inverse belongs to the cusp calculus of pseudodifferential operators (because the algebra of cusp-differential operators is closed under holomorphic functional calculus). We will explain how to get an index formula for Fredholm cusp-pseudodifferential operators which contains as a particular case the standard index formulas on manifolds with asymptotically cylindrical ends. For instance, one new contribution of the present work is an index formula for the Dirac operator for certain non-exact cusp metrics. One of our main results (Theorem 16) is an index formula for the spin Dirac operator for a metric conformal to the cusp metric g :

$$g_p := x^{2p}g$$

in the Fredholm case, for $p \geq 0$ and with the additional assumption that $\alpha(0)$ is closed.

In Section 1 we review the definition of Hochschild homology and a general method of computing the index of elliptic operators on closed manifolds. This method is inspired from homology and relies on residue traces. The cusp algebra of pseudodifferential operators is recalled in Section 2. We also review the necessary facts about the ideals and subquotients of the cusp algebra. In Section 3 we recall the residue functionals on the cusp algebra introduced in [39]. In Section 4 we compute the index in terms of the residue functionals as in [39], and we show that the local contribution is the same as the more familiar constant term in the supertrace of the heat density. Moreover we link the boundary term with the eta invariant. Finally in Section 5 we review the computation of the Hochschild homology groups of the cusp algebra from [39] and from subsequent papers.

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1. MOTIVATION: THE CASE OF CLOSED MANIFOLDS

Let $\Psi^z(X; E, F)$ denote the space of classical pseudodifferential operators of order $z \in \mathbb{C}$ over a compact, boundaryless manifold X , acting between sections of vector bundles $E, F \rightarrow X$. The principal symbol map gives rise to the short exact sequence:

$$0 \rightarrow \Psi^{z-1}(X; E, F) \hookrightarrow \Psi^z(X; E, F) \xrightarrow{\sigma_z} \mathcal{C}^\infty(S^*X, \text{End}(\pi^*E, \pi^*F)) \rightarrow 0$$

where S^*X is the sphere bundle inside the cotangent bundle T^*X for a fixed metric, and $\pi : S^*X \rightarrow X$ is the bundle projection. An operator $A \in \Psi^z(X; E, F)$ is called *elliptic* if $\sigma_k(A) \in \mathcal{C}^\infty(S^*X, \text{End}(\pi^*E, \pi^*F))$ is invertible. Such an operator is Fredholm (the meaning of which is unequivocal in this situation). Its index is computed by the Atiyah-Singer formula in terms of the K -theory element in $K_c^0(T^*X)$ represented by the principal symbol. Note that although the bundles E and F become isomorphic on S^*X after pull-back, they may be non-isomorphic on X if the Euler class of X is nonzero.

Let now $A \in \Psi^k(X; E)$ be an elliptic operator of order $k \in \mathbb{Z}$ acting on the sections of a fixed bundle E , in other words we assume $E \cong F$. One can give an interpretation of its index in terms of Hochschild homology (cf. [12, 13, 39, 45]). Namely, the operators of integral order form an algebra $\Psi^{\mathbb{Z}}(X)$ (we suppress the bundle E from the notation). Inside this algebra we have the bilateral ideal of smoothing operators, $\mathcal{I} = \Psi^{-\infty}(X)$ which fits into the short exact sequence

$$(2) \quad 0 \rightarrow \mathcal{I} \rightarrow \Psi^{\mathbb{Z}}(X) \rightarrow \mathcal{A} \rightarrow 0.$$

The quotient $\mathcal{A} := \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X)$ is isomorphic as a vector space to the algebra of formal Laurent series with values in $\mathcal{C}^\infty(S^*X, \text{End}(\pi^*E, \pi^*F))$, in fact these two algebras have isomorphic graded associated algebras (with respect to the order filtration).

Hochschild homology of an algebra with unit \mathfrak{A} is defined, in low dimensions, by the differential complex

$$\dots \rightarrow \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\partial_3} \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\partial_2} \mathfrak{A} \rightarrow 0$$

where

$$\begin{aligned} \partial_2(a_0 \otimes a_1 \otimes a_2) &= a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1, \\ \partial_1(a_0 \otimes a_1) &= a_0 a_1 - a_1 a_0. \end{aligned}$$

If $A \in \Psi^{\mathbb{Z}}(X)$ is elliptic, then by a Neumann series argument we see that its image in \mathcal{A} is invertible. We get the well known fundamental Hochschild 1-cycle,

$$(3) \quad h_A := [A] \otimes [A]^{-1} \in C_1(\mathcal{A}).$$

For topological algebras, it is more reasonable to consider instead the continuous homology, obtained by replacing the usual tensor product with the projective tensor product. In our case, however, the multiplication in the algebra \mathcal{A} is not jointly continuous. A more delicate completion is then needed for the space of Hochschild chains [9].

Using the spectral sequence of the Hochschild complex filtered by the total order, the Hochschild homology of \mathcal{A} was obtained in [11] and in an unpublished work of Wodzicki.

$$(4) \quad \text{HH}_k(\mathcal{A}) \cong H^{2n-k}(S^*X \times S^1).$$

Also, it is not hard to see that the operator trace map $\text{Tr} : \mathcal{I} \rightarrow \mathbb{C}$ descends to an isomorphism between $\text{HH}_0(\mathcal{I})$ and \mathbb{C} .

Now the sequence (2) gives rise to a long exact sequence in homology:

$$\text{HH}_1(\mathcal{A}) \xrightarrow{\delta} \text{HH}_0(\mathcal{I}) \rightarrow \text{HH}_0(\Psi^{\mathbb{Z}}(X)) \rightarrow \text{HH}_0(\mathcal{A}) \rightarrow 0.$$

This holds because the ideal \mathcal{I} is H -unital [58].

Let $B \in \Psi^{\mathbb{Z}}(X)$ be an inverse of A up to \mathcal{I} , thus $[B] = [A]^{-1} \in \mathcal{A}$. Then by definition, $\delta(h_A)$ equals the commutator $[A, B] \in \mathcal{I}$. But Calderòn's formula for the index reads precisely

$$\text{Ind}(A) = \text{Tr}[A, B]$$

for any inverse B of A modulo \mathcal{I} . Therefore

$$\text{Ind}(A) = \text{Tr} \circ \delta(h_A).$$

To get a local expression for the index in terms of $[A]$, let us recall the construction of the Guillemin–Wodzicki noncommutative residue. Let $Q \in \Psi^1(X)$ be a self-adjoint, strictly positive pseudodifferential operator. Its complex powers belong to the calculus of pseudodifferential operators. It follows that for all entire maps $A : \mathbb{C} \rightarrow \Psi^k(X)$, $k \in \mathbb{C}$, the function

$$Z(A(z); z) := \text{tr}(A(z)Q^{-z})$$

is well-defined for $\Re(z) > k + n$ and extends meromorphically to \mathbb{C} with simple poles. (See [5] for the technical issues related to holomorphic functions with values in the space of pseudodifferential operators.) The residue

$$\text{Tr}_R(A(0)) := \text{Res}_{z=0} Z(A; z)$$

is independent of Q , depends only on $A(0)$ and defines a trace functional on $\Psi^{\mathbb{Z}}(X)$. This functional vanishes on \mathcal{I} and descends to an isomorphism from $\text{HH}_0(\Psi^{\mathbb{Z}}(X))$ to \mathbb{C} . Another notable feature of Tr_R is its local character, namely the fact that it is expressible in terms of the full symbol expansion of A . We compute below the index using this functional.

From Calderón’s formula, $\text{Ind}(A) = Z([A, B]; 0)$. Next, for $\Re(z) > n$,

$$\begin{aligned} Z([A, B]; z) &= \text{Tr}((ABQ^{-z} - BAQ^{-z})) \\ &= \text{Tr}((ABQ^{-z} - AQ^{-z}B)) && \text{by the trace property} \\ &= zZ\left(A \frac{B - Q^{-z}BQ^z}{z}; z\right). \end{aligned}$$

Define $[\log Q, B]$ as the value at $z = 0$ of $(B - Q^{-z}BQ^z)/z$. Although we shall not use this in what follows, let us make the following remark. Let us define $\log Q$ using functional calculus for unbounded, self-adjoint operators. Then $[\log Q, B]$ is the commutator of $\log Q$ and B . From the above discussion on the properties of the noncommutative residue, we finally get

$$\text{Ind}(A) = \text{Tr}_R(A[\log Q, B])$$

where B is any inverse of A modulo \mathcal{I} .

In homological terms, the derivation $[\log Q, \cdot]$ acts on the Hochschild complex by the usual actions of derivations on Hochschild homology. This action on Hochschild homology corresponds, under the identification (4), to the cup product with the generator of $H^1(S^1)$. Let us explain this in more detail. Let us denote by $[A]$ the class of A in the quotient $\mathcal{A} \simeq \Psi^{\mathbb{Z}}(X)/\mathcal{I}$, then $[A]^{-1} = [B]$ is defined and $h_A := [A] \otimes [A]^{-1}$ defines a class in $\text{HH}_0(\mathcal{A})$, Equation (3). With these notation, the residue trace is simply a constant multiple of pairing with the fundamental class of S^*M . Thus

$$(5) \quad \text{Ind}(A) = \text{Tr}_R(u_{\log Q} h_A)$$

does express the index as the integral of a certain volume form on S^*X locally defined in terms of $[A]$. The cohomology class of this form on S^*X is (by the Atiyah–Singer index formula) given by $(-1)^n p^* \mathcal{T}(X)$, where n is the dimension of X , $\mathcal{T}(X)$ is the Todd class of X , $p : S^*X \rightarrow X$ is the natural projection, and p^* is the map induces by p on cohomology.

A serious shortcoming of this point of view is the fact that the positive and negative spinor bundles on an even-dimensional closed spin manifold are in general not isomorphic. Thus we cannot hope to obtain directly the index formula for twisted Dirac operators in this way. As we will see, this problem disappears in the case of manifolds with boundary.

It is the above general discussion that we extend to the case of a compact manifold with boundary in such a way that (5) becomes a pseudodifferential index formula in that context, extending the Atiyah–Patodi–Singer–Melrose index theorem for b-Dirac operators. In this setting, every two bundles, between which an elliptic operator acts, must be isomorphic (see Remark 9) so without loss of generality we can work with operators acting on sections of a fixed bundle.

2. THE CUSP ALGEBRA, IDEALS AND QUOTIENTS

We shall use in this and in the following sections the geometric framework described in the Introduction. We shall also need the algebra $\Psi_c^{\mathbb{Z}}(X)$ of *cusp-pseudodifferential operators* defined by Melrose [34, 36, 35] (throughout this paper, we shall use only classical pseudodifferential operators and symbols). This algebra is a quantization of the algebra of cusp differential operators in the

same way in which the usual algebra of pseudodifferential operators on a boundaryless manifold is a quantization of the algebra of differential operators on that manifold. An alternative definition of the algebra $\Psi_c^{\mathbb{Z}}(X)$ is given in [23]. In [24], one can find a proof of Melrose's result that $\Psi_c^{\mathbb{Z}}(X)$ is closed under holomorphic functional calculus. This is essentially equivalent to the fact that if $T \in \Psi_c^m(X)$, $m \geq 0$, is elliptic and invertible as an unbounded operator, then $T \in \Psi_c^{-m}(X)$.

We look at the cusp calculus instead of the more familiar b-calculus for two reasons. First, unlike the b-calculus, the algebra $\Psi_c^{\mathbb{Z}}(X)$ is spectrally invariant, which allows us to consider complex powers inside the calculus. Secondly, every b-differential operator is a cusp differential operator (by a transcendental change of variables) but not *vice versa*, so our index theorem includes the b-index theorem of Melrose [35] for Dirac operators on asymptotically cylindrical manifolds. However, the b-calculus $\Psi_b^0(X)$ and the cusp calculus $\Psi_c^0(X)$ have isomorphic norm closures in the bounded operators on $L^2(X)$. In [38], the K-theory of the b-calculus is described. Since the K-theory is given in terms of the norm completion, the conclusions are equally valid for the cusp calculus.

It turns out that the cusp algebra $\Psi_c^{\mathbb{Z}}(X)$ has a nice structure of ideals. We shall be, in fact, more interested in understanding the subquotients of these ideals than in understanding the algebra $\Psi_c^{\mathbb{Z}}(X)$. Fortunately, these subquotients are much easier to define than the algebra $\Psi_c^{\mathbb{Z}}(X)$. We now proceed to define these ideals. We first need to fix some notation.

If $x \in C^\infty(X)$ is a global boundary defining function fixing the trivialization of the normal bundle then the Lie algebra of 'cusp vector fields' on X is

$$\mathcal{V}_c(X) := \{V \in C^\infty(X; TX), Vx \in x^2 C^\infty(X)\}.$$

The subspace $\dot{C}^\infty(X) \subset C^\infty(X)$ of functions vanishing to all orders at the boundary is preserved by the action of $\mathcal{V}_c(X)$. Furthermore, conjugation by any complex power x^z of a boundary defining function preserves the algebra $\Psi_c^{\mathbb{Z}}(X)$ of cusp pseudodifferential operators. Just as we consider the algebra of all integral order pseudodifferential operators, it is also convenient to consider

$$x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X) := \bigcup_{k \in \mathbb{Z}} \bigcup_{m \in \mathbb{Z}} x^k \Psi_c^m(X) \quad \text{and} \quad \mathcal{I} := x^\infty \Psi_c^{-\infty}(X) = \bigcap_{k \in \mathbb{Z}} \bigcap_{m \in \mathbb{Z}} x^k \Psi_c^m(X).$$

The algebra \mathcal{I} turns out to be an H-unital ideal of $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$. Let $\mathcal{A} := x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)/\mathcal{I}$. Then we obtain a long exact sequence in Hochschild cohomology in which the boundary map ∂ can be interpreted as an index map [45] in the sense that $\partial([A] \otimes [A]^{-1}) = \text{Ind}(A) \in \mathbb{Z}$.

To define the other ideals we are interested in, let us first notice that the ideal $\mathcal{I} := x^\infty \Psi_c^{-\infty}(X)$ was defined as the residual ideal for the joint filtration by order and by boundary power. When considered separately, each of these filtrations gives rise, in turn, to ideals and associated quotients as follows

$$\begin{aligned} \mathcal{I}_\partial &:= x^{-\mathbb{Z}}\Psi_c^{-\infty}(X)/\mathcal{I}, & x^{-\mathbb{Z}}\Psi_c^{-\infty}(X) &:= \bigcup_{k \in \mathbb{Z}} \left(\bigcap_{m \in \mathbb{Z}} x^k \Psi_c^m(X) \right), \\ \mathcal{I}_\sigma &:= x^\infty \Psi_c^{\mathbb{Z}}(X)/\mathcal{I}, & x^\infty \Psi_c^{\mathbb{Z}}(X) &:= \bigcap_{k \in \mathbb{Z}} \left(\bigcup_{m \in \mathbb{Z}} x^k \Psi_c^m(X) \right), \\ (6) \quad \mathcal{A}_\sigma &:= \mathcal{A}/\mathcal{I}_\sigma, & \mathcal{A}_\partial &:= \mathcal{A}/\mathcal{I}_\partial, \quad \text{and} \quad \mathcal{A}_{\partial, \sigma} := \mathcal{A}/(\mathcal{I}_\sigma + \mathcal{I}_\partial). \end{aligned}$$

These algebras give rise to the following commutative diagram in which both the lines and the columns are exact.

$$(7) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & x^\infty \Psi_c^{\mathbb{Z}}(X) & \longrightarrow & \mathcal{I}_\sigma \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & x^{-\mathbb{Z}} \Psi_c^{-\infty}(X) & \longrightarrow & x^{-\mathbb{Z}} \Psi_c^{\mathbb{Z}}(X) & \longrightarrow & \mathcal{A}_\sigma \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}_\partial & \longrightarrow & \mathcal{A}_\partial & \longrightarrow & \mathcal{A}_{\partial,\sigma} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The filtrations given by order and, respectively, by the vanishing order at the boundary give rise to the principal symbol map and to the indicial map. The first is completely analogous to the principal symbol map in the boundaryless case and reduces to it over the interior as follows. The choice of a cusp structure defines a modified tangent bundle ${}^cTX \simeq TX$ such that $\mathcal{V}_c(X) = \mathcal{C}^\infty(X; {}^cTX)$. If ${}^cT^*X$ is the dual bundle, then the symbol map gives rise, for each m , to a short exact sequence

$$0 \longrightarrow \Psi_c^{m-1}(X) \hookrightarrow \Psi_c^m(X) \xrightarrow{\sigma_m} \mathcal{G}^m({}^cT^*X) \longrightarrow 0,$$

where $\mathcal{G}^m({}^cT^*X) := \{a \in \mathcal{C}^\infty({}^cT^*X \setminus 0); \text{homogeneous of degree } m\}$.

Let $\Psi_{\text{sus}}^m(\partial X)$ be the ‘suspended algebra’ of pseudodifferential operators introduced in [37]. It consists of the pseudodifferential operators on $\mathbb{R} \times \partial X$ that are translation-invariant in \mathbb{R} and have convolution kernels vanishing rapidly with all derivatives at infinity. Similarly, the indicial map gives rise, for each m , to a short exact sequence

$$0 \longrightarrow x \Psi_c^m(X) \hookrightarrow \Psi_c^m(X) \xrightarrow{\text{In}} \Psi_{\text{sus}}^m(\partial X) \longrightarrow 0.$$

Let us fix an operator $Q \in \Psi_c^1(X)$ that is elliptic and positive with principal symbol $q = \sigma_1(Q)$. Let ${}^cS^*X = ({}^cT^*X \setminus 0)/\mathbb{R}^+$ be the sphere bundle associated to ${}^cT^*X$. Then $\mathcal{G}^m({}^cT^*X) \simeq \mathcal{C}^\infty({}^cS^*X)q^m$. We can put together all these isomorphisms in a continuous way to obtain isomorphisms

$$\mathcal{I}_\sigma \simeq \dot{\mathcal{C}}^\infty({}^cS^*X)[[q^{-1}], \quad \mathcal{A}_\sigma \simeq x^{-\mathbb{Z}} \mathcal{C}^\infty({}^cS^*X)[[q^{-1}],$$

where $[[y]]$ stands for Laurent series in y , i.e., for each element there is an upper bound on the powers of y^{-1} , but no lower bound. Below we shall use a similar notation for Laurent series in several variables.

Let us denote by $\Psi_{\text{sus}}^m(\partial X)$ the algebra of pseudodifferential operators on $\mathbb{R} \times \partial X$ that are translation invariant [37]. Similar descriptions of the algebras \mathcal{I}_∂ , \mathcal{A}_∂ , and $\mathcal{A}_{\partial,\sigma}$ in terms of $\Psi_{\text{sus}}^{\mathbb{Z}}(\partial X)$ will be given below in Equation (15).

The same results hold if we consider operators acting between sections of a vector bundle $E \rightarrow X$. This is true also of the results in the following sections. For simplicity, we shall not include E in our notation.

3. RESIDUE FUNCTIONALS

As in the case of smooth manifolds without boundary, see [18, 57], we consider functionals that arise as the residue of the analytic continuation of ‘zeta-type’ functions. Let us fix a function $x \in \mathcal{C}^\infty(X)$ defining the boundary $M = \partial X$ and a positive, elliptic, and invertible element $Q \in \Psi_c^1(X)$. For example, Q can be taken to be $(\Delta_c + 1)^{1/2}$, where Δ is the Laplacian associated to a cusp metric. Then $Q^z \in \Psi_c^z(X)$ [5, 39, 55].

In this section, we shall proceed to a large extent as in [39]. In particular, we shall need the following technical lemma from [39]. The proof given here is different.

Lemma 1. *Suppose that $A = A(\tau, z) \in x^p\Psi_c^m(X)$ is a holomorphic function on $\Omega \subset \mathbb{C}^2$, then the function $\text{Tr}(A(\tau, z)x^zQ^{-\tau})$ is defined and holomorphic in $\{\Re z > -p+1\} \cap \{\Re \tau > m + \dim X\} \cap \Omega$ and extends to a meromorphic function on Ω with at most simple poles on the surfaces $z = -p+1-j$ and $\tau = m - k + \dim X$, $j, k \in \mathbb{Z}_+$.*

Proof. The proof is standard, see [48], for example, so we shall be short. Using a partition of unity argument, we can assume $X = [0, 1] \times Y$, with Y smooth, without boundary. Let $S^m({}^cT^*X)$ be the space of classical symbols of order m and $S^\infty({}^cT^*X) := \cup_m S^m({}^cT^*X)$. Let us fix in what follows a map

$$(8) \quad \rho : S^\infty({}^cT^*X) \rightarrow \Psi_c^\infty(X), \quad \rho(S^m({}^cT^*X)) \subset \Psi_c^m(X), \quad \text{and} \quad \sigma_m(\rho(a)) = a + S^{m-1}({}^cT^*X).$$

(A map with these properties will be called a *quantization map*.) We can further assume that $X = \mathbb{R}^n$ and that ρ is the standard quantization corresponding to the product cusp metric. With this metric, the interior $(0, 1) \times \mathbb{R}^n$ is identified to $\mathbb{R} \times \mathbb{R}^n$ with the standard Euclidean metric and the formula for ρ becomes

$$(9) \quad \rho(a)u(p) = (2\pi)^{-n-1} \int_{\mathbb{R}^{n+1}} e^{i(p-q)\cdot\xi} a(p, \xi)u(q)dq d\xi.$$

To prove our result, it is enough to estimate $\text{Tr}(\rho(a_{z,\tau}))$, where $a_{z,\tau}(p, \xi) = p_1^z b(p, \xi)|\xi|^\tau$ for some symbol $b(p, \xi)$. (In our notation, we have $p = (p_1, p_2, \dots, p_n)$ with $p_1 = x$.) The result then follows using the relation

$$\text{Tr}(\rho(a)) = (2\pi)^{-n-1} \int_{\mathbb{R}^{n+1}} a(p, \xi) dp d\xi$$

and integration in polar coordinates in ξ . □

Let us fix the auxiliary operator Q and the boundary defining function x as in Lemma 1. We are interested in the following zeta-type function for a holomorphic family of operators $A : \mathbb{C}^2 \rightarrow x^p\Psi_c^m(X)$

$$Z(A; \tau, z) = Z_{x,Q}(A; \tau, z) := \text{Tr}(A(z, \tau)x^zQ^{-\tau}).$$

Lemma 1 shows that $\tau z Z_{x,Q}(A; \tau, z)$ is holomorphic in a neighborhood of $0 \in \mathbb{C}^2$. We shall examine the four functionals $\text{Tr}_{\partial,\sigma}(A), \widehat{\text{Tr}}_\partial(A), \widehat{\text{Tr}}_\sigma(A), \widehat{\text{Tr}}(A) \in \mathbb{C}$ defined by

$$(10) \quad \tau z Z(A; \tau, z) = \text{Tr}_{\partial,\sigma}(A) + \tau \widehat{\text{Tr}}_\partial(A) + z \widehat{\text{Tr}}_\sigma(A) + \tau z \widehat{\text{Tr}}(A) + \tau^2 W + z^2 W',$$

where W and W' are holomorphic near 0. It will be crucial in what follows that we allow A to be a *holomorphic function*. The functionals $\text{Tr}_{\partial,\sigma}(A), \widehat{\text{Tr}}_\partial(A), \widehat{\text{Tr}}_\sigma(A)$, and $\widehat{\text{Tr}}(A)$ are hence defined for A a holomorphic function $\mathbb{C}^2 \rightarrow x^p\Psi_c^m(X)$. Occasionally, A will be a constant function. For example, if $A_1 \in x^p\Psi_c^m(X)$, then we define $\text{Tr}_{\partial,\sigma}(A_1), \widehat{\text{Tr}}_\partial(A_1), \widehat{\text{Tr}}_\sigma(A_1)$, and $\widehat{\text{Tr}}(A_1)$ by regarding A_1 as a constant function.

For further reference, let us record the following simple lemma.

Lemma 2. *Let $A : \mathbb{C}^2 \rightarrow x^p\Psi_c^m(X)$ be a holomorphic function, then*

$$\widehat{\text{Tr}}(zA) = \widehat{\text{Tr}}_\partial(A) \quad \text{and} \quad \widehat{\text{Tr}}(\tau A) = \widehat{\text{Tr}}_\sigma(A).$$

Proof. The coefficients (functionals) $\text{Tr}_{\partial,\sigma}(A), \widehat{\text{Tr}}_\partial(A), \widehat{\text{Tr}}_\sigma(A), \widehat{\text{Tr}}(A) \in \mathbb{C}$ are uniquely determined by Equation (10). For example,

$$\widehat{\text{Tr}}(A) = \left[\partial_\tau \partial_z (\tau z Z(A; \tau, z)) \right]_{\tau=0, z=0}.$$

The lemma follows from this uniqueness by substituting in Equation (10) zA or τA for A and then comparing the coefficients. □

Note that $Z(A; \tau, z)$ is entire for $A \in \mathcal{I}$, and hence $\text{Tr}_{\partial,\sigma}, \widehat{\text{Tr}}_\partial$, and $\widehat{\text{Tr}}_\sigma$ descend to linear maps on \mathcal{A} by identifying an element in $x^p\Psi_c^m(X)$ with a constant function.

Lemma 3. *Assume $A(z, \tau) = A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$. The double residue $\text{Tr}_{\partial, \sigma}(A)$ of $Z(A; \tau, z)$ at $\tau = z = 0$ defines a trace functional on the algebra $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$. The value $\text{Tr}_{\partial, \sigma}(A)$ is independent of the choice of Q or x , vanishes on $\mathcal{I}_\sigma + \mathcal{I}_\partial$ and therefore descends to a trace functional on $\mathcal{A}_{\partial, \sigma}$.*

Proof. Let x' be another defining function for the boundary M of X . Then the function

$$B(\tau, z) = A - A(x'/x)^z$$

is holomorphic and vanishes at $z = 0$, which gives

$$Z_{x, Q}(A; \tau, z) - Z_{x', Q}(A; \tau, z) = \text{Tr}(Ax^z Q^{-\tau} - A(x')^z Q^{-\tau}) = z \text{Tr}(B(\tau, z)x^z Q^{-\tau}),$$

where $B(\tau, z)$ is entire, with values in $x^p\Psi_c^m(X)$. Thus Lemma 1 shows that the difference of the zeta functions $Z(A; \tau, z)$ for different choices of the boundary defining function x is regular at $z = 0$. Therefore the functional $\text{Tr}_{\partial, \sigma}(A)$ does not depend on the choice of the boundary defining function. A similar argument shows independence on the choice of Q .

To prove that $\text{Tr}_{\partial, \sigma}$ is a trace, consider $A, B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$. The trace property of Tr gives, for large real values of z and τ ,

$$(11) \quad Z_{x, Q}([A, B]; \tau, z) = \text{Tr}(A(B - x^z Q^{-\tau} B x^{-z} Q^\tau)x^z Q^{-\tau}).$$

Since the family $A(B - x^z Q^{-\tau} B x^{-z} Q^\tau)$ is holomorphic as a family of classical pseudodifferential operators and vanishes at $z = \tau = 0$, Lemma 1 shows that the double residue at $z = \tau = 0$ of the right-hand side of Equation (11) vanishes so $\text{Tr}_{\partial, \sigma}([A, B]) = 0$. \square

Before deriving an explicit formula for the functional $\text{Tr}_{\partial, \sigma}(A)$, let us consider the functional $\widehat{\text{Tr}}_\sigma$. Let $2X$ be the compact manifold without boundary obtained by doubling X across its boundary, as before. Then

$$x^\infty\Psi_c^{\mathbb{Z}}(X) \subset \Psi^{\mathbb{Z}}(2X)$$

is an ideal, consisting precisely of those elements of $\Psi^{\mathbb{Z}}(2X)$ that have Schwartz kernels supported in the set $X \times X \subset 2X \times 2X$. The smaller ideal $\Psi^{\mathbb{Z}}(X^\circ) \subset \Psi^{\mathbb{Z}}(2X)$ of operators with kernels supported in the interior of $X \times X$, i.e., in $X^\circ \times X^\circ$, is dense in $x^\infty\Psi_c^{\mathbb{Z}}(X)$. The Guillemin–Wodzicki residue trace for $2X$ is defined on the latter space; we shall denote it Tr_R .

Lemma 4. *The restriction Tr_σ of $\widehat{\text{Tr}}_\sigma$ to \mathcal{I}_σ is a trace functional. The map Tr_σ coincides with the extension by continuity of the Guillemin–Wodzicki residue trace Tr_R from $\Psi^{\mathbb{Z}}(X^\circ)$ to $x^\infty\Psi_c^{\mathbb{Z}}(X)$.*

Proof. If $A \in \mathcal{I}_\sigma$ then Ax^z is entire with values in \mathcal{I}_σ , as a family of fixed order. It follows that $Z_{x, Q}(A; \tau, z)$ is entire in z . Thus we only need consider the simpler function

$$Z_Q(A; \tau) = \text{Tr}(AQ^{-\tau}), \quad A \in \mathcal{I}_\sigma.$$

A particular case of Lemma 1 shows that this function meromorphic in τ , with at most a simple pole at $\tau = 0$. By (10), the residue of this function at 0 is simply $\widehat{\text{Tr}}_\sigma(A) = \text{Tr}_\sigma(A)$. This shows that there exists a function $U(\tau)$ holomorphic near 0 such that

$$(12) \quad \tau Z_Q(A; \tau) = \text{Tr}_\sigma(A) + \tau U(\tau), \quad \text{for } A \in \mathcal{I}_\sigma.$$

Let $A \in \Psi^{\mathbb{Z}}(X^\circ) \subset \Psi^{\mathbb{Z}}(2X)$. If Q were positive and elliptic in $\Psi^1(2X)$, then (12) would be exactly the definition of the Guillemin–Wodzicki residue trace. The definition of ellipticity in the cusp calculus implies that, for any given B with kernel supported in $X^\circ \times X^\circ$, there exists $Q' \in \Psi^1(2X)$ positive, elliptic, and satisfying $Q'B - QB \in \Psi^{-\infty}(2X)$. Since the residue trace vanishes on regularizing operators, we obtain that

$$\text{Tr}_\sigma(A) = \text{Tr}_R(A), \quad \text{for } A \in \Psi^{\mathbb{Z}}(X^\circ).$$

Since Tr_σ is continuous in the topology of \mathcal{I}_σ and Tr_R is a trace, it follows that Tr_σ is a trace as well. (This can also be proved by a simple computation similar to that of (11).) \square

Let $A \in \mathcal{I}_\sigma$ and $\sum_{j=-\infty}^m a_j$ be the full symbol expansion of A in a coordinate system. Lemma 4 and the usual formula for the Guillemin–Wodzicki residue then give

$$(13) \quad \text{Tr}_\sigma(A) = (2\pi)^{-n} \int_{cS^*X} a_{-n}^c \nu,$$

where ${}^c\nu$ is the symplectic volume form on ${}^cS^*X$.

Let us denote by $\Psi_{\text{sus}}^m(\partial X)$ the algebra of pseudodifferential operators on $\mathbb{R} \times \partial X$ that are translation invariant, as before. We fix a tubular neighborhood $V = M \times [0, \epsilon)$ of the boundary. Then a cutoff function with support in V allows us to identify $\Psi_{\text{sus}}^m(\partial X)$ with a subspace of $\Psi_c^{\mathbb{Z}}(X)$. Let $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$, so $A \in x^p\Psi_c^m(X)$ for some p and $m \in \mathbb{Z}$. Then

$$(14) \quad A \sim \sum_{j=-\infty}^{-p} x^{-j} A_j, \quad A_j \in \Psi_{\text{sus}}^m(\partial X).$$

Let us notice, for later use that for $p = 0$, A_0 is the *indicial operator* of A . We shall denote the indicial operator of A by $\mathcal{N}(A) := A_0$.

This gives

$$(15) \quad \mathcal{I}_\partial \simeq \Psi_{\text{sus}}^{-\infty}(\partial X)[[x], \quad \mathcal{A}_\partial \simeq \Psi_{\text{sus}}^{\mathbb{Z}}(\partial X)[[x], \quad \text{and } \mathcal{A}_{\partial, \sigma} \simeq \mathcal{C}^\infty({}^cS_{\partial X}^*X)[[x, q^{-1}].$$

In terms of the expansion in Equation (14), we can now write a formula for $\text{Tr}_{\partial, \sigma}(A)$.

Lemma 5. *Let $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$, $A \sim \sum_{j=-\infty}^{-p} x^{-j} A_j$ as in (14) and let $a_{k,l}$ be the term homogeneous of degree k in the asymptotic expansion of the symbol of A_l . Let ω_∂^{n-1} be the symplectic measure on $T^*M = T^*(\partial X)$. Then*

$$\text{Tr}_{\partial, \sigma}(A) = (2\pi)^{-n} \int_{{}^cS_{\partial X}^*X} a_{-n, -1} {}^c\nu_\partial,$$

where ${}^c\nu_\partial$ is the measure obtained by contracting the form $\omega_\partial^{n-1} d\xi$ with the radial vector field on ${}^cT_\partial^*X = T^*\partial X \times \mathbb{R}_\xi$.

Proof. Comparing the definitions of various functionals, it follows directly that $\text{Tr}_{\partial, \sigma}(A)$ is the residue at $z = 0$ of the meromorphic function $\text{Tr}_\sigma(x^z A)$. Thus our result follows from (13). \square

A similar argument gives

$$\widehat{\text{Tr}}_\sigma(A) = \lim_{\epsilon \downarrow 0} \left((2\pi)^{-n} \int_{{}^cS^*X \cap \{x > \epsilon\}} a_{-n} {}^c\nu + (\log \epsilon) \text{Tr}_{\partial, \sigma}(A) + \sum_{l > 0} \gamma_l \epsilon^{-l} \right),$$

where the constants γ_l are chosen to ensure that the limit exists. In particular, the functional $\widehat{\text{Tr}}_\sigma$ vanishes on \mathcal{I}_∂ and so defines a continuous functional on \mathcal{A}_σ that is independent of the choice of Q . If $x' = ax$, with $0 < a \in \mathcal{C}^\infty(X)$, is another boundary defining functions, then the difference of the resulting two boundary functionals is given by the formula

$$\widehat{\text{Tr}}_\sigma(A; x') - \widehat{\text{Tr}}_\sigma(A; x) = \text{Tr}_{\partial, \sigma}(A \log a), \quad A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X).$$

Let us now reverse the roles of x and Q and proceed to consider the functional $\widehat{\text{Tr}}_\partial$, defined on $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ by (10). We proceed as in [39]. Since $Z(A; \tau, z)$ is entire in z for any fixed τ and $A \in \mathcal{I}$, the functional $\widehat{\text{Tr}}_\partial$ descends to a linear map on the quotient $\mathcal{A}_\partial := \mathcal{A}/\mathcal{I}_\sigma$ (see Equation (6)). We shall denote by Tr_∂ the restriction of $\widehat{\text{Tr}}_\partial$ to \mathcal{I}_∂ . If $A \in \Psi_c^{-\infty}(X)$, then $Z(A; \tau, z)$ is entire in τ . It follows that $\text{Tr}_\partial(A)$ is the residue at $z = 0$ of $\text{Tr}(Ax^z)$. Recall from [37] that on the indicial algebra $\Psi_{\text{sus}}^{-\infty}(\partial X) \simeq \mathcal{S}(\mathbb{R}) \widehat{\otimes} \Psi^{-\infty}(\partial X)$ there is a trace functional given by integration of the trace of the indicial family

$$(16) \quad \overline{\text{Tr}}(B) = (2\pi)^{-1} \int_{\mathbb{R}} \text{Tr } \widehat{B}(\xi) d\xi.$$

Lemma 6. *The functional $\widehat{\text{Tr}}_\partial$ is a trace on \mathcal{I}_∂ , is independent of the choice of x or Q , and is determined in terms of the trace (16) by*

$$\widehat{\text{Tr}}_\partial(A) = \overline{\text{Tr}}(A_{-1}), \quad A \in \mathcal{I}_\partial,$$

if $A \sim \sum_{j=-\infty}^{-p} x^{-j} A_j$, with $A_j \in \Psi_{\text{sus}}^{-\infty}(\partial X)$.

Proof. This follows from a simple calculation based on the formula for the trace as the integral of the Schwartz kernel over the diagonal. \square

4. AN INDEX 1-COCYCLE AND INDEX FORMULAS

Recall that we consider operators acting on sections of a vector bundle E , although we do not include E in the notation. For any pair of elements $A, B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$, we examine the “regularized trace” at $\tau = 0, z = 0$ of the commutator $[A, B] := AB - BA$ and interpret it in terms of the functionals introduced in Equation (10)

$$(17) \quad \text{IF}(A, B) = \widehat{\text{Tr}}([A, B]), \quad A, B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X).$$

We shall denote by $H_c^m(X; E)$ the cusp Sobolev spaces on X . Recall that for $m \geq 0$, this space is defined as the domain of $(\Delta_c + 1)^{m/2}$, where Δ_c is the Laplacean associated to a cusp-metric on X . For $m \leq 0$, $H_c^m(X; E)$ can be defined as the dual of $H_c^{-m}(X; E)$ with pivot L^2 (see [2, 4] and the references therein for more results on the cusp Sobolev spaces, as well as on generalizations of these spaces and applications to boundary value problems and numerical methods. In particular, the various mapping properties needed below are easy results that can be found, for example, in [4]).

Lemma 7. *If $A \in x^p\Psi_c^m(X)$ is elliptic and invertible in \mathcal{A} (i.e. invertible modulo \mathcal{I}), then it defines a Fredholm operator $x^r H_c^s(X; E) \rightarrow x^{r+p} H_c^{s-m}(X; E)$, with index*

$$\text{Ind}(A) = \text{IF}(A, B), \quad \text{for any } B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X), \quad AB - I \in \mathcal{I}.$$

An elliptic element $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ that is invertible in \mathcal{A} will be called *fully elliptic* in what follows.

Proof. From the definition of B , we have that $[A, B] \in \mathcal{I}$, and hence $Z([A, B]; \tau, z)$ is holomorphic on \mathbb{C}^2 . By standard arguments, A acts on the above Sobolev spaces. Since the operators in \mathcal{I} are compact on every Sobolev space, it follows that A is Fredholm. The fact that $\text{Ind}(A)$ is the value at $\tau = 0, z = 0$ is nothing but Calderón’s formula. Indeed, if $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ is invertible modulo \mathcal{I} , then we can choose B to be the generalized inverse that satisfies

$$AB - \text{Id} = -p_{\text{coker } A}, \quad BA - \text{Id} = -p_{\text{ker } A},$$

where $p_{\text{ker } A}$ and $p_{\text{coker } A}$ are projections onto the null space and a complement to the range respectively. Thus $AB - \text{Id}$ and $BA - \text{Id}$ are both in \mathcal{I} and

$$\begin{aligned} \text{Ind}(A) &= \text{Tr}(p_{\text{ker } A}) - \text{Tr}(p_{\text{coker } A}) = -\text{Tr}(BA - \text{Id}) + \text{Tr}(AB - \text{Id}) \\ &= -\widehat{\text{Tr}}(BA - \text{Id}) + \widehat{\text{Tr}}(AB - \text{Id}) = \text{IF}(A, B). \end{aligned}$$

This completes the proof. \square

The previous lemma can be viewed as expressing the compatibility between the boundary map in Hochschild and cyclic homologies and the boundary (or index) map in K -theory for a particular cocycle, namely for the Fredholm trace. This compatibility is proved in general in [45].

For any fully elliptic $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ (i.e. elliptic and invertible in \mathcal{A}) we set

$$(18) \quad \eta(A) = 2\widehat{\text{Tr}}_{\partial}(A[\log x, A^{-1}]) \quad \text{and} \quad \overline{\text{AS}}(A) = \widehat{\text{Tr}}_{\sigma}([\log Q, A^{-1}]A).$$

In fact, $\eta(A)$ is defined for all invertible elements of \mathcal{A}_{∂} while $\overline{\text{AS}}(A)$ is defined for all invertible elements of \mathcal{A}_{σ} . Occasionally, we shall write $\eta(B) = 2\widehat{\text{Tr}}_{\partial}(B[\log x, B^{-1}])$ for an invertible $B \in \mathcal{A}_{\partial}$. The notation is supposed to suggest that $\overline{\text{AS}}(A)$ is a generalization of the Atiyah-Singer integrand involving a characteristic form, and that $\eta(A)$ is a generalization of the eta invariant. Formally, it is certainly the case that $\overline{\text{AS}}(A)$ only depends on the (full) symbol of A whereas $\eta(A)$ only depends on the (full) indicial family of A , i.e. on the respective images of A in the quotients \mathcal{A}_{σ} and \mathcal{A}_{∂} . However, both $\overline{\text{AS}}(A)$ and $\eta(A)$ depend on the choice of Q , and on the tacitly fixed cusp metric of the form (1) on X . To obtain more explicit formulas, we shall remove the ambiguity and we choose B and Q in terms of A in Proposition 12.

Proposition 8. *The index functional IF defined by (17) descends from $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ to a cocycle on \mathcal{A} and*

$$(19) \quad \text{IF}(A, B) = \widehat{\text{Tr}}_{\sigma}([\log Q, B]A) - \widehat{\text{Tr}}_{\partial}(A[\log x, B]).$$

In particular, if $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ is invertible in \mathcal{A} , then

$$(20) \quad \text{Ind}(A) = \text{IF}(A, A^{-1}) = \overline{\text{AS}}(A) - \eta(A)/2.$$

Proof. If $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ and $C \in \mathcal{I}$, then $\text{Tr}([A, C]) = 0$, and hence $\text{IF}(A, B) = \text{IF}(A, B + C)$. This shows that $\text{IF}(A, B)$ depends only on the classes of A and B in \mathcal{A} . The ideal property of \mathcal{I} and the fact that $\widehat{\text{Tr}}_{\partial}$ and $\widehat{\text{Tr}}_{\sigma}$ vanish on \mathcal{I} , implies that both bilinear functionals from Equation (19) descend to bilinear functionals on \mathcal{A} . Then

$$(21) \quad \begin{aligned} [A, B]x^z Q^{-\tau} &= A(B - x^z B x^{-z})x^z Q^{-\tau} + (Q^{\tau} B Q^{-\tau} - B)A x^z Q^{-\tau} \\ &\quad - [Q^{\tau} B Q^{-\tau}, A x^z Q^{-\tau}]. \end{aligned}$$

We have $B - x^z B x^{-z} = -z[\log x, B] + z^2 C_1(z)$ and $Q^{\tau} B Q^{-\tau} - B = \tau[\log Q, B] + \tau^2 C_2(\tau)$ where C_1, C_2 are entire functions of cusp operators of fixed order. By taking the trace in (21), we get

$$(22) \quad \begin{aligned} Z([A, B]; \tau, z) &= -Z(Az[\log x, B]; \tau, z) + Z(\tau[\log Q, B]A; \tau, z) \\ &\quad + Z(Az^2 C_1(z); \tau, z) + Z(\tau^2 C_2(\tau)A; \tau, z). \end{aligned}$$

From Lemma 1, we see that both $\widehat{\text{Tr}}(Az^2 C_1(z))$ and $\widehat{\text{Tr}}(\tau^2 C_2(\tau)A)$ vanish. Thus (22) gives

$$\widehat{\text{Tr}}([A, B]) = -\widehat{\text{Tr}}(Az[\log x, B]) + \widehat{\text{Tr}}(\tau[\log Q, B]A).$$

The proof is completed then by using the relations in Lemma 2. \square

Let us notice that assuming that our elliptic cusp pseudodifferential operator A on X is in $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ is not a loss of generality, because A must act between isomorphic bundles, in view of the following remark.

Remark 9. Let P be an elliptic cusp operator acting between sections of two smooth vector bundles E_+ and E_- over the connected compact manifold X with nonempty boundary. Then $E_+ \cong E_-$ over X . Indeed, the ellipticity assumption on P implies that E_+ and E_- have isomorphic pull-backs to the cusp-cosphere bundle ${}^c S^* X$. Recall now that if X is connected with nonempty boundary, then ${}^c S^*$ has a section; the principal symbol of P evaluated on this section gives an isomorphism between E_+ and E_- .

Let us make another simple, but important remark.

Remark 10. The Q and x terms appearing in the index formula of Proposition 8 are the same as the ones appearing in the definition of the functionals $\widehat{\text{Tr}}_{\partial}$ and $\widehat{\text{Tr}}_{\sigma}$ (Equation (10)).

We now proceed to simplify the index formula of Equation (20). We assume for simplicity that $A \in \Psi_c^1(X; E)$. Following [27, 29], we set

$$(23) \quad \begin{aligned} Q_1 &:= (AA^* + p_{\ker A^*})^{1/2}, \\ Q_2 &:= (A^*A + p_{\ker A})^{1/2}, \\ B &:= A^*Q_1^{-2} = Q_2^{-2}A^*, \end{aligned}$$

where p_V denotes the orthogonal projection onto the space V . We note the commutation relation

$$(24) \quad BQ_1^{-\tau} = Q_2^{-\tau}B.$$

For a meromorphic function $Z(\tau, z)$ with only simple poles as in Lemma 1 we shall denote $Z(\tau, z)_{\tau=0, z=0} = Z(0, 0)$ if Z is holomorphic in a neighborhood of $(0, 0)$, and, otherwise

$$Z(\tau, z)_{\tau=0, z=0} = \left[\partial_{\tau} \partial_z (\tau z Z(\tau, z)) \right]_{\tau=0, z=0}.$$

We use the above operator Q_1 in the definition (10) of the trace functionals to obtain

$$\begin{aligned}
\overline{\text{AS}}(A) &= \widehat{\text{Tr}}_\sigma([\log Q_1, B]A) \\
&= \widehat{\text{Tr}}((Q_1^\tau B Q_1^{-\tau} - B)A) && \text{by Lemma 2} \\
&= \text{Tr}((Q_1^\tau B Q_1^{-\tau} - B)A x^z Q_1^{-\tau})_{z=0, \tau=0} && \text{by definition} \\
&= (\text{Tr}([B, Q_1^{-\tau}]A x^z))_{z=0, \tau=0} && \text{by the trace property} \\
&= (\text{Tr}((Q_2^{-\tau} - Q_1^{-\tau})B A x^z))_{z=0, \tau=0} && \text{by Equation (24)} \\
(25) \quad &= (\text{Tr}((Q_2^{-\tau} - Q_1^{-\tau})x^z))_{z=0, \tau=0}.
\end{aligned}$$

The last equality follows since $(Q_2^{-\tau} - Q_1^{-\tau})(BA - \text{Id})x^z$ is a holomorphic family of operators in \mathcal{I} which vanishes at $\tau = 0$.

Remark 11. Note here that in the boundaryless case, from Proposition 8 and the above computation we recover the well-known identity

$$\text{Ind}(A) = \text{Tr}((Q_2^{-\tau} - Q_1^{-\tau}))_{\tau=0}.$$

However, one must assume that A acts on a fixed vector bundle, which excludes the case of the Dirac operator, in general. One can make sense of the formula without this assumption, as the difference of the traces of $Q_2^{-\tau}$ and $Q_1^{-\tau}$, namely $\text{Ind}(A) = \left(\text{Tr}(Q_2^{-\tau}) - \text{Tr}(Q_1^{-\tau}) \right)_{\tau=0}$.

Using (25), we can give a more familiar interpretation of the local term $\overline{\text{AS}}$: it is (not surprisingly) the integral on X of the *index density*, another local density defined in terms of heat kernel expansions. The word “local” means that, like $\overline{\text{AS}}(A)$ itself, the index density is a density which at every point depends only on the jets of the full symbol of A and of the metric at that point. Since this quantity is locally computable, we need no property of the heat operator on X to give it a meaning.

Proposition 12. *If $A \in \Psi_c^1(X; E)$, choose $Q = Q_1 = (AA^* + p_{\ker A^*})^{1/2}$ in the definition (10). Then*

$$\overline{\text{AS}}(A) = \left(\int_X a_0 \right)_{z=0},$$

where a_0 is the density locally defined on X as the constant term in the asymptotics as $t \rightarrow 0$ of the supertrace of the restriction to the diagonal of the distributional kernel of the heat operator (Equation 26). Moreover, the regularization is not necessary, in the sense that

$$\lim_{\epsilon \rightarrow 0} \int_{\{x > \epsilon\}} a_0$$

exists (and thus equals $\overline{\text{AS}}(A)$).

More precisely, let us denote by $k(x, y)$ the distribution kernel of a pseudodifferential operator of low order, and by “tr” (respectively “Str”) the trace (respectively the supertrace) of an endomorphism of a finite dimensional space. Moreover, denote by LIM the constant term in an asymptotic expansion. Then

$$\begin{aligned}
(26) \quad \text{Str} \left(\exp \left(-t \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}^2 \right) (x, x) \right) &= \text{tr} \left(\exp(-tA^*A)(x, x) - \exp(-tAA^*)(x, x) \right) \\
\text{and } a_0(x) &= \text{LIM}_{t \rightarrow 0} \text{tr} \left(\exp(-tA^*A)(x, x) - \exp(-tAA^*)(x, x) \right).
\end{aligned}$$

The density a_0 is a cusp density, *i.e.* a smooth density on X times x^{-2} . We do not claim that the diverging terms vanish; however their integrals on the slices $\{x = \text{constant}\}$ vanish.

Proof. The first statement is a particular case of [27, Proposition 16]; we recall the argument briefly. Let $q_1(\tau)$ and $q_2(\tau)$ denote the meromorphic extension of the restriction to the diagonal of the Schwartz kernel of $Q_1^{-\tau}$ and $Q_2^{-\tau}$:

$$q_j(\tau)(x) := Q_j^{-\tau}(x, x).$$

These meromorphic families of densities are easily seen to be regular at $\tau = 0$. Indeed, as in Lemma 1, the possible residue at $\tau = 0$ of $q_j(\tau)(x)$ can be computed in terms of the $-n^{\text{th}}$ homogeneous component of the full symbol of the identity operator:

$$\text{Res}_{\tau=0} q_j(\tau)(x) = \frac{1}{(2\pi)^n} \int_{S_x^* X} I_{-n}(x, \xi) d\xi.$$

But of course the full symbol of the identity is concentrated in homogeneity 0. Hence (25) gives

$$\overline{\text{AS}}(A) = \left(\int_X (q_2(0) - q_1(0)) x^z \right)_{z=0}.$$

By (25), the density $q_2(0) - q_1(0)$ is locally defined and depends only on the homogeneous components of the full symbol of A of homogeneity at least $-n$. Since the constant term in the heat expansion is also locally defined, we can consider that we work on a closed manifold (we can achieve this by deforming the operator A outside a ball centered at a fixed point p , and then using the so-called double construction, in which we glue two copies of X along the boundary to obtain a smooth manifold without boundary).

Then the relationship between complex powers and the heat semigroup is given by the Mellin transform:

$$\Gamma(\tau/2) Q_j^{-\tau} = \int_0^\infty t^{\tau/2-1} e^{-tQ_j^2} dt.$$

In particular, on the diagonal for $\Re(\tau) > n$, the Schwartz kernels are related by

$$\Gamma(\tau/2)(q_2(\tau)(x) - q_1(\tau)(x)) = \int_0^\infty t^{\tau/2-1} \text{tr}(e^{-t(A^*A + p_{\ker A})}(x, x) - e^{-t(AA^* + p_{\ker A^*})}(x, x)) dt.$$

We are interested in the residue at $\tau = 0$ of both sides. Since the Gamma function has a pole with residue 1, the left-hand side equals twice the zeta-function density from (25). The residue at the pole $\tau = 0$ in the integral from the right-hand side equals precisely twice the constant term in the pointwise supertrace of the heat asymptotics. Since $e^{-t(A^*A + p_{\ker A})} = e^{-tA^*A} + (e^t - 1)p_{\ker A}$, we see that the projections on $\ker A$, $\ker A^*$ do not contribute to this constant term. We conclude that $q_2(0) - q_1(0)$ equals the heat density a_0 defined above.

Let us now show that the density $f(\epsilon) := \int_{\{x=\epsilon\}} a_0$ is integrable as a function of ϵ (this is of course weaker than claiming that a_0 is integrable). We know that a_0 is a smooth multiple of $x^{-2} dx$. This gives the following Laurent expansion for f at $\epsilon = 0$:

$$f(\epsilon) = \epsilon^{-2} f_{-2} + \epsilon^{-1} f_{-1} + h(\epsilon)$$

where $f_j \in \mathbb{R}$ and h is smooth at $\epsilon = 0$.

By Lemma 6, the sub-leading term f_{-1} equals the coefficient of $z^{-1}\tau^0$ in the Laurent expansion of the meromorphic function $\text{Tr}((Q_2^{-\tau} - Q_1^{-\tau})x^z)$ at $\tau = 0, z = 0$. Arguing as in Equation (25),

$$\begin{aligned} \text{Res}_{z=0} \left(\text{Tr}((Q_2^{-\tau} - Q_1^{-\tau})x^z)_{\tau=0} \right) &= \text{Res}_{z=0} \left(\text{Tr}((Q_2^{-\tau} - Q_1^{-\tau})ABx^z)_{\tau=0} \right) \\ &= \text{Res}_{z=0} \left(\text{Tr}((Q_2^{-\tau}AB - BQ_2^{-\tau}A)x^z)_{\tau=0} \right) \\ &= \text{Res}_{z=0} \left(\text{Tr}(Q_2^{-\tau}(AB - BA)x^z)_{\tau=0} \right) \end{aligned}$$

(we can freely commute across x^z because every commutator with x^z vanishes at $z = 0$, so such a commutation introduces a factor of z which annihilates the residue at $z = 0$). This residue vanishes by Lemma 3 because $[A, B] \in \mathcal{I}$.

From Equation (16) for $\Re(\tau) > n$ and then by meromorphic extension, the leading term f_{-2} equals

$$(27) \quad \int_{\partial X} (q_2(0) - q_1(0)) = \frac{1}{2\pi} \left(\int_{\mathbb{R}} \text{Tr}(\mathcal{N}(Q_2)^{-\tau}(\xi) - \mathcal{N}(Q_1)^{-\tau}(\xi)) d\xi \right)_{\tau=0}.$$

Denote by A_0 the indicial operator $\mathcal{N}(A)$ of A (cf. Equation 14). For every fixed ξ , the operators

$$\mathcal{N}(Q_2)^{-\tau}(\xi) = (A_0^*(\xi)A_0(\xi))^{-\tau/2} \quad \text{and} \quad \mathcal{N}(Q_1)^{-\tau}(\xi) = (A_0(\xi)A_0^*(\xi))^{-\tau/2}$$

are conjugate via the invertible operator $A_0(\xi) \in \Psi^1(\partial X)$, that is,

$$\mathcal{N}(Q_1)^{-\tau} = A_0 \mathcal{N}(Q_2)^{-\tau} A_0^{-1}.$$

Hence the traces in (27) vanish for each ξ when $\Re(\tau) \gg 0$. The unique continuation property then gives $f_{-2} = 0$. \square

Let us now analyze the ‘eta’ contribution $\eta(A)$ from the index formula (20).

Proposition 13. *Let $A \in x^p \Psi_c^m(X, E)$ have invertible image in \mathcal{A}_∂ . The boundary contribution $\widehat{\text{Tr}}_\partial(A[\log x, A^{-1}])$ from the index formula (20) depends only on the indicial families of A and Q , and is given explicitly by*

$$\widehat{\text{Tr}}_\partial(A[\log x, A^{-1}]) = \frac{1}{\pi i} \left(\int_{\mathbb{R}} \text{Tr}(A_0 \partial_\xi(A_0^{-1}) Q_0^{-\tau}) d\xi \right)_{\tau=0}.$$

Proof. Notice that $[\log x, A^{-1}] \in x^{-p+1} \Psi_c^{-m-1}(X, E)$, therefore

$$A[\log x, A^{-1}] \in x \Psi_c^{-1}(X).$$

Set $A_0 := \mathcal{N}(x^{-p}A)$. Then

$$\mathcal{N}(x^{p-1}[\log x, A^{-1}]) = -i \partial_\xi(A_0^{-1}).$$

The formula follows from Equation (16) and Lemma 6 \square

Again, for this to have any significance we fix the auxiliary operator Q in terms of A . If $A \in \Psi_c^m(X, E)$, we take $Q = Q_1$ defined by Equation (23).

Lemma 14. *Let $M := \partial X$. Assume that A is a first-order cusp differential operator “of Dirac type” near the boundary, in the sense that the indicial operator $\mathcal{N}(A)$ satisfies*

$$\mathcal{N}(A)(\xi) = \nu(i\xi + D)$$

where ν is a unitary transformation of $E|_M$, and D is a self-adjoint first-order differential operator on $C^\infty(M, E)$. Then $\eta(A)$, defined with the auxiliary operator Q_1 from (23), equals the Atiyah-Patodi-Singer η -invariant of D .

Proof. Implicitly we have fixed a metric h on M . Define a cusp metric on the interior of X by

$$g = \frac{dx^2}{x^4} + h.$$

Then $\mathcal{N}(A^*) = A_0^* = (-i\xi + D)\nu^{-1}$ and $A_0^{-1} = A_0^* Q_0^{-2}$. From the definition, $\mathcal{N}(Q) = Q_0 = \nu(\xi^2 + D^2)^{1/2} \nu^*$. From Proposition 13, $\eta(A) = \eta(A, \tau)_{\tau=0}$, where

$$\begin{aligned} \eta(A, \tau) &= \frac{1}{\pi i} \int_{\mathbb{R}} \text{Tr}(-i\nu A_0^{-1} Q_0^{-\tau}) d\xi \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \text{Tr}((i\xi + D)(\xi^2 + D^2)^{-\frac{\tau}{2}-1}) d\xi \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \text{Tr}(D(\xi^2 + D^2)^{-\frac{\tau}{2}-1}) d\xi \end{aligned}$$

since the remainder is an integral odd in ξ . By decomposing the trace onto the eigenspaces of D we get

$$\begin{aligned} \eta(A, \tau) &= \frac{1}{\pi} \sum_{\lambda \in \text{Spec}(D)} \int_{\mathbb{R}} \lambda(\xi^2 + \lambda^2)^{-\frac{\tau}{2}-1} d\xi \\ &= \eta(D, \tau) \frac{h(\tau)}{\pi} \end{aligned}$$

where $h(\tau) := \int_{\mathbb{R}} (1 + \xi^2)^{-\frac{\tau}{2}-1} d\xi$. Clearly h is regular at $\tau = 0$ and $h(0) = \pi$. From (22) we know that $\eta(A, \tau)$ is regular at $\tau = 0$ (or in other words, the residue at 0 of the eta function is cobordism-invariant; in fact this residue always vanishes on closed manifolds). This gives the result. \square

Remark 15. Examples of operators A as in Lemma 14 are the Dirac operators associated to a product-type cusp metric

$$g = \frac{dx^2}{x^4} + g^M(x).$$

In even dimensions, the automorphism ν is *not* Clifford multiplication by the normal unit vector field (or equivalently, unit 1-form) to M . Indeed, let $\Sigma^+ \oplus \Sigma^-$ be the spin bundle over M . Close to the boundary, we can write the chiral Dirac operator D_g^+ associated to the metric g and the given spin structure as

$$D_g^+ = c(dx/x^2)(x^2\partial_x + D^M),$$

acting from Σ^+ to Σ^- . We need additionally to identify the positive and the negative spinor bundles through Clifford multiplication by a unit-length cusp vector field v in cTX (such a vector field exists by Remark 9). Then $c(v)D_g^+$ acts in $C^\infty(X, \Sigma^+)$ and satisfies the hypothesis of Lemma 14 for $\nu = c(v)c(dx/x^2)$ on M , and $D = D^M$. Note that for index purposes, multiplication by $c(v)$ does not matter.

Thus the index formula (20) has, for operators as in Lemma 14, the same form as the Atiyah-Patodi-Singer formula. It is more general since we assume neither that we have a product decomposition near the boundary like Atiyah-Patodi-Singer do, nor that the operator has “b-asymptotics” near the boundary as in Melrose [35]. We remark here that Taylor series in the cusp setting correspond to power series in the logarithm of the boundary defining function y in the b-setting, by the change of variables $y = \exp(-1/x)$.

For spin Dirac operators, we arrive at an index formula which generalizes that of Melrose [35]. We assume that g is as in Equation (1), namely, $g = a((x^{-2}dx + \alpha(x))^2 + g^M(x))$, $M = \partial X$. We assume that the function a was extended to a positive function on X .

Theorem 16. *Let X be a compact (even-dimensional) Spin manifold with boundary, with a Riemannian metric in the interior of the form $x^{2p}g$, with g as in Equation (1) and $p \geq 0$ a real parameter.*

Assume that the metric is closed in the sense that $\alpha(0)$ is a closed 1-form on $M = \partial X$. Assume moreover that the Dirac operator D^M on M with respect to the induced spin structure and the metric $g^M := g^M(0)$ is invertible. Then the Dirac operator on (X°, g_p) is essentially self-adjoint and Fredholm as an unbounded operator in $L^2(X^\circ, \Sigma, g_p)$; its kernel is independent of p , and the index of its chiral part D_g^+ equals

$$\text{Ind}(D_g^+) = \int_X \hat{A}(g_p) - \frac{1}{2}\eta(D^M).$$

Before proceeding to the proof of Theorem 16, we need to prove some Lemmas. We begin with the following lemma from [41] (see also [35, 48]).

Lemma 17. *The indicial operator $\mathcal{N}(D_g^+)$ of D_g^+ is invertible for all $\xi \in \mathbb{R}$ if, and only if, the boundary operator D^M is invertible.*

Proof. Recall that $g = a^2((x^{-2}dx + \alpha(x))^2 + g^M(x))$, with $a > 0$ a smooth function on X . The “conformal invariance” of the Dirac operator gives $D_g = a^{-\frac{n+1}{2}}Da^{\frac{n-1}{2}}$, where D is the Dirac operator associated to the metric $a^{-2}g$ (see also [1, 19, 20, 47]).

Therefore the normal operator of D_g equals

$$\mathcal{N}(D_g^+) = a^{-\frac{n+1}{2}}\mathcal{N}(D_{a^{-2}g}^+)a^{\frac{n-1}{2}}.$$

Clearly, $\mathcal{N}(D_g^+)$ is invertible if, and only if, $\mathcal{N}(D_{a^{-2}g}^+)$ is invertible. The normal operator of $D_{a^{-2}g}^+$ was computed in [41]

$$(28) \quad \mathcal{N}(D_{a^{-2}g}^+) = c(x^2\partial_x)(D^M + i\xi(1 - c(\alpha))).$$

Thus $\mathcal{N}(D_g^+)$ is invertible if, and only if, $D^M + i\xi(1 - c(\alpha))$ is invertible for all ξ . Now the self-adjoint part $D_0^M - i\xi c(\alpha)$ clearly commutes with the skew-adjoint component $i\xi$; thus

$$(D^M + i\xi(1 - c(\alpha)))^*(D^M + i\xi(1 - c(\alpha))) = (D^M - i\xi c(\alpha))^2 + \xi^2.$$

This is strictly positive for $\mathbb{R} \ni \xi \neq 0$, and is invertible for $\xi = 0$ if and only if D^M is invertible. \square

Lemma 18. *It is enough to prove Theorem 16 for $p = 0$ and $a \equiv 1$.*

Proof. The form \hat{A} is conformally invariant, so the right-hand side does not depend on p and a . We proved that D_g is fully elliptic (*i.e.* elliptic with invertible image in \mathcal{A}). Then by the main result of [42], the kernel of the Dirac operator (and thus also the index) does not change when we pass from g to g_p for positive p , and even less when we multiply the metric by the innocuous factor a . Let us recall the argument briefly: after a unitary conjugation, we can write

$$D_{g_p}^+ = x^{-p/2} D_g^+ x^{-p/2}.$$

Thus the map

$$\ker(D_g^+) \rightarrow \ker(D_{g_p}^+), \quad \psi \mapsto x^{p/2} \psi$$

is formally an isomorphism. On general L^2 spinors, the inverse of this map may be ill-defined; however, since our operators are fully elliptic, elements in their L^2 null-space decay faster than any power of x . \square

Let $\phi(x)$ be a smooth function which equals 0 near $x = 1$ and $\phi \equiv 1$ near $x = 0$. Consider the family of metrics on X given by

$$g_\epsilon := \left(\frac{dx}{x^2} + \epsilon \phi(x) \alpha \right)^2 + g^M.$$

This provides a smooth deformation of $g = g_1$ into $g_0 = dx^2/x^4 + g^M$ through closed cusp metrics; by Lemma 19 applied to g_ϵ , the associated Dirac operators are fully elliptic for every ϵ , thus the index is constant under this deformation. From Equation (20) and Proposition 12 applied to the operator $c(v)D_{g_\epsilon}^+$ (where $c(v)$ is the Clifford multiplication from Remark 15), we deduce that

$$(29) \quad \text{Ind}(D_{g_\epsilon}) = \int_X \hat{A}(g_\epsilon) - \frac{1}{2} \eta(c(v)D_{g_\epsilon}).$$

For $\epsilon = 0$, the metric g_0 is a product cusp metric near the boundary. By Lemma 14 and Remark 15, the term $\eta(c(v)D_{g_0})$ equals $\eta(D^M)$, the eta invariant of the metric g^M .

Lemma 19. *The contraction of the differential form $\hat{A}(g_\epsilon)$ with $x^2 \partial_x$ vanishes on $M \times [0, 1]$.*

Proof. For notational simplicity work with $\epsilon = 1$. Also for simplicity, we perform the change of variables $y := 1/x \in [1, \infty)$ to obtain

$$g = (dy - \phi(y)\alpha)^2 + g^M.$$

The inclusion of $M := \partial X \subset X$ defines an embedding $TM \subset TX$ of tangent spaces. Let $M \times [0, 1] \simeq U \subset X$ be a tubular neighborhood of M such that $\phi = 0$ outside U . The embedding $TM \subset TX$ then allows us to extend sections of M to be constant in the $t \in [0, 1]$ variable to obtain an inclusion $\Gamma(TM) \subset \Gamma(TU)$. Next, we regard $\alpha \in \Gamma(\Lambda^2 T^*X) \subset \text{End}(TX)$ using the Riemannian metric on X to identify T^*X with TX . This leads to the embedding

$$\Gamma(TM) \hookrightarrow \Gamma(TX), \quad U \mapsto \tilde{U} := U + \phi(y)\alpha(U)\partial_y.$$

We remark that \tilde{U} is perpendicular to ∂_y at every point. Note that

$$[\partial_y, \tilde{U}] = \phi'(y)\alpha(U)\partial_y.$$

We compute

$$[\tilde{U}, \tilde{V}] = [U, V] + \phi(y)U(\alpha(V) - V(\alpha(U)))\partial_y = \widetilde{[U, V]} + \phi(y)d\alpha(U, V).$$

The fact that α is closed is therefore equivalent to

$$[\tilde{U}, \tilde{V}] = \widetilde{[U, V]}.$$

We claim that

$$(30) \quad \nabla_{\tilde{U}} \partial_y = 0.$$

Indeed, ∂_y has constant length 1 so $g(\nabla_{\tilde{U}}\partial_y, \partial_y) = 0$. Moreover, for every vector fields U, V on M (constant in y) we have

$$g(\tilde{U}, \partial_y) = 0, \quad g(\tilde{U}, \tilde{V}) = g^M(U, V), \quad [\partial_y, \tilde{U}] \perp \tilde{V}, \quad \text{and} \quad [\tilde{U}, \tilde{V}] \perp \partial_y.$$

From the Cartan formula for the Levi-Civita connection we see that $g(\nabla_{\tilde{U}}\partial_y, \tilde{V}) = 0$. Thus $\nabla_{\tilde{U}}\partial_y = 0$ as claimed. This implies that

$$(31) \quad R_{\tilde{U}\tilde{V}}\partial_y = 0.$$

We now show that Equation (31) implies $\partial_{y \lrcorner} \text{tr}(R^{2k}) = 0$.

We shall compute $\partial_{y \lrcorner} \text{tr}(R^{2k})$ using an orthonormal frame $\{Y_j\}$ on M and the induced orthonormal frame $\{\partial_y, \tilde{Y}_j\}$ on $M \times [0, 1]$. We denote the curvature coefficients in this frame by

$$R_{ijkl} := \langle R_{\tilde{Y}_i \tilde{Y}_j} \tilde{Y}_k, \tilde{Y}_l \rangle, \quad R_{y j k l} := \langle R_{\partial_y \tilde{Y}_j} \tilde{Y}_k, \partial_y \rangle,$$

etc. For the contraction $\partial_{y \lrcorner} \text{tr}(R^{2k})$ to be nonzero, we need the contribution of a curvature endomorphism $R_{y j} := R_{\partial_y \tilde{Y}_j}$ for some j . However, by (31) and the curvature identities, $R_{y j k l} = 0$. Thus any nonzero contribution must come from a monomial which contains $R_{y j y k}$ or $R_{y j k y}$ for some k . Now for the trace of such a monomial to be nonzero, at least one coefficient y must appear among the last two coefficients of another curvature term appearing in the monomial. Moreover, for the product in the exterior algebra to be nonzero, y is not allowed to appear among the first two coefficients of such a term. Then again by (31), such a coefficient vanishes, so the lemma follows. \square

We are ready now to prove Theorem 16.

Proof. By Lemma 17, the Dirac operator of our metric is fully elliptic and hence it is Fredholm. For $p \geq 0$, D_p is essentially self-adjoint (see [41, 42]) and its domain is precisely the Sobolev space $x^p H_c^1(X, \Sigma)$. So there is no ambiguity when speaking of the index of D^+ .

By Lemma 18, it is enough to prove the theorem for $p = 0$ and $a \equiv 1$ near the boundary.

We now make an additional simplification. Let $\phi(x)$ be a smooth function which equals 0 near $x = 1$ and $\phi \equiv 1$ near $x = 0$, as above. Let us deform smoothly g as follows: we linearly deform the family of metrics $g^M(x)$ into the constant family g^M for $x \in [0, 1]$, and we linearly deform $\alpha(x)$ into $\phi(x)\alpha$ (recall that we denoted $g^M = g^M(0)$, $\alpha = \alpha(0)$). This deformation leads to a deformation of the Dirac operator through Fredholm operators, since the normal operator depends only on g^M and α at the boundary so it is left unchanged. We deduce that the index does not change during this deformation. The 'eta' boundary contribution depends only on g^M and α so it is also left unchanged. From the abstract index formula of proposition 8, we conclude that the integral of the local density is also left unchanged. Thus it is enough to assume in the rest of the proof that the metric near the boundary equals

$$g = \left(\frac{dx}{x^2} + \phi(x)\alpha \right)^2 + g^M,$$

where α, g^M are independent of x .

By Lemma 19, we see that the volume form component of $\hat{A}(g_\epsilon)$ does not depend on ϵ , since it is supported outside the support of ϕ . Thus $\text{Ind}(D_{g_\epsilon})$ (which is constant in ϵ) is computed by our abstract index formula as an integral independent of ϵ , plus a boundary term. It follows that the boundary term itself is constant in ϵ and can be computed at $\epsilon = 0$, where it yields $\eta(D^M)$. The local density is independent of ϵ so it can be computed at $\epsilon = 1$ where it gives $\hat{A}(g)$. This completes the proof. \square

5. HOMOLOGY GROUPS RESULTS

The spectral sequence methods of [11] allow us to determine the Hochschild homology groups of most of the algebras introduced in the previous section as appropriate cohomology groups. Morita invariance of Hochschild homology shows the calculations are the same for scalar algebras and for

algebras of operators acting between sections of a smooth vector bundle $E \rightarrow X$. The results from [39] are

$$\begin{aligned}
& \mathrm{HH}_p(\mathcal{I}_\sigma) \simeq H^p({}^c S^* X \times \mathbb{S}_\sigma), \\
& \mathrm{HH}_p(\mathcal{A}_\sigma) \simeq H_{\mathrm{rel}}^p({}^c S^* X \times \mathbb{S}_\sigma) \oplus H^p({}^c S_{\partial X}^* X \times \mathbb{S}_\sigma), \\
& \mathrm{HH}_p(\mathcal{I}_\partial) \simeq \begin{cases} \mathbb{C} & \text{for } k = 0, 1, \\ 0 & \text{otherwise} \end{cases} \\
(32) \quad & \mathrm{HH}_p(\mathcal{A}_\partial) \simeq \begin{cases} H^p({}^c S_{\partial X}^* X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) / \mathbb{C} & \text{for } p = 1, 2, \\ H^p({}^c S_{\partial X}^* X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) & \text{otherwise} \end{cases} \\
& \mathrm{HH}_p(\mathcal{A}_{\partial, \sigma}) \simeq H^p({}^c S_{\partial X}^* X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) \\
& \mathrm{HH}_p(\mathcal{A}) \simeq \begin{cases} \mathbb{C} \oplus H_{\mathrm{rel}}^p({}^c S^* X \times \mathbb{S}_\sigma) \oplus H^p({}^c S_{\partial X}^* X \times \mathbb{S}_\sigma) / \mathbb{C} & \text{for } p = 1 \\ H_{\mathrm{rel}}^p({}^c S^* X \times \mathbb{S}_\sigma) \oplus H^p({}^c S_{\partial X}^* X \times \mathbb{S}_\sigma) & \text{otherwise.} \end{cases}
\end{aligned}$$

Inspired by [39], similar results were obtained in [9, 10, 28, 29, 40]. The paper [29] computes the Hochschild homology for the fibered cusp algebra, of which the cusp algebra is a particular case. We will therefore omit the proofs of the above homology computations and focus our attention on the fact that in each case, HH_0 is 1-dimensional for X has dimension ≥ 2 and connected (to ensure that $S^* X$ is connected). The isomorphism of these spaces to \mathbb{C} is realized by suitable trace functionals. These trace functionals are:

- for \mathcal{I} the ordinary trace,
- for \mathcal{A}_σ , \mathcal{A}_∂ and $\mathcal{A}_{\partial, \sigma}$ a “double residue trace” $\mathrm{Tr}_{\partial, \sigma}$, which we define by analytic continuation,
- for \mathcal{I}_σ the Guillemin–Wodzicki residue trace denoted Tr_σ , and
- for \mathcal{I}_∂ a functional Tr_∂ induced from the trace functional $\widehat{\mathrm{Tr}}$ defined in [37] on $\Psi_{\mathrm{sus}}^Z(Y)$ for any boundaryless manifold Y (its definition is recalled below).

Analytic continuation arguments give extensions of the last two functionals to $\widehat{\mathrm{Tr}}_\sigma$ on \mathcal{A}_σ and $\widehat{\mathrm{Tr}}_\partial$ on \mathcal{A}_∂ . These are not trace functionals. Rather, commutation with the operators $\log Q$ and $\log x$ defines derivations, and the Hochschild boundaries of these functionals are given by

$$\begin{aligned}
(33) \quad & (\partial \mathrm{Tr}_\partial)(A, B) = \widehat{\mathrm{Tr}}_\partial([A, B]) = \mathrm{Tr}_{\partial, \sigma}(A[\log Q, B]) = (i_{\log Q} \mathrm{Tr}_{\partial, \sigma})(A, B) \\
& (\partial \mathrm{Tr}_\sigma)(A, B) = \widehat{\mathrm{Tr}}_\sigma([A, B]) = -\mathrm{Tr}_{\partial, \sigma}(A[\log x, B]) = -(i_{\log x} \mathrm{Tr}_{\partial, \sigma})(A, B),
\end{aligned}$$

for all $A, B \in \mathcal{A}$. The ideals \mathcal{I}_σ and \mathcal{I}_∂ are H-unital and, in (33), $\partial \mathrm{Tr}_\sigma$ and $\partial \mathrm{Tr}_\partial$ represent the images under the respective boundary maps of the functionals Tr_σ and Tr_∂ in the long exact sequences

$$\begin{aligned}
0 & \longrightarrow \mathrm{HH}^0(\mathcal{A}_{\partial, \sigma}) \longrightarrow \mathrm{HH}^0(\mathcal{A}_\sigma) \longrightarrow \mathrm{HH}^0(\mathcal{I}_\sigma) \xrightarrow{\partial} \mathrm{HH}^1(\mathcal{A}_{\partial, \sigma}) \longrightarrow \dots \\
0 & \longrightarrow \mathrm{HH}^0(\mathcal{A}_{\partial, \sigma}) \longrightarrow \mathrm{HH}^0(\mathcal{A}_\partial) \longrightarrow \mathrm{HH}^0(\mathcal{I}_\partial) \xrightarrow{\partial} \mathrm{HH}^1(\mathcal{A}_{\partial, \sigma}) \longrightarrow \dots
\end{aligned}$$

These maps give rise to an analog of the index formula (20). Indeed their sum gives a homotopy invariant of invertible elements of $\mathcal{A}_{\partial, \sigma}$, which we call the “boundary index”:

$$\mathrm{Ind}_\partial(A) = \mathrm{Bf}(A, A^{-1}), \quad \mathrm{Bf}(A, B) = (-i_{\log x} \mathrm{Tr}_{\partial, \sigma} + i_{\log Q} \mathrm{Tr}_{\partial, \sigma})(A, B).$$

This represents an obstruction for lifting an invertible element $A \in \mathcal{A}_{\partial, \sigma}$ to an invertible element of \mathcal{A} .

The index functional itself, defined as the image of Tr under the boundary map $\partial : \mathrm{HH}^0(\mathcal{I}) \rightarrow \mathrm{HH}^1(\mathcal{A})$, can be expressed in terms of derivations in Hochschild homology

$$\mathrm{IF}(A, B) = \widehat{\mathrm{Tr}}_\sigma([\log Q, B]A) - \widehat{\mathrm{Tr}}_\partial(A[\log x, B]).$$

The cyclic homology groups are obtained from the SBI exact sequence [13, 21, 31, 56] using the computations of Hochschild homology groups.

The homological approach to index formulas on singular or non-compact spaces has the advantage that it shows (based on excision in periodic cyclic homology) that the index can be expressed as the sum of an interior, local term and a term that depends only on the behavior of our operator at infinity. The local term can then be identified with the Atiyah-Singer integrand as in the present paper. The term at infinity will also depend on suitable indicial operators. In general, these indicial operators are defined using groupoids [3, 23, 46].

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