ON PLURICANONICAL LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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Abstract. We give a short proof of the fact that compact pluricanonical locally conformally Kähler manifolds have parallel Lee form.

1. Introduction

A locally conformally Kähler (lcK) manifold is a complex manifold (M, J) together with a Hermitian metric g which is conformal to a Kähler metric in the neighbourhood of every point. The logarithmic differentials of the conformal factors glue up to a globally defined closed 1-form θ, called the *Lee form*, such that the fundamental 2-form $\Omega := g(J\cdot, \cdot)$ satisfies

$$\mathrm{d}\Omega = 2\theta \wedge \Omega. \tag{1}$$

When θ is parallel with respect to the Levi-Civita connection ∇ of g, the lcK manifold (M, J, g) is called *Vaisman*. G. Kokarev introduced in the context of harmonic maps [6] the seemingly larger class of *pluricanonical lcK manifolds*, defined as those lcK manifolds (M, g, J) for which the covariant derivative of the metric dual $\xi := \theta^\sharp$ of the Lee form anti-commutes with the complex structure J:

$$\nabla_{JX}\xi = -J\nabla_X\xi, \qquad \forall X \in \mathrm{T}M. \tag{2}$$

In their recent preprint [10], L. Ornea and M. Verbitsky announce the proof of the following result:

Theorem 1. *Every compact pluricanonical lcK manifold (M, J, g) is Vaisman.*

The arguments given in [10] are based on an impressive amount of previous results by numerous authors. Among these we mention: the classification of complex surfaces by Kodaira, the classification of complex surfaces of Kähler rank 1 by Chiose and Toma [3] and Brunella [2], some results by M. Kato concerning subvarieties of Hopf manifolds [5], the classification of surfaces carrying Vaisman metrics by Belgun [1], as well as several previous results by Ornea, Verbitski and their collaborators [4], [8], [9].

As a matter of fact, while were not able to follow their arguments in detail, we discovered instead that Theorem 1 can be proved in a more direct way. Our idea is based on the

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observation that on a pluricanonical manifold the metric duals ξ and $J\xi$ of θ and $J\theta$ commute, and this eventually shows that some partial Laplacian (in the analists sense) of the square norm of $\nabla\xi$ is larger than or equal to the square norm of $\nabla\xi\circ\nabla\xi$. By looking at a point where $|\nabla\xi|^2$ is maximal, this shows that $\nabla\xi\equiv 0$. The next section contains the details of our proof.

In the final section we remark that our argument shows the existence, in every complete lcK manifold which is pluricanonical but not Vaisman, of a totally geodesic embedded holomorphic curve isometric either to the euclidean plane or to a flat cylinder.

2. Proof of Theorem 1

It is well known that on Hermitian manifolds, the exterior derivative of the fundamental 2-form determines its covariant derivative. The formula for the covariant derivative of J determined by (1) is (see e.g. [7]):

$$\nabla_X J = X\wedge J\theta + JX\wedge\theta, \qquad \forall X\in \mathrm{T}M, \tag{3}$$

where if α is a 1-form, $J\alpha$ denotes the 1-form defined by $(J\alpha)(Y) := -\alpha(JY)$ for all tangent vectors Y, and $X\wedge\alpha$ is the endomorphism of the tangent bundle defined by

$$(X\wedge\alpha)(Y) := g(X,Y)\alpha^\sharp - \alpha(Y)X.$$

Consider the symmetric endomorphism $S := \nabla\xi$. We need to show that, under the compactness assumption, (2) implies the vanishing of S. By (3) we have

$$\nabla_X\xi = SX, \qquad \nabla_X(J\xi) = JSX + \theta(X)J\xi + \theta(JX)\xi - |\theta|^2 JX, \tag{4}$$

which by lowering the indices also reads

$$\nabla_X\theta = (SX)^\flat, \qquad \nabla_X(J\theta) = (JSX)^\flat + X\lrcorner(\theta\wedge J\theta - |\theta|^2\Omega) \tag{5}$$

Since $SJ=-JS$ by (2), the endomorphism JS is symmetric. From (5) we thus get

$$\mathrm{d}(J\theta) = 2(\theta\wedge J\theta - |\theta|^2\Omega), \tag{6}$$

$$\mathcal{L}_\xi g = 2g(S\cdot,\cdot), \qquad \mathcal{L}_{J\xi} g = 2g(JS\cdot,\cdot). \tag{7}$$

Taking a further exterior derivative in (6) and using (1) yields

$$0 = \mathrm{d}^2(J\theta) = 2\left(-\theta\wedge \mathrm{d}(J\theta) - \mathrm{d}(|\theta|^2)\wedge\Omega - |\theta|^2\mathrm{d}\Omega\right) = -2\mathrm{d}(|\theta|^2)\wedge\Omega,$$

whence $|\theta|^2$ is constant on M. We thus obtain for every tangent vector X

$$0 = X(|\xi|^2) = 2g(\nabla_X\xi,\xi) = 2g(SX,\xi) = 2g(S\xi,X),$$

showing that $S\xi = 0$ (and therefore also $SJ\xi = 0$ using (2)). From (4) we thus get $\nabla_{J\xi}\xi = \nabla_\xi(J\xi) = \nabla_{J\xi}(J\xi) = \nabla_\xi\xi = 0$, and in particular

$$[\xi, J\xi] = 0. \tag{8}$$

We note for later use that the distribution $\{\xi, J\xi\}$ is integrable, and its integral leaves are totally geodesic.

Cartan's formula shows that on every lcK manifold

$$\mathcal{L}_{J\xi}\Omega = \mathrm{d}(J\xi \lrcorner \Omega) + J\xi \lrcorner \mathrm{d}\Omega = -\mathrm{d}\theta + 2J\xi \lrcorner (\theta \wedge \Omega) = 0. \tag{9}$$

Moreover, on pluricanonical manifolds, equation (6) gives

$$\mathcal{L}_{\xi}\Omega = \mathrm{d}(\xi \lrcorner \Omega) + \xi \lrcorner \mathrm{d}\Omega = \mathrm{d}(J\theta) + 2\xi \lrcorner (\theta \wedge \Omega) = 0. \tag{10}$$

From (7) and (10) we infer

$$\mathcal{L}_{\xi} J = 2JS, \qquad \mathcal{L}_{J\xi} J = -2S. \tag{11}$$

We notice that (8) implies $[\mathcal{L}_{\xi}, \mathcal{L}_{J\xi}] = \mathcal{L}_{[\xi, J\xi]} = 0$, and thus from (11):

$$\mathcal{L}_{\xi} S = -\frac{1}{2}\mathcal{L}_{\xi}\mathcal{L}_{J\xi} J = -\frac{1}{2}\mathcal{L}_{J\xi}\mathcal{L}_{\xi} J = -\mathcal{L}_{J\xi}(JS) = 2S^2 - J\mathcal{L}_{J\xi} S, \tag{12}$$

which (after composing with J on the left) also reads

$$\mathcal{L}_{J\xi} S = J\mathcal{L}_{\xi} S - 2JS^2. \tag{13}$$

Taking a further Lie derivative in (12) and using (11) yields

$$\begin{aligned}\mathcal{L}_{J\xi}\mathcal{L}_{\xi} S &= 2S\mathcal{L}_{J\xi} S + 2(\mathcal{L}_{J\xi} S)S + 2S\mathcal{L}_{J\xi} S - J\mathcal{L}^2_{J\xi} S \\ &= 4S\mathcal{L}_{J\xi} S + 2(\mathcal{L}_{J\xi} S)S - J\mathcal{L}^2_{J\xi} S.\end{aligned}$$

Similarly, from (13) and (11) we obtain:

$$\begin{aligned}\mathcal{L}_{\xi}\mathcal{L}_{J\xi} S &= 2JS\mathcal{L}_{\xi} S + J\mathcal{L}^2_{\xi} S - 4JS^3 - 2J(\mathcal{L}_{\xi} S)S - 2JS\mathcal{L}_{\xi} S \\ &= J\mathcal{L}^2_{\xi} S - 4JS^3 - 2J(\mathcal{L}_{\xi} S)S \\ &= J\mathcal{L}^2_{\xi} S - 8JS^3 - 2(\mathcal{L}_{J\xi} S)S.\end{aligned}$$

Comparing the last two equations and using $\mathcal{L}_{\xi}\mathcal{L}_{J\xi} = \mathcal{L}_{J\xi}\mathcal{L}_{\xi}$ we obtain

$$J(\mathcal{L}^2_{\xi} S + \mathcal{L}^2_{J\xi} S) = 4S\mathcal{L}_{J\xi} S + 4(\mathcal{L}_{J\xi} S)S + 8JS^3. \tag{14}$$

We compose with $-SJ$ to the left and take the trace in the above equation:

$$\mathrm{tr}(S(\mathcal{L}^2_{\xi} S + \mathcal{L}^2_{J\xi} S)) = -4\mathrm{tr}(SJS(\mathcal{L}_{J\xi} S)) - 4\mathrm{tr}(SJ(\mathcal{L}_{J\xi} S)S) + 8\mathrm{tr}(S^4) = 8\mathrm{tr}(S^4),$$

from the trace identity and the hypothesis $SJ = -JS$. Using this we compute:

$$\begin{aligned}(\mathcal{L}^2_{\xi} + \mathcal{L}^2_{J\xi})(\mathrm{tr}(S^2)) &= \mathrm{tr}\left((\mathcal{L}^2_{\xi} S)S + 2(\mathcal{L}_{\xi} S)^2 + S(\mathcal{L}^2_{\xi} S) + (\mathcal{L}^2_{J\xi} S)S + 2(\mathcal{L}_{J\xi} S)^2 + S(\mathcal{L}^2_{J\xi} S)\right) \\ &= 2\mathrm{tr}((\mathcal{L}_{\xi} S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi} S)^2) + 2\mathrm{tr}\left(S(\mathcal{L}^2_{\xi} S) + S(\mathcal{L}^2_{J\xi} S)\right) \\ &= 2\mathrm{tr}((\mathcal{L}_{\xi} S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi} S)^2) + 16\mathrm{tr}(S^4).\end{aligned}$$

By taking the Lie derivative of the relation $g(S\cdot, \cdot) = g(\cdot, S\cdot)$ with respect to ξ and using (7) we immediately get $g(\mathcal{L}_{\xi} S\cdot, \cdot) = g(\cdot, \mathcal{L}_{\xi} S\cdot)$, i.e., the endomorphism $\mathcal{L}_{\xi} S$ is symmetric. Taking now the Lie derivative of the relation $SJ + JS = 0$ with respect to ξ and using (12) we obtain that $\mathcal{L}_\xi S$ anti-commutes with J. Finally, (13) shows that the symmetric part of $\mathcal{L}_{J\xi}S$ is $J\mathcal{L}_\xi S$ and its skew-symmetric part is $-2JS^2$. The previous relation thus reads

$$\begin{aligned}(\mathcal{L}_\xi^2+\mathcal{L}_{J\xi}^2)(\mathrm{tr}(S^2)) &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2)+2\mathrm{tr}((\mathcal{L}_{J\xi}S)^2)+16\mathrm{tr}(S^4)\\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2)+2\mathrm{tr}((\mathcal{L}_\xi S)^2)-4S^4)+16\mathrm{tr}(S^4)\\ &= 4\mathrm{tr}((\mathcal{L}_\xi S)^2)+8\mathrm{tr}(S^4).\end{aligned}$$

We use now the compactness assumption: there exists a point $x_{max}\in M$ where $\mathrm{tr}(S^2)$, the square norm of S, attains its supremum. At x_{max} the left hand side of the equation above is non-positive, while the right hand side is non-negative (since we have seen that $\mathcal{L}_\xi S$ is symmetric). We deduce that $\mathrm{tr}(S^4)$ – and thus S itself – both vanish at x_{max}, so S vanishes identically. This is the conclusion of Theorem 1.

3. Non-compact pluricanonical manifolds

Our method of proof extends partially to the case where the pluricanonical manifold (M,J,g) is complete but not compact.

Theorem 2. *Let (M,J,g) be a complete pluricanonical manifold which is not Vaisman. Then there exists a totally geodesic holomorphic curve embedded in M and isometric either to the standard $\mathbb{R}^2$ or to a flat cylinder $\mathbb{R}^2/l\mathbb{Z}$ for some radius $l>0$.*

Proof. The leaves of the foliation tangent to the totally geodesic distribution $\{\xi, J\xi\}$ are totally geodesic holomorphic curves, and consist of complete flat surfaces. The universal cover of such a surface is isometric to the euclidean plane, hence each leaf is isomorphic (as Kähler manifold) to either $\mathbb{R}^2$, a flat cylinder, or a flat torus.

Over a compact leaf, the endomorphism S vanishes by the same argument as in the last paragraph of the proof of Theorem 1. So if (M,J,g) is not Vaisman, we must have at least one non-compact leaf, hence either a flat $\mathbb{R}^2$ or a cylinder isometrically and holomorphically embedded in M. ☐

We do not know whether there exist complete non-compact pluricanonical manifolds which are not Vaisman.

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